VARIATIONS ON PROPERTIES OF ULTRAFILTERS RELATED TO COMPACTNESS OF TOPOLOGICAL SPACES AND TO MODEL-THEORETICAL PRINCIPLES

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5. General Topology
10. Set theory and foundations

Many properties of ultrafilters can be equivalently stated as properties of their ultrapowers. For example, for λ a regular cardinal, an ultrafilter is λ-decomposable if and only if the ultrapower $\prod_D \langle \lambda, < \rangle$ contains an element greater than all former elements of λ.

Thus, for $\lambda > \mu$ regular cardinals, the property "*every λ-decomposable ultrafilter is μ-decomposable*" can be reformulated in model-theoretical terms as follows: "*for every ultrapower extension $\mathfrak{B}$ of $\langle \lambda, < \rangle$, if $\mathfrak{B}$ contains an element greater than all former elements of λ then $\mathfrak{B}$ contains an element $< d(\mu)$ but greater than all former elements of μ*".

If in the above italicized statement we consider elementary extensions of $\langle \lambda, < \rangle$ rather than ultrapower extensions, we get a stronger property, namely "*the model $\langle \lambda, < \rangle$ has an expansion $\mathfrak{A}$ in a language with at most λ new symbols such that whenever $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$ and $\mathfrak{B}$ has an element x such that $\mathfrak{B} \models \gamma < x$ for every $\gamma < \lambda$, then $\mathfrak{B}$ has an element y such that $\mathfrak{B} \models \alpha < y < \mu$ for every $\alpha < \mu$*".

We show that the above model-theoretical property has many equivalent reformulations both in set-theoretical terms and in topological terms.

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