Parabolic sheaves, root stacks and the Kato-Nakayama space

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Outline

Parabolic sheaves as sheaves on “stacks of roots”, and log geometry.

Partly joint with A. Vistoli, and Carchedi-Scherotzke-Sibilla.

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Parabolic sheaves (on a curve)

Let X be a compact Riemann surface.

Narasimhan-Seshadri correspondence: there is a bijection

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$$\updownarrow$$
$$\{\text{degree 0 stable (holomorphic) vector bundles on } X\}.$$

(via local systems)

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(via local systems)

What about the non-compact case?

Let $x_1, \ldots, x_k \in X$, and consider $X \setminus \{x_1, \ldots, x_k\}$.

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(Mehta-Seshadri, Deligne)

The "parabolic" structure is meant to encode the action of the small loops around the punctures.

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$\sim$ eigenspaces and eigenvalues of the matrix corresponding to a small loop $\gamma \in \pi_1(X \setminus \{x_1, \ldots, x_k\})$ around the puncture x_i.

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If $D = x_1 + \cdots + x_k$ (divisor on X), by taking inverse images along $E \to E|_D$, a parabolic bundle can be seen as

$$E(-D) \subset F_h \subset \cdots \subset F_1 = E$$

with weights $0 \leq a_1 < \cdots < a_h < 1$.

We can generalize and allow sheaves and maps

$$E \otimes \mathcal{O}(-D) \to F_h \to \cdots \to F_1 = E$$

whose composition $E(-D) \to E$ is multiplication by the section 1_D of the line bundle $\mathcal{O}(D)$, and weights $0 \leq a_1 < \cdots < a_h < 1$.

This definition makes sense for any variety X with an effective Cartier divisor $D \subseteq X$ (Maruyama-Yokogawa).

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One defines morphisms, subsheaves, kernels, cokernels, etc..
$\rightsquigarrow$ a nice category of parabolic sheaves.

Parabolic sheaves are “best” defined on an arbitrary logarithmic scheme.

Log schemes (K. Kato, Fontaine-Illusie)

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More concretely: if P is a monoid, a symmetric monoidal functor $L \colon P \to \mathrm{Div}(X)$ sends

$$p \mapsto (L_p, s_p)$$

with isomorphisms

$$L_p \otimes L_q \cong L_{p+q}$$

carrying $s_p \otimes s_q$ to s_{p+q}.

If $D \subseteq X$ is an eff. Cartier divisor we get a log scheme (X, D): take the symmetric monoidal functor $\mathbb{N} \to \mathrm{Div}(X)$ sending 1 to $(\mathcal{O}(D), 1_D)$.

If D has r irreducible components $D_1, \dots, D_r$ and is simple normal crossings you might want to "separate the components" with the functor $\mathbb{N}^r \to \mathrm{Div}(X)$ sending e_i to $(\mathcal{O}(D_i), 1_{D_i})$ (Iyer-Simpson, Borne).

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Example: start with X non-proper, compactify to $X \subseteq \overline{X}$ with SNC complement $D = \overline{X} \setminus X = D_1 \cup \ldots \cup D_r$,

and take $(\overline{X}, (D_1, \ldots, D_r))$.

How to think about this

To visualize the log scheme $(X, L\colon A \to \mathrm{Div}_X)$, think about the stalks of the sheaf A.

There is a largest open subset $U \subseteq X$ where $A_p = 0$ (might be empty).
In the "divisorial" case, $U = X \setminus D$.

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To visualize the log scheme $(X, L\colon A \to \mathrm{Div}_X)$, think about the stalks of the sheaf A.

There is a largest open subset $U \subseteq X$ where $A_p = 0$ (might be empty).
In the "divisorial" case, $U = X \setminus D$.

More generally A is locally constant on a stratification ($\sim$ discrete data).

Example: $X = \mathbb{A}^2$, $D = \{xy = 0\}$. The stalks of the sheaf A are

$$\begin{array}{lll} 0 & \text{on} & \mathbb{A}^2 \setminus \{xy = 0\} \\ \mathbb{N} & \text{on} & \{xy = 0\} \setminus \{(0,0)\} \\ \mathbb{N}^2 & \text{on} & \{(0,0)\}. \end{array}$$

Parabolic sheaves (on log schemes)

A parabolic sheaf on X with respect to D

$$E \otimes \mathcal{O}(-D) \to F_h \to \cdots \to F_1 = E$$

with rational weights $0 \leq a_1 < \cdots < a_h < 1$ with common denominator n

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can be seen as a diagram

$$\begin{array}{cccccc} -1 & -a_h & \cdots & -a_2 & -a_1 & 0 \\ & & & & & \\ E \otimes \mathcal{O}(-D) \longrightarrow & F_h \longrightarrow & \cdots \longrightarrow & F_2 \longrightarrow & F_1 \longrightarrow & E \end{array}$$

of sheaves placed in the interval $[-1, 0]$, with maps going in the positive direction.

We can fill the (possible) “gaps” in $[-1,0] \cap \frac{1}{n}\mathbb{Z}$ by “looking at the sheaf on the left”, and

extend out of $[-1,0]$ by tensoring with powers of $\mathcal{O}(D)$.

(so that $E_{q+1} \cong E_q \otimes \mathcal{O}(D)$ for every $q \in \frac{1}{n}\mathbb{Z}$)

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We obtain a functor

$$\frac{1}{n}\mathbb{Z}\to \mathrm{Qcoh}(X)$$

where there is one arrow $a\to b$ in $\frac{1}{n}\mathbb{Z}$ if and only if $a\leq b$ (i.e. there is $p\in\frac{1}{n}\mathbb{N}$ such that $a+p=b$).

Let X be a log scheme with log structure $L\colon P \to \mathrm{Div}(X)$, and choose an index $n \in \mathbb{N}$ ($\sim$ common denominator of the weights).

Denote by $\frac{1}{n}P^{\mathrm{wt}}$ the category with objects the elements of $\frac{1}{n}P^{gp}$ and arrows $a \to b$ elements $p \in \frac{1}{n}P$ such that $a + p = b$.

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Definition (Borne-Vistoli)

A parabolic sheaf on X with weights in $\frac{1}{n}P$ is a functor $E\colon \frac{1}{n}P^{\mathrm{wt}} \to \mathrm{Qcoh}(X)$ together with isomorphisms

$$E_{a+p} \cong E_a \otimes L_p \text{ for any } a \in \tfrac{1}{n}P^{gp} \text{ and } p \in P$$

(that satisfy some compatibility properties).

Example

If the log structure on X is given by a snc divisor $D = D_1 + D_2$ with 2 irreducible components and we take $n = 2$,

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then a parabolic sheaf can be seen as

$$\begin{tikzcd}
E_{(0,0)} \otimes \mathcal{O}(-D_1) \arrow[r] & E_{(-\frac{1}{2},0)} \arrow[r] & E_{(0,0)} \\
E_{(0,-\frac{1}{2})} \otimes \mathcal{O}(-D_1) \arrow[r] \arrow[u] & E_{(-\frac{1}{2},-\frac{1}{2})} \arrow[r] \arrow[u] & E_{(0,-\frac{1}{2})} \arrow[u] \\
E_{(0,0)} \otimes \mathcal{O}(-D) \arrow[r] \arrow[u] & E_{(-\frac{1}{2},0)} \otimes \mathcal{O}(-D_2) \arrow[r] \arrow[u] & E_{(0,0)} \otimes \mathcal{O}(-D_2) \arrow[u]
\end{tikzcd}$$

in the "negative unit square", and extended outside by tensoring with powers of $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$.

Root stacks (Olsson, Borne-Vistoli)

Take a log scheme X with log structure $L\colon A \to \mathrm{Div}_X$, and $n \in \mathbb{N}$.

The *n-th root stack* $\sqrt[n]{X}$ parametrizes liftings

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & \mathrm{Div}_X \\ & \mathrm{Div}_X \arrow[u, "\wedge n"'] \end{tikzcd}$$

where $\wedge n\colon \ \mathrm{Div}_X \to \mathrm{Div}_X$ is given by

$$(L, s) \mapsto (L^{\otimes n}, s^{\otimes n}).$$

- If the log structure is induced by an irreducible Cartier divisor $D \subseteq X$, the stack $\sqrt[n]{X}$ parametrizes n-th roots of the divisor D.

  That is, pairs (L, s) such that $(L, s)^{\otimes n} \cong (\mathcal{O}(D), 1_D)$.

- If X is a compact Riemann surface and $D = x_1 + \ldots + x_k$, then $\sqrt[n]{X}$ is an orbifold with coarse moduli space X, and stabilizer $\mathbb{Z}/n\mathbb{Z}$ over the punctures x_i.

Root stacks are tame Artin stacks, Deligne–Mumford in good cases (for example if $\mathrm{char}(k) = 0$).

Theorem (Borne-Vistoli)

Let X be a log scheme with log structure $A \to \mathrm{Div}_X$. There is an equivalence between

parabolic sheaves on X with weights in $\frac{1}{n}A$, and

quasi-coherent sheaves on the root stack $\sqrt[n]{X}$.

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The "pieces" E_a of the parabolic sheaves are obtained (roughly) as eigensheaves for the action of the stabilizers of $\sqrt[n]{X}$.

The infinite root stack

(with A. Vistoli)

If $n \mid m$, there is a projection morphism

$$\sqrt[m]{X} \to \sqrt[n]{X}$$

that corresponds to raising to the $\frac{m}{n}$-th power.
This gives a projective system of algebraic stacks.

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Definition

The infinite root stack of X is the inverse limit $\sqrt[\infty]{X} = \varprojlim_n \sqrt[n]{X}$.

The stack $\sqrt[\infty]{X}$ parametrizes compatible systems of roots of all orders. It is not algebraic, but it has local presentations as a quotient stack.

If X is a compact Riemann surface with the log structure induced by the divisor $D = x_1 + \cdots + x_k$, the infinite root stack $\sqrt[\infty]{X}$

- looks like X outside of D, and
- there is a stabilizer group $\widehat{\mathbb{Z}}$ at each of the points x_i.

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Theorem (-, Vistoli)

There is an equivalence between quasi-coherent sheaves on $\sqrt[\infty]{X}$ and parabolic sheaves on X with arbitrary rational weights.

($\rightsquigarrow$ moduli spaces for parabolic sheaves)

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Theorem (-, Vistoli)

Every isomorphism $\sqrt[\infty]{X} \cong \sqrt[\infty]{Y}$ of stacks comes from a unique isomorphism of log schemes $X \cong Y$.

The Kato-Nakayama space

From now on consider schemes locally of finite type over $\mathbb{C}$.
Let X be a log scheme.

There is an "underlying topological space" X_{log} with a surjective map $\tau\colon X_{log} \to X$.

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The (reduced) fiber of $\sqrt[\infty]{X} \to X$ over x is $B\widehat{\mathbb{Z}}^k$, where k is the same number.

Note that $S^1 = B\mathbb{Z}$, and so $\widehat{(S^1)^k} \cong B\widehat{\mathbb{Z}}^k$.

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Theorem (Carchedi, Scherotzke, Sibilla, -)

There is a canonical map of topological stacks

$$\Phi_X \colon X_{log} \to \sqrt[\infty]{X}_{top}$$

that induces an equivalence upon profinite completion.

The description of the map is easier if one interprets X_{log} itself as parametrizing “roots” of a certain kind.

X_{log} as a root stack

As the stack $\sqrt[n]{X}$ parametrizes

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it turns out $\sqrt[n]{X}_{top}$ parametrizes

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & {[\mathbb{C}/\mathbb{C}^\times]_X} \\ & {[\mathbb{C}/\mathbb{C}^\times]_X.} \arrow[u, "\wedge n"'] \end{tikzcd}$$

(note $\mathrm{Div}_X \sim [\mathbb{A}^1/\mathbb{G}_m]_X$).

A way to map to something that dominates every morphism $\wedge n\colon [\mathbb{C}/\mathbb{C}^\times]_X \to [\mathbb{C}/\mathbb{C}^\times]_X$ is to (in some sense) extract a logarithm.

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Consider the stack $X_{\mathbb{H}}$ that parametrizes

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & {[\mathbb{C}/\mathbb{C}^\times]_X} \\ & {[\mathbb{H}/\mathbb{C}^+]_X} \arrow[u, "exp"'] \end{tikzcd}$$

where $\mathbb{H} = \mathbb{R}_{\geq 0} \times \mathbb{R}$ and *exp* is induced by $\mathbb{H} \to \mathbb{C}$ given by

$$(x, y) \mapsto x \cdot e^{iy}$$

and by the exponential $\mathbb{C}^+ \to \mathbb{C}^\times$.

For every n we have a factorization

$$\begin{tikzcd} {[\mathbb{C}/\mathbb{C}^\times]} & \\ & {[\mathbb{C}/\mathbb{C}^\times]} \arrow[ul, "\wedge n"'] \\ {[\mathbb{H}/\mathbb{C}^+]} \arrow[uu, "\exp"] \arrow[ur, "\phi_n"'] & \end{tikzcd}$$

where $\phi_n \colon [\mathbb{H}/\mathbb{C}^+] \to [\mathbb{C}/\mathbb{C}^\times]$ is given by $\mathbb{H} \to \mathbb{C}$

$$(x, y) \mapsto (\sqrt[n]{x}, y/n) \mapsto \sqrt[n]{x} \cdot e^{i\frac{y}{n}}$$

and by $\mathbb{C}^+ \to \mathbb{C}^\times$ given by $z \mapsto e^{\frac{z}{n}}$.

Now the diagram

$$
\begin{tikzcd}
A \arrow[r] \arrow[rrd, dotted, bend right=15] \arrow[rdd, dashed, bend right] & {[\mathbb{C}/\mathbb{C}^\times]_X} & \\
& & {[\mathbb{C}/\mathbb{C}^\times]_X} \arrow[lu, "\wedge n"'] \\
& {[\mathbb{H}/\mathbb{C}^+]_X} \arrow[uu, "exp"] \arrow[ru, "\phi_n"'] &
\end{tikzcd}
$$

gives a natural transformation $X_{\mathbb{H}} \to \sqrt[n]{X}_{top}$.

These are compatible and give $X_{\mathbb{H}} \to \sqrt[\infty]{X}_{top} = \varprojlim_n \sqrt[n]{X}_{top}$.

Theorem ((in progress) -, Vistoli)

The topological space X_{log} represents the stack $X_{\mathbb{H}}$.

The morphism $X_{log} \to \sqrt[\infty]{X}_{top}$ that we obtain is the one mentioned before.

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The morphism $X_{log} \to \sqrt[\infty]{X}_{top}$ that we obtain is the one mentioned before.

This is related to

- real roots of the log structure, i.e. diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d] & {[\mathbb{C}/\mathbb{C}^{\times}]_X} \\ A_{\mathbb{R}_{\geq 0}} \arrow[ur, dashed] & \end{tikzcd}$$

- parabolic sheaves with real weights.

Thank you for your attention!