COUNTING INFINITE POINT-SETS

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ULTRAMATH2008 - *”Ultrafilters and Ultraproducts in Mathematics”.* Pisa, June 1^{st}-7^{th}, 2008.

Euclid's Common Notions

1. *Things equal to the same thing are also equal to one another.*
2. *And if equals be added to equals, the wholes are equal.*
3. *And if equals be subtracted from equals, the remainders are equal.*
4. *Things applying* [exactly] *onto one another are equal to one another.*
5. *The whole is greater than the part.*

NB We translate $\epsilon\phi\alpha\rho\mu o\zeta o\nu\tau\alpha$ by *"applying* [exactly] *onto"*, instead of the usual *"coinciding with"*. This translation seems to give a more appropriate rendering of the Euclidean usage of the verb $\epsilon\phi\alpha\rho\mu o\zeta\epsilon\iota\nu$, which refers to superposition of congruent figures.

The presence of the fourth and fifth principles among the Common Notions in the original Euclid's treatise is controversial, notwithstanding the fact that they are explicitly accepted in the fundamental commentary by Proclus to Euclid's Book I, where all the remaining statements included as axioms by Pappus and others are rejected as spurious additions.

We consider the five principles on a par, since all of them can be viewed as basic assumptions for any reasonable theory of magnitudes.

The 1^{st} Euclidean principle for collections

- *Things equal to the same thing are also equal to one another*

essentially states that "having equal sizes" is an equivalence. We write $A \approx B$ when A and B are equinumerous (have equal sizes). The first Euclidean principle becomes

E1 (Equinumerosity Principle)

$$A \approx C,\ B \approx C \implies A \approx B.$$

2^{nd} and 3^{rd} Euclidean principles for collections

- *And if equals be added to equals, the wholes are equal*
- *And if equals be subtracted from equals, the remainders are equal*

addition and subtraction are “compatible” with equinumerosity. For collections, sum and difference naturally correspond to disjoint union and relative complement:

E2 (Sum Principle)

$$A \approx A',\ B \approx B',\ A \cap B = A' \cap B' = \emptyset \Longrightarrow A \cup B \approx A' \cup B'$$

E3 (Difference Principle)

$$A \approx A',\ B \approx B',\ B \subseteq A,\ B' \subseteq A' \Longrightarrow A \setminus B \approx A' \setminus B'$$

The 4^{th} Euclidean principle for collections

- *Things applying* [exactly] *onto one another are equal to one another*

> . . . the [fourth] Common Notion . . . is intended to assert that superposition is a legitimate way of proving the equality of two figures . . . or . . . to serve as an *axiom of congruence*. ([5], p.225).

i.e. "appropriately faithful" transformations (*congruences*) preserve sizes: it is a criterion for being equinumerous.

Equinumerosity vs. equipotency

• Cantor: all and only biunique transformations are size-preserving.

• Cardinal arithmetic: $\mathfrak{a} + \mathfrak{b} = \max(\mathfrak{a}, \mathfrak{b})$ whenever the latter is infinite.

No cancellation law, hence 3^{rd} principle **E3** fails (*a fortiori no subtraction*)

Isometries vs. congruences

Even the isometries of Euclidean geometry work only for *special classes of bounded geometrical figures*.

• Banach-Tarski: a ball can be partitioned into six pieces that can be used to rebuild two balls identical to the original one

Without any structure, the 3^{rd} (and 5^{th}) common notion can be saved only by restricting the meaning of *"applying* [exactly] *onto"* to comprehend only "natural transformations", such as permutations and repetitions of components of n-tuples, embeddings in higher dimensions, and similars.

Natural congruences

A notion of congruence appropriate for the 4^{th} Euclidean principle might include all "natural transformations" that map tuples to tuples having the same sets of components

- Two tuples are congruent if their respective sets of components coincide.
- A natural congruence is an injective function mapping tuples to congruent tuples.

E4a (Natural Congruence Principle)

$X \approx T[X]$ for all natural congruences T.

Generalized Substitutions

A notion of congruence appropriate for the 4^{th} Euclidean principle might include also all "generalized substitutions" that, fixed a function $f : \mathbb{N} \to \mathbb{N}$, take any m-tuple $x = (x_1, \ldots, x_m)$ and replace the component a_i of a fixed n-tuple $a = (a_1, \ldots, a_n)$ by $x_{f(i)}$, whenever possible:

$$S_f^a(x) = (y_1, \ldots, y_n) \quad \text{where} \quad y_i = \begin{cases} x_{f(i)} & \text{if } 1 \leq f(i) \leq m \\ a_i & \text{otherwise.} \end{cases}$$

E4b (Generalized Substitution Principle)

$$\{1, \ldots, m\} \subseteq f[\{1, \ldots, n\}] \implies A \approx S_f^a[A]$$

for all sets A of m-tuples and any n-tuple a.

More congruences?

When some algebraic or geometric structure is added, it may be possible to have more congruences, naturally connected with this structure.

However a wider class of "isometries" is admissible only after "appropriately restricting" their domains of application. In fact any transformation T with an infinite orbit

$$\Gamma = \{x, Tx, T^2x, \ldots\}$$

maps Γ onto a proper subset of Γ, so T is not a "congruence" for Γ itself.

An important example is that of finite dimensional spaces over wellordered lines, where suitably restricted translations and homotheties can be taken as isometries.

The 5^{th} Euclidean principle for collections

- *The whole is greater than the part*

Say that A is smaller than B, written $A \prec B$, when A is equinumerous to a proper subset of B

$$A \prec B \iff A \approx A' \subset B$$

Comparison of sizes must be consistent with equinumerosity. So the fifth principle becomes

E5 (Ordering Principle)

$$A \subset B \approx B' \implies A \not\approx B \;\&\; A \prec B'$$

The problem of comparability

Homogeneous magnitudes are usually arranged in a linear ordering.

• Cardinalities of infinite sets are always comparable, thanks to Zermelo's Axiom of Choice.

The followig strengthening of the Ordering Principle would be most wanted (but it may exceed ZFC!)

E5b (Total Ordering Principle)

Exactly one of the following relations always holds:

$$A \prec B, \quad A \approx B, \quad B \prec A$$

A weaker alternative could be requiring **E5b** only for a transitive extension of the relation $\prec$.

Restricted isometries

An interesting point of view considers equinumerosity as witnessed by an appropriate family of "restricted isometries":

IP (Isometry Principle) *There exists a group of transformations $\mathcal{T}$ such that*

$$A \approx B \iff \exists T \in \mathcal{T} \;\; A \subseteq \mathrm{dom}\, T \;\&\; B = T[A].$$

Remark: **IP2** implies both the Half Cantor Principle HCP of [2],

$$A \approx B \implies |A| = |B|$$

and half of the ordering principle **E5**

$$A \subset B \approx B' \implies A \prec B'$$

The algebra of numerosities

Measuring size amounts to associating suitable "numbers" (numerosities) to the equivalence classes of equinumerous collections. Sum and ordering of numerosities can be naturally defined *à la Cantor*

(sum) $\mathfrak{n}(X) + \mathfrak{n}(Y) = \mathfrak{n}(X \cup Y)$ whenever $X \cap Y = \emptyset$;

(ord) $\mathfrak{n}(X) \leq \mathfrak{n}(Y)$ if and only if $X \preceq Y$.

thanks to the principles **E2**, **E3**, and **E5a**.

A "satisfactory" algebra of numerosities should also comprehend a product, so as to obtain (the non-negative part of) a (discretely) ordered ring.

(This condition was in fact the starting point of the theory outlined in [2].)

The product of numerosities

One could view the notion of measure as originating from the length of lines, and later extended to higher dimensions by means of products. In classical geometry, a product of lines is usually intended as the corresponding rectangle. So one could use Cartesian products in defining the product of numerosities.

The natural "arithmetical" idea that multiplication is an iterated addition of equals is consistent with the "geometrical" idea of rectangles, because the Cartesian product $A \times B$ can be naturally viewed as the union of "B-many disjoint copies" of A

$$A \times B = \bigcup_{b \in B} A_b, \text{ where } A_b = \{(a,b) \mid a \in A\}.$$

- But is A_b a "faithful copy" of A?

The Product Principle

- Let $A = \{b, (b,b), ((b,b),b), \ldots, (((\ldots,b),b),b), \ldots\}$

$A_b = A \times \{b\}$ is a proper subset of A, so (the numerosity of) the singleton $\{b\}$ is not an identity w.r.t. (the numerosity of) A.

A disjointness constraint, stronger than that of the Sum Principle **E2**, *e.g.*

$$TC(A) \cap TC(B) = TC(A') \cap TC(B') = \emptyset$$

has to be put in the following

PP (Product Principle)

$$A \approx A',\ B \approx B' \implies A \times B \approx A' \times B'$$

An “Axiom der Beschränkung”

We can avoid the introduction of restrictions on products by considering only finite dimensional point sets, *i.e.* subsets of the n-dimensional spaces $\mathbb{E}_n(\mathcal{L})$ built over any “line” $\mathcal{L}$, where “paradoxical” sets of the kind of A cannot appear.

It amounts to assuming an “Axiom der Beschränkung”, similar to that commonly used in admitting *only well-founded sets*.

Finite dimensional point-sets

- Fix a "base line" $\mathcal{L}$ (an arbitrary set or class)

- $\mathbb{E}_n(\mathcal{L})$ = the n-dimensional Euclidean space over $\mathcal{L}$, *i.e.* the collection of *all* n*-tuples of elements of* $\mathcal{L}$.

- n-dimensional point-set (over $\mathcal{L}$) = subset of $\mathbb{E}_n(\mathcal{L})$

- given point-sets $X \in \mathbb{E}_h(\mathcal{L})$ and $Y \in \mathbb{E}_k(\mathcal{L})$, identify the Cartesian product $X \times Y$ with the $(h+k)$-dimensional point-set obtained by concatenation, *i.e.* $\{(z_1, \ldots, z_{h+k}) \mid (z_1, \ldots, z_h) \in X, (z_{h+1}, \ldots, z_{h+k}) \in Y\}$

Full families of point-sets over $\mathcal{L}$

We consider families of point-sets over $\mathcal{L}$, which are sufficiently rich so as to make Euclid's principles work. Call

$$\mathbb{W} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}(\mathbb{E}_n(\mathcal{L}))$$

a full family of point-sets over $\mathcal{L}$ if

- $X \subseteq Y \in \mathbb{W} \implies X \in \mathbb{W}$
- $X, Y \in \mathbb{W} \implies X \times Y \in \mathbb{W}$
- $X \in \mathbb{W} \implies T[X] \cap \mathbb{E}_n(\mathcal{L}) \in \mathbb{W}$ for every natural congruence T

Numerosity on a full family of point-sets

Definition. Let $\mathbb{W}$ be a full family of point-sets over $\mathcal{L}$. An equinumerosity relation for $\mathbb{W}$ is an equivalence $\approx$ satisfying the following conditions for all $X, Y \in \mathbb{W}$:

(e1) $X \approx Y \iff X \setminus Y \approx Y \setminus X$

(e2) $X \approx X' \Longrightarrow X \times Y \approx X' \times Y$

(e3) $X \approx \{x\} \times X$ for all $x \in \mathcal{L}$

(e4) $X \approx T[X]$ for every natural congruence T

(e5a) $X \subset Y \approx Y' \Longrightarrow \exists X' \subset Y' \;\; X \approx X' \not\approx Y'$.

The equinumerosity $\approx$ is Euclidean if **(e5a)** is strengthened to

(e5b) for all X, Y, exactly one of the conditions $X \approx Y$, $X \prec Y$, $Y \prec X$ holds .

Theorem. *Any equinumerosity relation satisfies the five Euclidean principles* **E1**-**E5a** *together with the product principle* **PP**. *Moreover the Total Ordering Principle* **E5b** *is fulfilled if and only if the equinumerosity is Euclidean.*

• finite point-sets receive their "number of elements" as numerosities:

Proposition. *Let A, B be finite. Then*

$$A \approx B \iff |A| = |B|$$

Moreover, if X is infinite, then $A \prec X$.

The algebra of numerosities

A surjective map $\mathfrak{n} : \mathbb{W} \to \mathfrak{N}$ is a numerosity function corresponding to the equinumerosity relation $\approx$ if

$$\mathfrak{n}(X) = \mathfrak{n}(Y) \iff X \approx Y.$$

Define $+, \cdot$ and $<$ on $\mathfrak{N}$ by

(sum) $\mathfrak{n}(X) + \mathfrak{n}(Y) = \mathfrak{n}(X \cup Y)$ whenever $X \cap Y = \emptyset$;

(prod) $\mathfrak{n}(X) \cdot \mathfrak{n}(Y) = \mathfrak{n}(X \times Y)$ for all X, Y;

(ord) $\mathfrak{n}(X) < \mathfrak{n}(Y)$ if and only if $X \prec Y$.

Theorem. *The structure $\langle \mathfrak{N}, +, \cdot, < \rangle$ is a positive subsemiring of a partially ordered discrete ring, and $\mathbb{N}$ can be taken as an initial segment of $\mathfrak{N}$. $\mathfrak{N}$ is the positive part of a discretely ordered ring if and only if $\mathfrak{n}$ is Euclidean.*

The natural series

Let $\mathbb{T} = \{t_a \mid a \in \mathcal{L}\}$ be a family of indeterminates over $\mathbb{Q}$, indexed by $\mathcal{L}$. Let $R \subseteq \mathbb{Z}[[\mathbb{T}]]$ be the set of all formal series in $\mathbb{T}$ of bounded degree (*i.e.* the subring generated by the homogeneous series).

- To each point $x = (x_1, \ldots, x_d) \in \mathbb{E}_d(\mathcal{L})$ associate the monomial

$$t_x = t_{x_1} \ldots t_{x_d}$$

- The natural series of the point set $X \subseteq \bigcup_{d \leq n} \mathbb{E}_d(\mathcal{L})$ is the formal sum

$$\overset{X}{\sum} = \sum_{x \in X} t_x \in R$$

- natural series behave well w.r.t. disjoint unions and Cartesian products: $\overset{X}{\sum} + \overset{Y}{\sum} = \overset{X \cup Y}{\sum} + \overset{X \cap Y}{\sum}$ and $\overset{X \times Y}{\sum} = \overset{X}{\sum} \cdot \overset{Y}{\sum}$

- Let $\overline{\sigma}$ be obtained from $\sigma \in R$ by replacing each monomial by the corresponding squarefree monomial and summing up the coefficients, and put

$$\sigma \leq^* \tau \iff \overline{\sigma} \leq \overline{\tau} \quad \text{(coefficientwise)}$$

- Let $\mathfrak{i}$ be the ideal of R generated by $\{\sigma - \overline{\sigma} \mid \sigma \in R\}$, and let $\mathfrak{R} = R/\mathfrak{i}$ be the corresponding (partially ordered) quotient ring.
- Let $R_{\mathbb{W}}$ be the subring of R generated by the natural series σ^X for $X \in \mathbb{W}$, and let $\mathfrak{R}_{\mathbb{W}} = R_{\mathbb{W}}/\mathfrak{i} \cap R_{\mathbb{W}}$ be the corresponding (partially ordered) quotient ring.

Theorem. *Let $\approx$ be an equinumerosity relation on a full family $\mathbb{W}$. There exists a prime ideal $\mathfrak{p} \supseteq \mathfrak{i} \cap R_{\mathbb{W}}$ of the ring $R_{\mathbb{W}}$ such that, for all $X, Y \in \mathbb{W}$,*

$$X \approx Y \iff \sigma^X - \sigma^Y \in \mathfrak{p}$$

moreover

$$\sigma^X <^* \sigma^Y \Longrightarrow X \prec Y$$

The equinumerosity $\approx$ is Euclidean if and only if for all $X, Y \in \mathbb{W}$ there exists $Z \in \mathbb{W}$ such that

$$\sigma^X - \sigma^Y \pm \sigma^Z \in \mathfrak{p}$$

Nonstandard numbers as numerosities

Let $\mathcal{I} = [\mathcal{L}]^{<\omega}$ be the family of all finite subsets of $\mathcal{L}$.

For $\sigma \in R$ and $F \in \mathcal{I}$ let

$$\sigma_F = \sum_{G \subseteq F} \xi_G,$$

where ξ_F is the coefficient of the monomial $t_F = \prod_{a \in F} t_a$ in the squarefree series $\overline{\sigma}$, so σ_F is ''the value of σ on the characteristic function of F''. Put

$$\Phi(\sigma) = \langle \sigma_F \mid F \in \mathcal{I} \rangle$$

The map $\Phi : R \to \mathbb{Z}^{\mathcal{I}}$ is a ring homomorphism whose kernel includes $\imath$, and $\sigma <^* \tau \Longrightarrow \Phi\sigma < \Phi(\tau)$.

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathcal{I}$, let $\mathbb{W}$ be the family of all finite dimensional point sets over $\mathcal{L}$, and for $X \in \mathbb{W}$ put

$$\mathfrak{n}_{\mathcal{U}} = \Phi(\sigma^X) \mod \mathcal{U}$$

Corollary 1. *The map* $\mathfrak{n}_{\mathcal{U}} : \mathbb{W} \to Z^{\mathcal{I}}_{\mathcal{U}}$ *is a numerosity function, whose range* $\mathfrak{N}_{\mathcal{U}} \subseteq \mathbb{N}^{\mathcal{I}}_{\mathcal{U}}$ *is a semiring of nonstandard integers. The equinumerosity* $\approx_{\mathcal{U}}$ *corresponding to* $\mathfrak{n}_{\mathcal{U}}$ *is Euclidean if and only if the range* $\mathfrak{N}_{\mathcal{U}}$ *is an initial segment of* $\mathbb{N}^{\mathcal{I}}_{\mathcal{U}}$.

The countable line

Let $\mathcal{L}$ be countable, and let $\mathcal{I} = [\mathcal{L}]^{<\omega}$. Then for any ultrafilter $\mathcal{V}$ on $\mathbb{N}$ there exist ultrafilters $\mathcal{U}$ on $\mathcal{I}$ such that $\mathbb{N}^{\mathcal{I}}_{\mathcal{U}} \cong \mathbb{N}^{\mathbb{N}}_{\mathcal{V}}$.

A nonprincipal ultrafilter $\mathcal{V}$ on $\mathbb{N}$ is Euclidean if every polynomially bounded function $f : \mathbb{N} \to \mathbb{N}$ is $\mathcal{V}$-equivalent to a nondecreasing function. Notice that

$$\mathcal{V} \text{ Ramsey} \Longrightarrow \mathcal{V} \text{ Euclidean} \Longrightarrow \mathcal{V} \text{ P-point}$$

Corollary 2. *The numerosity function $\mathfrak{n}_{\mathcal{U}}$ is Euclidean if and only if the set of numerosities $\mathfrak{N}_{\mathcal{U}}$ is isomorphic to a proper initial segment of the ultrapower $\mathbb{N}^{\mathbb{N}}_{\mathcal{V}}$ modulo an Euclidean ultrafilter $\mathcal{V}$.*

Countable pointsets on arbitrary lines

Let $|\mathcal{L}| = \kappa$ be uncountable, let $\mathcal{I} = [\mathcal{L}]^{<\omega}$, and let $\mathbb{W}$ be the family of all countable point sets over $\mathcal{L}$.

A nonprincipal ultrafilter $\mathcal{U}$ on $\mathcal{I}$ is countably Euclidean if there exists an Euclidean ultrafilter $\mathcal{V}$ on $\mathbb{N}$ such that

$$\mathbb{N}^{[X]^{<\omega}}_{\mathcal{U}_X} \cong \mathbb{N}^{\mathbb{N}}_{\mathcal{V}} \quad \text{for all countable } X \subseteq \mathcal{L}$$

where $\mathcal{U}_X$ is the ultrfilter induced by $\mathcal{U}$ on $[X]^{<\omega}$.

Corollary 3. *The numerosity function $\mathfrak{n}_{\mathcal{U}}$ is Euclidean if and only if the ultrafilter $\mathcal{U}$ is countably Euclidean, and so there exists an Euclidean ultrafilter $\mathcal{V}$ over $\mathbb{N}$ such that the set of numerosities $\mathfrak{N}_{\mathcal{U}}$ is isomorphic to an initial segment of the ultrapower $\mathbb{N}^{\mathbb{N}}_{\mathcal{V}}$*

The real line

Let $\mathbb{W}_0$ be the family of all countable point sets over R, and let $\mathbb{W}$ be the family of all point sets over $\mathbb{R}$.

• A countably Euclidean ultrafilter $\mathcal{U}$ on $\mathcal{I} = [\mathbb{R}]^{<\omega}$ is weakly Euclidean if for each uncountable $\kappa \leq \mathfrak{c}$ there exists an ultrafilter $\mathcal{U}_\kappa$ on κ such that

$$\mathbb{N}^{[X]^{<\omega}}_{\mathcal{U}_X} \cong \mathbb{N}^{\kappa}_{\mathcal{U}_\kappa} \quad \text{for all } X \in [\mathbb{R}]^{\kappa}$$

where $\mathcal{U}_X$ is the ultrfilter induced by $\mathcal{U}$ on $[X]^{<\omega}$.

• If there exist a countably Euclidean ultrafilter, then there exist a weakly Euclidian ultrafilter $\mathcal{U}$ such that the numerosity function $\mathfrak{n}_{\mathcal{U}}$ defined on $\mathbb{W}$ is continuous w.r.t. normal approximations.

• We conjecture that no such numerosity on $\mathbb{W}$ can be Euclidean

Set theoretic commitments

• Assuming the Continuum Hypothesis every filter on $[\mathbb{N}]^{<\omega}$ can be refined to a countably Ramsey ultrafilter on $[\mathbb{R}]^{<\omega}$.

• The existence of Euclidean ultrafilters is independent of ZFC. A sufficient condition is $\mathfrak{c} = \mathrm{cov}(\mathfrak{B})$

• If there exists Euclidean ultrafilters on $\mathbb{N}$, then there exist countably Euclidean ultrafilters on $[\omega_n]^{<\omega}$.

• A sufficient condition for the existence of countably Euclidean ultrafilters on $[\lambda]^{<\omega}$ is that both $\kappa^{\aleph_0} = \kappa^+$ and $\square_\kappa$ hold for all singular cardinals $\kappa < \lambda$ of countable cofinality.

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