SOME ASPECTS OF ULTRAFILTER CONVERGENCE IN TOPOLOGY

DIRK HOFMANN

In this talk we wish to present ultrafilter characterisations of special classes of continuous maps between topological spaces, and, if time permits, how this work led to the development of a general setting presenting *spaces as categories*.

Our interest was motivated by the work of G. Janelidze and M. Sobral on (Grothendieck) descent theory in finite topological spaces, where, according to them, descriptions of various kind of descent maps "become very simple and natural as soon as they are expressed in the language of finite preorders". The preorder relation of a (finite) space is just the point convergence shadow of the ultrafilter convergence relation, hence one might expect to obtain results about arbitrary spaces by "just" replacing point convergence by ultrafilter convergence. Following this idea, we will present characterisations of the classes of (effective) descent maps, triquotient maps and local homeomorphisms in terms of lifting properties of chains of convergent ultrafilters [1, 3]. For instance, triquotient maps were introduced by E. Michael as those continuous maps $f : X \to Y$ for which there exists a mapping $(\_)^\sharp : \mathcal{O}X \to \mathcal{O}Y$ between the frames of open subsets which satisfies certain conditions. For finite spaces X and Y, f is a triquotient map if and only if each chain $y_n \to \cdots \to y_0$ of convergent points in Y is the image of a chain $x_n \to \cdots \to x_0$ of convergent points in X. For arbitrary spaces, we show that f is triquotient if and only if each (possibly infinite) chain $\cdots \mathfrak{Y}_2 \to \mathfrak{y}_1 \to y$ in Y of higher order ultrafilters is the image of a chain $\cdots \mathfrak{X}_2 \to \mathfrak{x}_1 \to x$ in X.

Our work with topological spaces presented as convergence structures shaped the idea that *topological spaces are categories*, and therefore can be studied using notions and techniques from (enriched) Category Theory. Here we consider the points of a space X as the objects of our category, and interprete the convergence $\mathfrak{x} \to x$ of an ultrafilter $\mathfrak{x}$ on X to a point $x \in X$ as a morphism in X. Under this perspective, a wide range of categorical concepts and theorems can be rephrased for (topological) spaces. In the second part of this talk we will present some of these and show their usefulness for the study of, for instance, injective spaces [2, 5, 4].

This talk is based on joint work with M.M. Clementino (University of Coimbra) and W. Tholen (York University, Toronto).

References

[1] M.M. Clementino and D. Hofmann. Triquotient maps via ultrafilter convergence. *Proc. Amer. Math. Soc.*, 130(11):3423–3431, 2002.

[2] M.M. Clementino and D. Hofmann. Lawvere completeness in Topology. Technical report, University of Aveiro, 2007, arXiv:math.CT/0704.3976.

[3] M.M. Clementino, D. Hofmann, and G. Janelidze. Local homeomorphisms via ultrafilter convergence. *Proc. Amer. Math. Soc.*, 133(3):917–922, 2005.

[4] D. Hofmann. Injective spaces via adjunction. Technical report, University of Aveiro, 2008, arXiv:math.CT/0804.0326.

[5] D. Hofmann and W. Tholen. Lawvere completion and separation via closure. Technical report, University of Aveiro, 2008, arXiv:math.CT/0801.0199.

Departamento de Matemática, Universidade de Aveiro, Portugal.
E-mail address: `dirk@ua.pt`