A definable nonstandard universe

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Introduction

How "**non-constructive**" is the use of nonstandard methods?

1. Nonstandard universes depend on the **Ultrafilter Existence Theorem** (UET) — a consequence of the Axiom of Choice (AC).

2. Infinite hyperintegers define **Lebesgue non-measurable sets** (Luxemburg 1973) — whose existence is independent from ZF (Solovay 1970).

Popular opinion **conjectured**: "*There is no definable nonstandard universe.*"

But:

1. UET $\not\Rightarrow$ AC (Banaschewski 1983)

2. One can **define** a nonstandard model of the **reals** in ZFC (Kanovei and Shelah 2004).

3. One can **define** a fully-fledged **nonstandard universe** in ZFC (H. 2008).

Terminology

The **superstructure** $V(M)$ over a set M is

$$V(M) := \bigcup_{n\in\mathbb{N}_0} V_n(M)$$

wherein

$$V_0(M) := M,$$

$$\forall n \in \mathbb{N} \quad V_n(M) := V_{n-1}(M) \cup \mathcal{P}\left(V_{n-1}(M)\right).$$

We shall always treat **reals as atoms**:

$v \notin r$ for all $r \in \mathbb{R}$ and every set v.

Let $\mathcal{L}_{V(\mathbb{R})}$ denote the language with

- one **constant symbol** $\dot{v}$ **for each** $v \in V(\mathbb{R})$,
- one **binary relation** $\dot{\in}$.

$V(\mathbb{R})$ is an $\mathcal{L}_{V(\mathbb{R})}$-structure:

$v^{V(\mathbb{R})} = v$ for all $v \in V(\mathbb{R})$, and $\in^{V(\mathbb{R})} = \in$.

Main result

Theorem (assuming ZFC)
There is a **definable set** ${}^*\mathbb{R}$ and a **definable embedding** ${}^* : V(\mathbb{R}) \hookrightarrow V({}^*\mathbb{R})$ such that ${}^* : V_n(\mathbb{R}) \hookrightarrow V_n({}^*\mathbb{R})$ for all $n \in \mathbb{N}_0$ and such that * is a **nonstandard embedding**. This means:

1. **Transfer Principle.** For all $\in$-formulae $\phi(v_1, \ldots, v_n)$ with bounded quantifiers and $a_1, \ldots, a_n \in V(\mathbb{R})$,
$$\phi[a_1, \ldots, a_n] \Leftrightarrow \phi\left[{}^*a_1, \ldots, {}^*a_n\right].$$

2. **Internal Definition Principle.** For all internal sets B_0, all internal $b_1, \ldots, b_n$ and all $\in$-formulae $\phi(v_0, \ldots, v_n)$, the set $\{x \in B_0 \ : \ \phi[x, b_1, \ldots, b_n]\}$ is internal.

3. **Countable Saturation Principle.** Let $C_n \neq \emptyset$ be an internal set and $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$.

Herein, a is **internal** if and only if $a \in {}^*A$ for some $A \in V(\mathbb{R})$.

Kanovei and Shelah's construction I

Definitions

- $A :=$
  $\{a : \beth_1 \to \mathcal{P}(\mathbb{N}) : a\,[\beth_1]$ ultrafilter on $\mathbb{N}\}$
- $D_a := a\,[\beth_1]$ for all $a \in A$
- $X \subset \mathbb{N}^A$ has **finite support** if and only if there is a finite $u \subset A$ such that for all $x, y \in \mathbb{N}^A$,
  $x \restriction u = y \restriction u \Rightarrow (x \in X \Leftrightarrow y \in X)$.
  u is then called a *finite support* of X.
- $\mathcal{H} := \left\{X \subset \mathbb{N}^A \;:\; X \text{ has finite support}\right\}$
- $<$ denotes the **lexicographical linear ordering** of A, based on the lexicographical ordering of $\mathcal{P}(\mathbb{N})$.

Lemma (Kanovei and Shelah 2004)

- $\mathcal{H}$ is an **algebra** of subsets of $\mathbb{N}^A$.
- Every $X \in \mathcal{H}$ has a $\subseteq$-**minimal support**, denoted $\|X\|$.
  Proof. $\|X\| = \bigcap \{u : u \text{ finite support of } X\}$.

Kanovei and Shelah's construction II

Definition (almost-all notation)

For $U \subset \mathcal{P}(\mathbb{N})$ and a well-formed formula $\Phi(i)$,

$$U \ i \ \Phi(i) :\Leftrightarrow \{i \in \mathbb{N} \ : \ \Phi(i)\} \in U.$$

Definition (iterated ultrafilter)

For $n \in \mathbb{N}$ and $u := \{a_1, \dots, a_n\} \subset A$ with $a_1 < \cdots < a_n$,

$$D_u := \left\{ X \subseteq \mathbb{N}^u : \begin{array}{c} D_{a_n} \ k_n \ \dots \ D_{a_1} \ k_1 \\ (k_1, \dots, k_n) \in X \end{array} \right\},$$

$$D := \left\{ X \in \mathcal{H} \ : \ \begin{array}{c} \{x \upharpoonright \|X\| \ : \ x \in X\} \\ \in D_{\|X\|} \end{array} \right\}.$$

Lemma (Kanovei and Shelah 2004)

D is an ultrafilter in the algebra $\mathcal{H}$.

A chain of bounded ultrapowers I

We construct a chain of $\mathcal{L}_{V(\mathbb{R})}$-structures $\mathcal{M}_\alpha := \left(M_\alpha, \left\{v^{\mathcal{M}_\alpha} : v \in V(\mathbb{R})\right\}, \in^{\mathcal{M}_\alpha}\right)$ $(\alpha \leq \aleph_1)$:

Base step. $\mathcal{M}_0 := (V(\mathbb{R}), \{v : v \in V(\mathbb{R})\}, \in)$.

Successor step. For $\alpha < \aleph_1$, $\mathcal{M}_{\alpha+1}$ is $\mathcal{M}_\alpha^{\mathbb{N}^A}/D$ (the **bounded** D**-ultrapower** of $\mathcal{M}_\alpha$):

- $M_{\alpha+1}$ is the set of all $\left[(x_g)_g\right]_D$ such that $\left\{g \in \mathbb{N}^A : \mathcal{M}_\alpha \models x_g \dot{\in} \dot{V_n(\mathbb{R})}\right\} \in D$ for some $n \in \mathbb{N}$,
- $\in^{\mathcal{M}_{\alpha+1}}$ is defined such that

$$\mathcal{M}_{\alpha+1} \models \left[(x_g)_g\right]_D \dot{\in} \left[(y_g)_g\right]_D$$
$$\Leftrightarrow \left\{g \in \mathbb{N}^A \;\; \mathcal{M}_\alpha \models x_g \dot{\in} y_g\right\} \in D$$

- the **canonical embedding** is denoted by $e_{\alpha,\alpha+1} : \mathcal{M}_\alpha \hookrightarrow \mathcal{M}_{\alpha+1}$, $x \mapsto \left[(x)_{g \in \mathbb{N}^A}\right]_D$,
- $e_{\gamma,\alpha+1} := e_{\alpha,\alpha+1} \circ e_{\gamma,\alpha}$ for all $\gamma < \alpha$, and $e_{\alpha,\alpha} := \mathrm{id}$.

Since D is an ultrafilter, $\in^{\mathcal{M}_{\alpha+1}}$ is well-defined.

A chain of bounded ultrapowers II

Limit step. For all limit ordinals λ, $\mathcal{M}_\lambda$ is the **direct limit** of all $\mathcal{M}_\alpha$, $\alpha < \lambda$:

- $M_\lambda :=$
  $$\left\{ (\alpha, x) : \begin{array}{cc} \alpha < \lambda, & x \in M_\alpha, \\ \forall \gamma < \alpha & x \notin e_{\gamma,\alpha}[M_\gamma] \end{array} \right\},$$
- $\in^{\mathcal{M}_\lambda}$ is defined such that
  $$\begin{aligned} & \mathcal{M}_\lambda \models (\alpha_0, x_0) \dot{\in} (\alpha_1, x_1) \\ \Leftrightarrow & \mathcal{M}_{\alpha_0 \vee \alpha_1} \models e_{\alpha_0, \alpha_0 \vee \alpha_1}(x_0) \dot{\in} e_{\alpha_1, \alpha_0 \vee \alpha_1}(x_1), \end{aligned}$$
- for $\alpha < \lambda$, $e_{\alpha,\lambda}$ is defined such that
  - $e_{\alpha,\lambda}(x) = (\alpha, x)$ if $x \notin e_{\gamma,\alpha}[M_\gamma]$ for all $\gamma < \alpha$,
  - $e_{\alpha,\lambda}(x) = (\gamma, y)$ if γ is the smallest ordinal $< \alpha$ such that there exists some $y \in M_\gamma$ with $x = e_{\gamma,\alpha}(y)$.

Interpretation of constants. $v^{\mathcal{M}_\alpha} = e_{0,\alpha}(v)$ for all $\alpha \leq \aleph_1$ and $v \in V(\mathbb{R})$.

From $\mathcal{M}_{\aleph_1}$ we will get the **internal universe**.

A chain of bounded ultrapowers III

Lemma (Transfer between $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$)
For all $\in$-formulae $\phi(v_1, \ldots, v_n)$ with bounded quantifiers and $\alpha < \beta \leq \aleph_1$,

$$\mathcal{M}_\beta \models \phi\left[e_{\alpha,\beta}(y_1), \ldots, e_{\alpha,\beta}(y_n)\right] \Leftrightarrow \mathcal{M}_\alpha \models \phi\left[y_1, \ldots, y_n\right]$$

for all $y_1, \ldots, y_n \in M_\alpha$.
Proof idea. Ordinal induction in β, based on the Łoś Theorem for bounded ultrapowers.

Lemma (Boundedness)
For all $y \in M_{\aleph_1}$, there exists some $n \in \mathbb{N}_0$ such that $y \in e_{0,\aleph_1}(V_n(\mathbb{R}))$.
Proof idea. Ordinal induction in β yields for all $\beta \leq \aleph_1$ and $y \in M_\beta$ some $n \in \mathbb{N}_0$ such that $y \in e_{0,\beta}(V_n(\mathbb{R}))$.

Lemma (Countable Saturation)
$\mathcal{M}_{\aleph_1}$ is $\aleph_1$-saturated.
Proof idea. Diagonal argument, using

- the boundedness of elements of $M_{\aleph_1}$ and
- the regularity of $\aleph_1$.

The nonstandard embedding

Let ${}^*\mathbb{R} := e_{0,\aleph_1}(\mathbb{R})$ be the set of **hyperreals**.

Define an **embedding** $M_{\aleph_1} \hookrightarrow V({}^*\mathbb{R})$ via $\in$-**recursion**:

$$\forall x \in {}^*\mathbb{R} \qquad j(x) = x$$

and

$$j(B) = \{j(a) : a \in M_{\aleph_1}, \quad \mathcal{M}_{\aleph_1} \models a \dot{\in} B\}$$

for all $B \in M_{\aleph_1} \setminus {}^*\mathbb{R}$.

Then, ${}^* := j \circ e_{0,\aleph_1}$ is a **definable embedding** $V(\mathbb{R}) \hookrightarrow V({}^*\mathbb{R})$.

(Note: ${}^*\mathbb{R} = j \circ e_{0,\aleph_1}(\mathbb{R}) = {}^*(\mathbb{R})$.)

$V({}^*\mathbb{R})$ is now an $\mathcal{L}_{V(\mathbb{R})}$-**structure** itself, through canonical interpretation:

- For all $x, y \in V({}^*\mathbb{R})$, let $V({}^*\mathbb{R}) \models x \dot{\in} y$ if and only if $x \in y$ and $y \notin {}^*\mathbb{R}$,
- Let $v^{V({}^*\mathbb{R})} = {}^*v$ for all $v \in V(\mathbb{R})$

The internal universe

The range of j is the internal universe:

Lemma (Internality)

1. $j\left[M_{\aleph_1}\right]$ is a transitive subclass of $V\left({}^*\mathbb{R}\right)$.
2. For all $x \in V\left({}^*\mathbb{R}\right)$, $x \in j\left[M_{\aleph_1}\right]$ if and only if there exists some $y \in V(\mathbb{R}) \setminus \mathbb{R}$ such that $x \in {}^*y$.

Proof.

1. By construction.
2. "$\Rightarrow$". By boundedness of elements of $M_{\aleph_1}$.
   "$\Leftarrow$". By transitivity of $j\left[M_{\aleph_1}\right]$.

Corollary (Transfer between $\mathcal{M}_{\aleph_1}$ and $V\left({}^*\mathbb{R}\right)$)
For all $\in$-formulae $\phi\left(v_1, \ldots, v_n\right)$ with bounded quantifiers and all $y_1, \ldots, y_n \in M_{\aleph_1}$,

$$V\left({}^*\mathbb{R}\right) \models \phi\left[j\left(y_1\right), \ldots, j\left(y_n\right)\right]$$
$$\Leftrightarrow \mathcal{M}_{\aleph_1} \models \phi\left[y_1, \ldots, y_n\right].$$

Proof. Formulae with bounded quantifiers are absolute with respect to transitive submodels.

Proof of the main result I

We prove the following:

Theorem (assuming ZFC)

1. **Definability.** ${}^*\mathbb{R}$ and ${}^* : V(\mathbb{R}) \hookrightarrow V({}^*\mathbb{R})$ are definable.
2. **Faithfulness to Superstructure Hierarchy.** ${}^* : V_n(\mathbb{R}) \hookrightarrow V_n({}^*\mathbb{R})$ for all $n \in \mathbb{N}_0$.
3. **Transfer Principle.** For all $\mathcal{L}_{V(\mathbb{R})}$-formulae ϕ with bounded quantifiers
$$V(\mathbb{R}) \models \phi \Leftrightarrow V({}^*\mathbb{R}) \models \phi.$$
4. **Internal Definition Principle.** For all internal sets B_0, all internal $b_1, \ldots, b_n$ and all $\in$-formulae $\phi(v_0, \ldots, v_n)$,
$\{x \in B_0 \ : \ \phi[x, b_1, \ldots, b_n]\}$ is internal.
5. **Countable Saturation Principle.** If $C_n \neq \emptyset$ is internal and $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$.

Proof of the main result II

Proof sketch.

1. Taking bounded ultrapowers with respect to D **preserves** definability.

2. $e_{0,\aleph_1}[V_n(\mathbb{R})] = e_{0,\aleph_1}(V_n(\mathbb{R}))$ by **Extensionality** in $M_{\aleph_1}$, so by definition of j,

$$\begin{aligned} {}^*[V_n(\mathbb{R})] =& j\left[e_{0,\aleph_1}[V_n(\mathbb{R})]\right] = j\left[e_{0,\aleph_1}(V_n(\mathbb{R}))\right] \\ =& j\left(e_{0,\aleph_1}(V_n(\mathbb{R}))\right) = {}^*V_n(\mathbb{R}). \end{aligned}$$

3.
   - Transfer lemma between $\mathcal{M}_0$ and $\mathcal{M}_{\aleph_1}$
   - Transfer lemma between $\mathcal{M}_{\aleph_1}$ and $V({}^*\mathbb{R})$

4.
   - Transfer the Axiom Scheme of **Separation** for $V_{N-1}(\mathbb{R})$ to an Axiom Scheme of Separation for $V_{N-1}({}^*\mathbb{R})$
   - Use the transitivity of $V_N({}^*\mathbb{R})$

5. $j[M_{\aleph_1}]$ is the internal universe.
   Let $\{B_n\}_n \subset M_{\aleph_1}$ be such that for all $n \in \mathbb{N}$, $C_n = j(B_n) = j[B_n]$.
   $M_{\aleph_1}$ is $\aleph_1$-**saturated**, so let $b \in \bigcap_n B_n$.
   Then $j(b) \in C_n$ for all $n \in \mathbb{N}$.

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This talk establishes the existence of a definable (over **ZFC**), countably saturated nonstandard enlargement of the superstructure over the reals. This nonstandard universe is obtained as the union of an inductive chain of bounded ultrapowers (i.e. bounded with respect to the superstructure hierarchy). The underlying ultrafilter is the one constructed by Kanovei and Shelah [2004]

$(C_n)_{n\in\mathbb{N}}$: decreasing sequence of internal nonempty sets.

A: set of continuum-length sequences of subsets of $\mathbb{N}$

Let $X \in \mathcal{H}$ and $x \in \mathbb{N}^A$. Then, membership in X can be decided, uniformly in $x \in \mathbb{N}^A$, by only looking at the values of x on the same finite subset u of A.

For any $X \in \mathcal{H}$, there is a finite set $\|X\| \subset A$ such that $\|X\|$ is a finite support of X, and no proper subset $u \subsetneq \|X\|$ is a finite support of X.

Proof of the Minimality Lemma: $\|X\| := \bigcap \{u \ : \ u \text{ finite support of } X\}$.

$U\ i\ \Phi(i)$ means: for U-almost all i, $\Phi(i)$ holds.

$\mathcal{M}_0$ is an $\mathcal{L}_{V(\mathbb{R})}$-structure with domain $M_0 = V(\mathbb{R})$, where v interprets the constant symbol $\dot{v}$ and $\in$ interprets the binary relation symbol $\dot{\in}$.

For the recursive definition of j, recall that reals are treated as atoms.