Matrici invertibili

Verificare che la matrice $A = \begin{pmatrix} 3 & -3 \\ 1 & -2 \end{pmatrix}$ è invertibile e calcolarne l'inversa

$$\det A = \begin{vmatrix} 3 & -3 \\ 1 & -2 \end{vmatrix} = -6 - (-3) = -3$$

$\det A \neq 0 \Rightarrow A$ invertibile

$$A^{\#} = \begin{pmatrix} -2 & -1 \\ 3 & 3 \end{pmatrix} \qquad A^{-1} = \frac{1}{\det(A)} (A^{\#})^{t}$$

$$A^{-1} = -\frac{1}{3} \begin{pmatrix} -2 & 3 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -1 \\ \frac{1}{3} & -1 \end{pmatrix}$$

Verifichiamo che $A \cdot A^{-1} = A^{-1} A = I_2$

$$\begin{pmatrix} 3 & -3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -1 \\ +\frac{1}{3} & -1 \end{pmatrix} = \begin{pmatrix} \cancel{3}\frac{2}{\cancel{3}} - \cancel{3}\frac{1}{\cancel{3}} & 3(-1) - 3(-1) \\ 1\left(\frac{2}{3}\right) - 2\left(\frac{1}{3}\right) & 1(-1) - 2(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{3} & -1 \\ \frac{1}{3} & -1 \end{pmatrix} \begin{pmatrix} 3 & -3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\cancel{3}}\cancel{3} - 1(1) & \frac{2}{\cancel{3}}(-\cancel{3}) - 1(-2) \\ \frac{1}{\cancel{3}}\cancel{3} - 1(1) & \frac{1}{\cancel{3}}(-\cancel{3}) - 1(-2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Risolvere il sistema

$$\begin{cases} 3x - 3y = 2 \\ x - 2y = +3 \end{cases} \qquad \begin{pmatrix} 3 & -3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ +3 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \end{pmatrix} \quad \text{per } A \text{ invertibile}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -1 \\ \frac{1}{3} & -1 \end{pmatrix} \begin{pmatrix} 2 \\ +3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}2 - 1(+3) \\ \frac{1}{3}2 - 1(+3) \end{pmatrix} = \begin{pmatrix} \frac{4}{3} - 3 \\ \frac{2}{3} - 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} \\ -\frac{7}{3} \end{pmatrix}$$

$$\begin{cases} x = -\frac{5}{3} \\ y = -\frac{7}{3} \end{cases}$$

Verificare che la matrice $A=\begin{pmatrix} 2 & +1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{pmatrix}$ è invertibile e calcolarne l'inversa

$$\det A = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ -2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} =$$

$$= 2(-3+2) - (-3-2) + (1+1) = -2+5+2 = 5$$

Regola di Sarrus

$$\det A = \left|\begin{array}{ccc|cc} 2 & 1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & -3 & 1 & -2 \end{array}\right| = -6+1+2-(-1-4-3) =$$

$$= -3+8 = 5$$

$\det A \neq 0 \Rightarrow A$ invertibile

$$A^{\#} = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ -2 & -3 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 & 4 & -3 \\ 5 & -5 & 5 \\ 2 & -3 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A}\,{}^{t}(A^{\#}) = \frac{1}{5}\begin{pmatrix} -1 & 5 & 2 \\ 4 & -5 & -3 \\ -3 & 5 & 1 \end{pmatrix} = \begin{pmatrix} -1/5 & 1 & 2/5 \\ 4/5 & -1 & -3/5 \\ -3/5 & 1 & 1/5 \end{pmatrix}$$

Verifichiamo $A \cdot A^{-1} = A^{-1} \cdot A = I_3$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{pmatrix}\begin{pmatrix} -1/5 & 1 & 2/5 \\ 4/5 & -1 & -3/5 \\ -3/5 & 1 & 1/5 \end{pmatrix} =$$

$$= \begin{pmatrix} \left(-\frac{2}{5}+\frac{4}{5}+\frac{3}{5}\right) & (2-1-1) & \left(4/5-\frac{3}{5}-\frac{1}{5}\right) \\ \left(-\frac{1}{5}+\frac{4}{5}-\frac{3}{5}\right) & (1-1+1) & \left(\frac{2}{5}-\frac{3}{5}+\frac{1}{5}\right) \\ \left(-\frac{1}{5}-\frac{8}{5}+\frac{9}{5}\right) & (1+2-3) & \left(\frac{2}{5}+\frac{6}{5}-\frac{3}{5}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{5} & 1 & 2/5 \\ 4/5 & -1 & -3/5 \\ -3/5 & 1 & 1/5 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{pmatrix} =$$

$$\begin{pmatrix} \left(-\cancel{\frac{2}{5}}+1+\cancel{\frac{2}{5}}\right) & \left(-\frac{1}{5}+1-\frac{4}{5}\right) & \left(\frac{1}{5}+1-\frac{6}{5}\right) \\ \left(\frac{8}{5}-1-\frac{3}{5}\right) & \left(\frac{4}{5}-1+\frac{6}{5}\right) & \left(-\frac{4}{5}-1+\frac{9}{5}\right) \\ \left(-\frac{6}{5}+1+\frac{1}{5}\right) & \left(-\frac{3}{5}+1-\frac{2}{5}\right) & \left(\cancel{\frac{3}{5}}+1-\cancel{\frac{3}{5}}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Determinare le soluzioni del seguente sistema

$$\begin{cases} 2x+y-z=3 \\ x+y+z=1 \\ x-2y-3z=4 \end{cases} \qquad \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/5 & 1 & 2/5 \\ 4/5 & -1 & -3/5 \\ -3/5 & 1 & 1/5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \left(-\frac{3}{5}+1+\frac{8}{5}\right) \\ \left(\cancel{\frac{12}{5}}-1-\cancel{\frac{12}{5}}\right) \\ \left(-\frac{9}{5}+1+\frac{4}{5}\right) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{cases} x=2 \\ y=-1 \\ z=0 \end{cases}$$

Dati i punti $A(3,1,1)$ $B(-3,-2,1)$ $C(1,\frac{3}{2},2)$ $D(2,0,\frac{2}{3})$ verificare che sono complanari e determinare eq. parametriche e cartesiana del piano che li contiene

$\overrightarrow{AB}=(-6,-3,0)$ $\quad\overrightarrow{AC}=(-2,\frac{1}{2},1)$ $\quad\overrightarrow{AD}=(-1,-1,-\frac{1}{3})$

$$\begin{vmatrix} -6 & -2 & -1 \\ -3 & 1/2 & -1 \\ 0 & 1 & -1/3 \end{vmatrix} = 1+3-6+2=0$$

I vettori $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ sono lin. dip., quindi i 4 punti sono complanari eq. par. $\pi: \begin{cases} x = 3 - 6t - 2s \\ y = 1 - 3t + \frac{1}{2}s \\ z = 1 \qquad + s \end{cases}$, $s, t \in \mathbb{R}$

$\begin{cases} x = 3 + 2h - 4K \\ y = 1 + h + K \\ z = 1 \qquad + 2K \end{cases}$ $\quad \begin{vmatrix} (x-3) & 2 & -4 \\ (y-1) & 1 & 1 \\ (z-1) & 0 & 2 \end{vmatrix} = 0$

$-3t = h \qquad s = 2K$

$(x-3)\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - (y-1)\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} + (z-1)\begin{vmatrix} 2 & -4 \\ 1 & 1 \end{vmatrix} = 0$

$\cancel{2}(x-3) - \overset{2}{\cancel{4}}(y-1) + \overset{3}{\cancel{6}}(z-1) = 0$

$x - \underline{3} - 2y + \underline{2} + 3z - \underline{3} = 0$

$\pi: x - 2y + 3z - 4 = 0$

Determinare la distanza di $P_0(2, -1, -1)$ da π e la sua proiezione ortogonale su π

$d(P_0, \pi) = \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|2+2-3-4|}{\sqrt{1+4+9}} = \dfrac{3}{14}\sqrt{14}$

r retta per P_0 $\perp$ a π

$r: \begin{cases} x = 2 + t \\ y = -1 - 2t \\ z = -1 + 3t \end{cases}$ $\quad r \cap \pi$

sostituiamo le eq. parametriche di r in π.

$(2+t) - 2(-1-2t) + 3(-1+3t) - 4 = 0$

$\cancel{2} + \underline{t} + \cancel{2} + \underline{4t} - 3 + \underline{9t} - \cancel{4} = 0$

$t = \frac{3}{14} \quad H\left(2 + \frac{3}{14}; -1 - \frac{3}{7}; -1 + \frac{9}{14}\right) = \left(\frac{31}{14}; -\frac{10}{7}; -\frac{5}{14}\right)$

$H\left(\frac{31}{14}; -\frac{10}{7}; -\frac{5}{14}\right)$ verifichiamo $\overline{P_0H} = d(P_0, \pi)$

$\overline{P_0H} = \sqrt{\left(\frac{31}{14} - 2\right)^2 + \left(-\frac{10}{7} + 1\right)^2 + \left(-\frac{5}{14} + 1\right)^2} = \sqrt{\frac{9}{(14)^2} + \frac{9}{49} + \frac{(9)^2}{(14)^2}} =$

$= \dfrac{\sqrt{9 + 36 + 81}}{14} = \dfrac{\sqrt{126}}{14} = \dfrac{3\sqrt{14}}{14}$

Rotazione di Centro $C(c_x, c_y)$ e angolo θ

$\theta > 0$ se la rotazione è in verso antiorario

$\theta < 0$ se la rotazione è in verso orario

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x - c_x \\ y - c_y \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$(\tilde{x}, \tilde{y})$ coordinate del punto (x, y) che ha subito la rotazione.

Rotazione di centro $C(-1, 2)$ di un angolo $\alpha = \frac{\pi}{6}$ in verso orario. $\theta = -\frac{\pi}{6}$

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos(-\frac{\pi}{6}) & -\sin(-\frac{\pi}{6}) \\ \sin(-\frac{\pi}{6}) & \cos(-\frac{\pi}{6}) \end{pmatrix} \begin{pmatrix} x+1 \\ y-2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

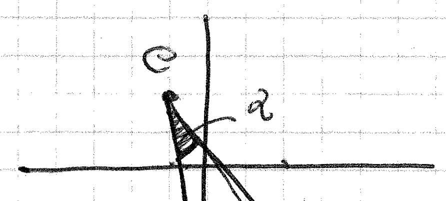
$$\begin{cases} \tilde{x} = \frac{\sqrt{3}}{2}(x+1) + \frac{1}{2}(y-2) - 1 \\ \tilde{y} = -\frac{1}{2}(x+1) + \frac{\sqrt{3}}{2}(y-2) + 2 \end{cases}$$

$$\begin{cases} \tilde{x} = \frac{\sqrt{3}}{2}x + \frac{1}{2}y + \frac{\sqrt{3}}{2} - 2 \\ \tilde{y} = -\frac{1}{2}x + \frac{\sqrt{3}}{2}y - \sqrt{3} + \frac{3}{2} \end{cases}$$

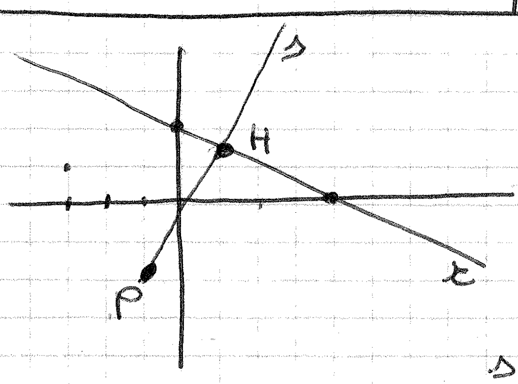
Se $P(2; -2)$ il suo ruotato è il punto

$Q\left(\sqrt{3} - 1 + \frac{\sqrt{3}}{2} - 2; \; -1 - \sqrt{3} - \sqrt{3} + \frac{3}{2}\right)$

$Q\left(\frac{3}{2}\sqrt{3} - 3; \; -2\sqrt{3} + \frac{1}{2}\right)$



Determinare il simmetrico (o riflesso) del punto $P(-1, -2)$ rispetto alla retta r $x + 2y = 4$



Determiniamo la retta s passante per P ortogonale ad r

$$s \begin{cases} x = -1 + t \\ y = -2 + 2t \end{cases} \quad t \in \mathbb{R} \quad \text{eq. par.}$$

$s: 2x - y + q = 0 \quad -2 + 2 + q = 0 \quad q = 0$

$s: 2x - y = 0$

Determiniamo H proiezione ortogonale di P su r

$P = r \cap s$

$$\begin{cases} x + 2y = 4 \\ 2x - y = 0 \end{cases} \quad \begin{cases} 5x = 4 \\ y = 2x \end{cases} \quad \begin{cases} x = \frac{4}{5} \\ y = \frac{8}{5} \end{cases} \quad H\left(\frac{4}{5}; \frac{8}{5}\right)$$

$$s\begin{cases} x = -1 + t \\ y = -2 + 2t \end{cases} \quad r: x + 2y = 4 \; ; \; -1 + t - 4 + 4t = 4 \; ; \; t = \frac{9}{5}$$

$$\begin{cases} x = -1 + \frac{9}{5} = \frac{4}{5} \\ y = -2 + \frac{18}{5} = \frac{8}{5} \end{cases} \quad H\left(\frac{4}{5}; \frac{8}{5}\right)$$

Il simmetrico di P rispetto alla retta r è un punto Q tale che H risulti punto medio del segmento PQ

$A(x_1, y_1)$ $B(x_2, y_2)$ $P(-1, -2)$ $H\left(\frac{4}{5}, \frac{8}{5}\right)$ $Q(q_x, q_y)$

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \quad \begin{cases} \frac{q_x - 1}{2} = \frac{4}{5} \\ \frac{q_y - 2}{2} = \frac{8}{5} \end{cases} \quad \begin{cases} q_x = 1 + \frac{8}{5} = \frac{13}{5} \\ q_y = \frac{16}{5} + 2 = \frac{26}{5} \end{cases}$$

Il punto $Q\left(\frac{13}{5}; \frac{26}{5}\right)$ è il simmetrico di $P(-1, -2)$ rispetto alla retta $r: x + 2y = 4$

Scrivere le equazioni della riflessione rispetto alla retta $r: x + 2y = 4$.

Sia $P(a, b)$ un generico punto del piano. Determiniamo la sua proiezione ortogonale H sulla retta r.

$$\begin{cases} x = a + t \\ y = b + 2t \end{cases} \quad s \text{ passante per } P \text{ e } \perp r$$

$2x - y + q = 0$ $\quad 2a - b + q = 0$ $\quad s: 2x - y - (2a - b) = 0$

$$H\begin{cases} x + 2y = 4 \\ 2x - y = 2a - b \end{cases} \quad \begin{cases} x + 2y = 4 \\ 4x - 2y = 4a - 2b \end{cases} \quad \begin{cases} 5x = 4a - 2b + 4 \\ 5y = 8 - (2a - b) \end{cases}$$

$$\begin{cases} x = \dfrac{4a-2b+4}{5} \\ y = \dfrac{-2a+b+8}{5} \end{cases} \qquad H\left(\dfrac{4a-2b+4}{5}; \dfrac{-2a+b+8}{5}\right)$$

$$a+t+2b+4t=4 \qquad t = \frac{4-(a+2b)}{5}$$

$$\begin{cases} x = a + \dfrac{4-(a+2b)}{5} = \dfrac{4a-2b+4}{5} \\ y = b + 2\,\dfrac{4-(a+2b)}{5} = \dfrac{-2a+b+8}{5} \end{cases}$$

Determiniamo infine $Q(q_x, q_y)$

$$\begin{cases} \dfrac{q_x+a}{2} = \dfrac{4a-2b+4}{5} \\ \dfrac{q_y+b}{2} = \dfrac{-2a+b+8}{5} \end{cases} \qquad \begin{cases} q_x = \dfrac{8a-4b+8}{5} - a \\ q_y = \dfrac{-4a+2b+16}{5} - b \end{cases}$$

$$\begin{cases} q_x = \dfrac{3a-4b+8}{5} \\ q_y = \dfrac{-4a-3b+16}{5} \end{cases} \qquad \begin{cases} \tilde{x} = \dfrac{3}{5}x - \dfrac{4}{5}y + \dfrac{8}{5} \\ \tilde{y} = -\dfrac{4}{5}x - \dfrac{3}{5}y + \dfrac{16}{5} \end{cases}$$

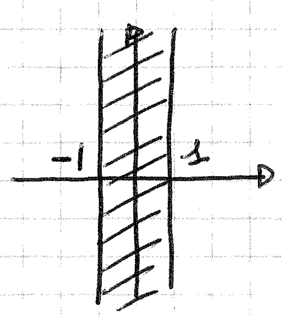
$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 8/5 \\ \frac{16}{5} \end{pmatrix}$$

equazione della riflessione rispetto alla retta r –

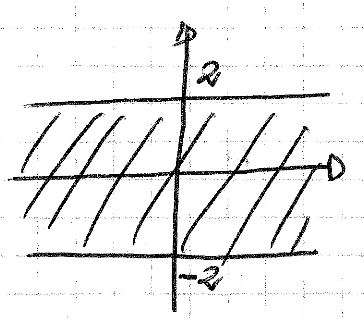
Determinare i campi di esistenza delle seguenti espressioni.

$$\frac{\sqrt{1-|x|} + \sqrt{2-|y|}}{x^2+y^2} \qquad \begin{cases} 1-|x| \geq 0 \\ 2-|y| \geq 0 \\ x^2+y^2 \neq 0 \end{cases} \qquad \begin{cases} |x| \leq 1 \\ |y| \leq 2 \\ x^2+y^2 \neq 0 \end{cases}$$

$|x| \leq 1$



$|y| \leq 2$

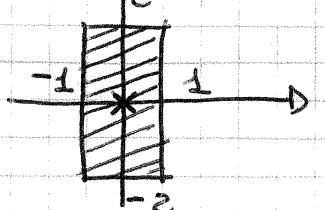


$x^2+y^2 \neq 0$

$(x,y) \neq (0,0)$

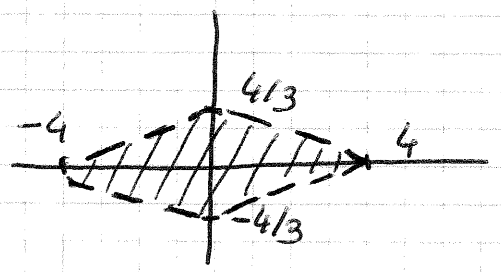
$\mathbb{R}^2 \setminus \{(0,0)\}$

$$\begin{cases} |x| \le 1 \\ |y| \le 2 \\ x^2+y^2 \neq 0 \end{cases}$$

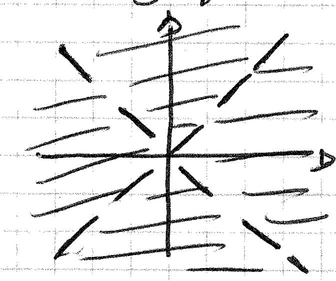


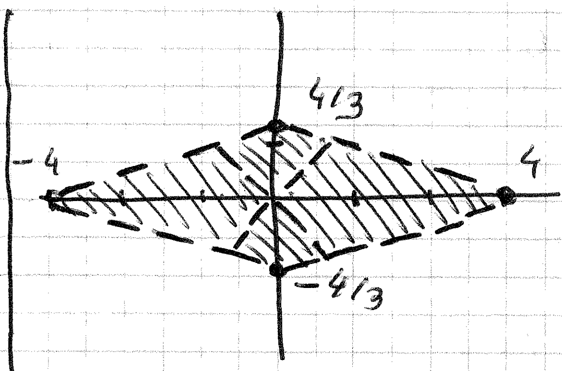
$$\log \frac{(4-|x|-3|y|)}{x^2-y^2} \qquad \begin{cases} 4-|x|-3|y| > 0 \\ x^2-y^2 \neq 0 \end{cases} \qquad \begin{cases} |x|+3|y| < 4 \\ (x-y)(x+y) \neq 0 \end{cases}$$

$|x|+3|y|<4$

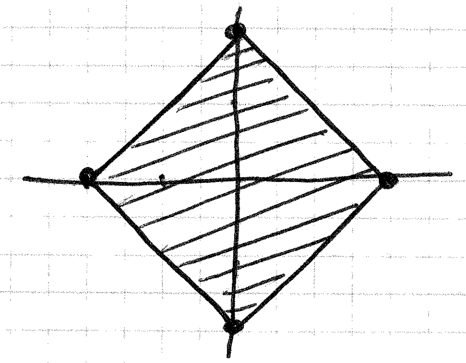


$x^2-y^2 \neq 0$



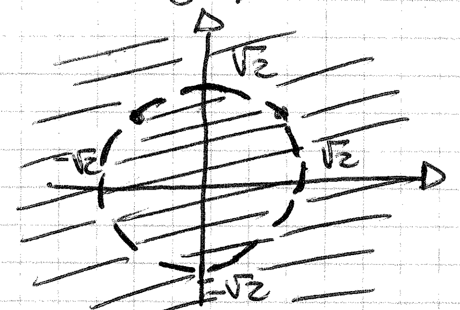


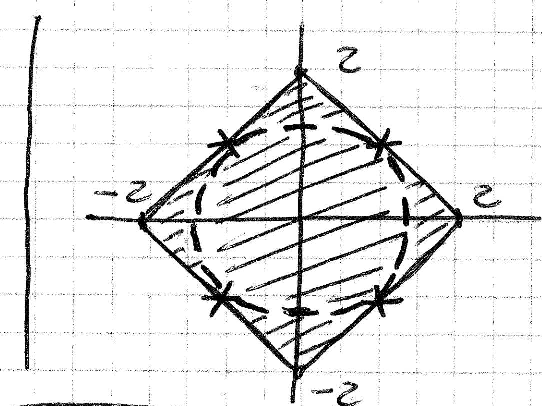
$$\frac{\sqrt{2-|x|-|y|}}{x^2+y^2-2} \qquad \begin{cases} 2-|x|-|y| \geq 0 \\ x^2+y^2 \neq 2 \end{cases} \qquad \begin{cases} |x|+|y| \le 2 \\ x^2+y^2 \neq 2 \end{cases}$$



$|x|+|y| \le 2$

$x^2+y^2 \neq 2$





Dato il parallelogramma di vertici $A(1,2,0)$ $B(3,-1,0)$ $C(0,-2,0)$ $D(-2,1,0)$, calcolare $\cos\alpha$ e $\operatorname{sen}\beta$ dove $\alpha = \hat{A}$ e $\beta = \hat{B}$

$\vec{AB} = (2,-3,0)$ $\quad \vec{AD}(-3,-1,0)$

$\vec{AB} \cdot \vec{AD} = |\vec{AB}|\,|\vec{AD}| \cos\hat{A}$

$\cos\hat{A} = (\vec{AB} \cdot \vec{AD}) / (|\vec{AB}|\,|\vec{AD}|)$

$\vec{AB} \cdot \vec{AD} = (2,-3,0) \cdot (-3,-1,0) = -3$

$|\vec{AB}| = \sqrt{13} \qquad |\vec{AD}| = \sqrt{10}$

$\cos\hat{A} = -\frac{3}{130}\sqrt{130}$

9)

$\vec{BA} = (-2,3,0)$ $\quad \vec{BC}(-3,-1,0)$

$|\vec{BA} \times \vec{BC}| = |\vec{BA}|\,|\vec{BC}| \operatorname{sen} B$

$\vec{BA} \times \vec{BC} = \left(\begin{vmatrix} 3 & 0 \\ -1 & 0 \end{vmatrix}; -\begin{vmatrix} -2 & 0 \\ -3 & 0 \end{vmatrix}; \begin{vmatrix} -2 & 3 \\ -3 & -1 \end{vmatrix} \right)$

$= (0,0,9) \qquad |\vec{BA} \times \vec{BC}| = 9$

$\operatorname{sen}\hat{\beta} = |\vec{BA} \times \vec{BC}| / (|\vec{BA}|\,|\vec{BC}|)$

$= \frac{9}{\sqrt{13}\sqrt{10}} = \frac{9}{130}\sqrt{130}$