$\frac{|x-5|\cdot|x+5|}{|x|} < \varepsilon$, $\forall x \in$ all'intorno di $x_0=5$ di ampiezza $\frac{1}{2}$, risulta:

$|x-5|<\frac{1}{2} \iff -\frac{1}{2} < x-5 < \frac{1}{2} \iff 5-\frac{1}{2} < x < 5+\frac{1}{2} \iff \frac{9}{2} < x < \frac{11}{2} \iff$

$\frac{1}{\frac{11}{2}} < \frac{1}{x} < \frac{1}{\frac{9}{2}} \iff \frac{2}{11} < \frac{1}{x} < \frac{2}{9} \Rightarrow \boxed{\frac{1}{|x|} < \frac{2}{9}}$;

$\frac{9}{2}+5 < x+5 < \frac{11}{2}+5 \iff \frac{19}{2} < x+5 < \frac{21}{2} \Rightarrow \boxed{|x+5| < \frac{21}{2}}$;

perciò

$\left|\frac{25-x^2}{x}\right| = \frac{|x+5|\cdot|x+5|}{|x|} < |x-5|\cdot\frac{21}{2}\cdot\frac{2}{9} = \frac{7}{3}|x-5|$ per $\forall x\in\mathbb{R}$

tale che $|x-5|<\frac{1}{2}$. Fissato ε, $\delta = \min\left[\frac{1}{2}, \frac{3}{7}\varepsilon\right]$,

risulta: $\left|\frac{25-x^2}{x}\right| < \frac{7}{3}\cdot|x-5| \le \frac{7}{3}\cdot\frac{3}{7}\varepsilon = \varepsilon$, $\forall x\in\mathbb{R}$

tale che $|x-5|<\delta$.

Problema 2

(a) Il termine dominante si mette in evidenza.

Fila A $\lim_{n\to+\infty} \sqrt[n]{3^n\left[\underset{\downarrow 0}{\frac{\log(n^3)}{3^n}}+\underset{\downarrow 0}{\frac{\sin n^n}{3^n}}+\underset{\downarrow 0}{\frac{3n^{2/3}}{3^n}}+2\right]} =$

$= \lim_{n\to+\infty} 3\cdot\sqrt[n]{2} = 3\cdot 1 = 3$ $\left(\lim_{\substack{n\to+\infty\\ a>0}}\sqrt[n]{a}=1\right)$

Fila B $\lim_{n\to+\infty} \sqrt[n]{4^n\left[\underset{\downarrow 0}{\frac{\log(n^5)}{4^n}}+\underset{\downarrow 0}{\frac{\cos n!}{4^n}}+3+2\underset{\downarrow 0}{\frac{n^{5/7}}{4^n}}\right]} =$

$= \lim_{n\to+\infty} 4\cdot\sqrt[n]{3} = 4\cdot 1 = 4$

$= 5 =$

Fila C

$$\lim_{n\to+\infty} \sqrt[n]{5^n \cdot \left[\frac{\log(n^7)}{5^n} + \frac{\sin n^2}{5^n} + \frac{3 \cdot n^{3/4}}{5^n} + 2 \right]} =$$

$$\downarrow 0 \qquad \downarrow 0 \qquad \downarrow 0$$

$$= \lim_{n\to+\infty} 5 \cdot \sqrt[n]{2} = 5 \cdot 1 = 5$$

Fila B

$$\lim_{n\to+\infty} \sqrt[n]{6^n \left[\frac{\log(n)}{6^n} + \frac{\cos n!}{6^n} + \frac{3 \cdot n^{3/5}}{6^n} + 5 \right]} =$$

$$\downarrow 0 \qquad \downarrow 0 \qquad \downarrow 0$$

$$= \lim_{n\to+\infty} 6 \cdot \sqrt[n]{5} = 6 \cdot 1 = 6$$

(6) Usiamo che: $\lim_{\substack{n\to+\infty \\ a_n \to 0 \\ b_n \to +\infty}} (1+a_n)^{b_n} = e^{\lim_{n\to+\infty} a_n \cdot b_n}$ (se esiste)

Fila A

$$\lim_{n\to+\infty} \left(1 + \sin \frac{5}{n^3}\right)^{\sqrt{n^6+3n^3+11}} \quad [(1+0)^{+\infty}] =$$

$$= e^{\lim_{n\to+\infty} \left(\sin \frac{5}{n^3}\right) \sqrt{n^6+3n^3+11}} = e^5 = \boxed{e^5} \text{ perché}$$

$$\lim_{n\to+\infty} \frac{\left(\sin \frac{5}{n^3}\right)}{\frac{5}{n^3}} \cdot \frac{5}{n^3} \cdot \sqrt{n^6+3n^3+11} =$$

$$= \lim_{\frac{5}{n^3}\to 0} \frac{\sin \frac{5}{n^3}}{\frac{5}{n^3}} \cdot \lim_{n\to+\infty} \frac{5}{n^3} \cdot \sqrt{n^6\left(1+\frac{3}{n^3}+\frac{11}{n^6}\right)} = 1 \cdot 5 \cdot \lim_{n\to+\infty} \frac{n^3}{n^3} = 5$$

$$\downarrow 0 \quad \downarrow 0$$

Fila B $\displaystyle\lim_{n\to+\infty}\left(1+\arctan\frac{7}{n}\right)^{\sqrt{n^2+2n+3}} =$

$\displaystyle = e^{\lim_{n\to+\infty}\left(\arctan\frac{7}{n}\right)\cdot\sqrt{n^2+2n+3}} = e^{7}$ perché

$$\lim_{n\to+\infty}\frac{\left(\arctan\frac{7}{n}\right)}{\frac{7}{n}}\cdot\frac{7}{n}\cdot\sqrt{n^2+2n+3} =$$

$$= \lim_{\frac{7}{n}\to 0}\frac{\arctan\frac{7}{n}}{\frac{7}{n}}\cdot\lim_{n\to+\infty}\frac{7}{n}\cdot\sqrt{n^2\left(1+\underset{\downarrow 0}{\frac{2}{n}}+\underset{\downarrow 0}{\frac{3}{n^2}}\right)} =$$

$$= 1\cdot 7\cdot\lim_{n\to+\infty}\frac{n}{n} = 7$$

Fila C $\displaystyle\lim_{n\to+\infty}\left(1+\tan\frac{4}{n^2}\right)^{\sqrt{n^4+2n^2+5}} =$

$\displaystyle = e^{\lim_{n\to+\infty}\left(\tan\frac{4}{n^2}\right)\cdot\sqrt{n^4+2n^2+5}} = e^{4}$ perché

$$\lim_{n\to+\infty}\frac{\tan\frac{4}{n^2}}{\frac{4}{n^2}}\cdot\frac{4}{n^2}\cdot\sqrt{n^4+2n^2+5} = \lim_{\frac{4}{n^2}\to 0}\frac{\tan\frac{4}{n^2}}{\frac{4}{n^2}}\cdot$$

$$\cdot\lim_{n\to+\infty}\frac{4}{n^2}\cdot\sqrt{n^4\left(1+\underset{\downarrow 0}{\frac{2}{n^2}}+\underset{\downarrow 0}{\frac{5}{n^4}}\right)} = 1\cdot 4\cdot\lim_{n\to+\infty}\frac{n^2}{n^2} = 4$$

Fila B $\lim_{n\to+\infty}\left(1+\arcsin\frac{3}{n}\right)^{\sqrt{n^2+n+2}} = \quad =7=$

$= e^{\lim_{n\to+\infty}\left(\arcsin\frac{3}{n}\right)\cdot\sqrt{n^2+n+2}} = e^3$ perché

$$\lim_{n\to+\infty}\frac{\arcsin\frac{3}{n}}{\frac{3}{n}}\cdot\frac{3}{n}\cdot\sqrt{n^2+n+2} =$$

$$= \lim_{\frac{3}{n}\to 0}\frac{\arcsin\frac{3}{n}}{\frac{3}{n}}\cdot\lim_{n\to+\infty}\frac{3}{n}\cdot\sqrt{n^2\Big(1+\underbrace{\tfrac{1}{n}}_{\downarrow 0}+\underbrace{\tfrac{2}{n^2}}_{\downarrow 0}\Big)} = 1\cdot 3\cdot\lim_{n\to+\infty}\frac{n}{n} = 3$$

Problema 3

Fila A $\lim_{x\to 0}\left[1+(\sin 2x)^2\right]^{\frac{1}{(e^{5x}-1)\arctan 3x}} =$

$$= \lim_{x\to 0} e^{\frac{1}{(e^{5x}-1)\arctan 3x}\log\left[1+(\sin 2x)^2\right]} =$$

$$= \lim_{x\to 0} e^{\frac{\log[1+(\sin 2x)^2]}{(e^{5x}-1)\arctan 3x}} = e^{\lim_{x\to 0}\frac{\log[1+(\sin 2x)^2]}{(e^{5x}-1)\arctan 3x}} = e^{\frac{4}{15}}$$

perché $\lim_{x\to 0}\frac{\log[1+(\sin 2x)^2]}{(\sin 2x)^2}\cdot\frac{(\sin 2x)^2}{(2x)^2}\cdot\frac{4x^2}{\frac{(e^{5x}-1)}{5x}\cdot 5x\cdot\frac{\arctan 3x}{3x}\cdot 3x} =$

$$= \lim_{x\to 0}\frac{\log[1+(\sin 2x)^2]}{(\sin 2x)^2}\cdot\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)^2\cdot\frac{4}{15}\cdot\frac{1}{\lim_{x\to 0}\frac{e^{5x}-1}{5x}\cdot\lim_{x\to 0}\frac{\arctan 3x}{3x}} =$$

$$= 1\cdot 1^2\cdot\frac{4}{15}\cdot\frac{1}{1\cdot 1} = \frac{4}{15} \quad\Rightarrow$$

$$f(x)=\begin{cases}\left[1+(\sin 2x)^2\right]^{\frac{1}{(e^{5x}-1)\arctan 3x}} & \text{per } x\neq 0\\ e^{\frac{4}{15}} & \text{per } x=0\end{cases}$$

$=8=$

Fila B (come fila A)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\frac{\log[\{1+(\sin 3x)^2]}{(e^{2x}-1)\arcsen 5x}} = e^{\frac{9}{10}} \quad \text{perché}$$

$$\lim_{x\to 0} \frac{\log[1+(\sin 3x)^2]}{(\sin 3x)^2}\cdot\frac{(\sin 3x)^2}{(3x)^2}\cdot\frac{9x^2}{\frac{e^{2x}-1}{2x}\cdot 2x\cdot\frac{\arcsen 5x}{5x}\cdot 5x} = \frac{9}{10}$$

$\to 1 \quad \to 1 \quad \to 1 \quad \to 1$

$$\Rightarrow f(x)=\begin{cases}\left[1+(\sin 3x)^2\right]^{\frac{1}{(e^{2x}-1)\arcsen 5x}} & \text{per } x\neq 0\\ e^{\frac{9}{10}} & \text{per } x=0\end{cases}$$

Fila C (come fila A)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\frac{\log[1+(\sin 4x)^2]}{(e^{3x}-1)\arctan 2x}} = e^{\frac{8}{3}} \quad \text{perché}$$

$$\lim_{x\to 0} \frac{\log[1+(\sin 4x)^2]}{(\sin 4x)^2}\cdot\frac{(\sin 4x)^2}{(4x)^2}\cdot\frac{\overset{8}{\cancel{16}}x^2}{\frac{e^{3x}-1}{3x}\cdot 3x\cdot\frac{\arctan 2x}{2x}\cdot 2x} = \frac{8}{3}$$

$\to 1 \quad \to 1 \quad \to 1 \quad \to 1$

$$\Rightarrow f(x)=\begin{cases}\left[1+(\sin 4x)^2\right]^{\frac{1}{(e^{3x}-1)\arctan 2x}} & \text{per } x\neq 0\\ e^{\frac{8}{3}} & \text{per } x=0\end{cases}$$

Fila B (come fila A)

$$\lim_{x\to0} f(x) = \lim_{x\to0} e^{\frac{\log[1+(\sin 5x)^2]}{(e^{4x}-1)\arcsin 2x}} = e^{\frac{25}{8}} \quad \text{perché}$$

$$\lim_{x\to0} \underbrace{\frac{\log[1+(\sin 5x)^2]}{(\sin 5x)^2}}_{\to 1} \cdot \underbrace{\frac{(\sin 5x)^2}{(5x)^2}}_{\to 1} \cdot \frac{25x^2}{\underbrace{\frac{e^{4x}-1}{4x}}_{\downarrow 1} \cdot 4x \cdot \underbrace{\frac{\arcsin 2x}{2x}}_{\downarrow 1} \cdot 2x} = \frac{25}{8}$$

$$\Rightarrow f(x) = \begin{cases} [1+(\sin 5x)^2]^{\frac{1}{(e^{4x}-1)\arcsin 2x}} & \text{per } x \neq 0 \\ e^{\frac{25}{8}} & \text{per } x = 0 \end{cases}$$

Problema 4

Fila A a) Sia ~~...~~ $z = x+iy$ dove $x, y \in R$

$\Rightarrow \bar{z} = x - iy$; $z \cdot \bar{z} = (x+iy)\cdot(x-iy) = x^2 - i^2y^2 = x^2+y^2$

$(1-i)^8 = [(1-i)^2]^4 = (1-2i+i^2)^4 = (1-2i-1)^4 = (-2i)^4 = 2^4 i^4 = 16$

$\Rightarrow x^2+y^2+4(x-iy) = 16-4i-4$

$x^2+y^2+4x-4yi = 12-4i \Rightarrow \begin{cases} x^2+y^2+4x = 12 \\ -4y = -4 \end{cases} \Rightarrow \boxed{y=1}$

$x^2+1+4x-12=0$

$x^2+4x-11=0$

$x_{1,2} = -2 \pm \sqrt{15}$ $\to x_1 = -2+\sqrt{15}$ $y_1 = 1$

$\to x_2 = -2-\sqrt{15}$ $y_2 = 1$

$\Rightarrow z_1 = -2+\sqrt{15}+i$; $z_2 = -2-\sqrt{15}+i$

b) $z^3 - 8 = 0$; $(z-2)(z^2+2z+4) = 0$

$z-2=0$ $z^2+2z+4=0$

$\boxed{z_1 = 2}$ $\Delta = 1-4 = -3$ $z_{2,3} = -1 \pm \sqrt{3}i \Rightarrow z_2 = -1+\sqrt{3}i$

$\to z_3 = -1-\sqrt{3}i$

Fila B a) $z = x+iy$; $x,y \in \mathbb{R}$ $= 10 =$

$\Rightarrow |z| = \sqrt{x^2+y^2}$, $\bar{z} = x - iy$ $\Rightarrow |z|^2 = x^2+y^2$

$(1+i)^6 = \left[(1+i)^2\right]^3 = (1+2i+i^2)^3 = (1+2i-1)^3 = (2i)^3 = 8i^3 = -8i$

$\Rightarrow \cancel{\sqrt{x^2+y^2} + 2(x-iy) = -8i + 24}$

$\cancel{\sqrt{x^2+y^2} + 2x - 2yi = 24 - 8i} \Rightarrow \cancel{x^2}$

$\Rightarrow x^2+y^2+2(x-iy) = -8i+24$

$x^2+y^2+2x-2yi = 24-8i \Rightarrow \begin{cases} x^2+y^2+2x = 24 \\ -2y = -8 \Rightarrow \boxed{y=4} \end{cases}$

$x^2+16+2x = 24$

$x^2+2x-8=0$

$x_{1,2} = -1 \pm 3 \rightarrow x_1 = 2 \quad y_1 = 4$

$\rightarrow x_2 = -4 \quad y_2 = 4$

$\Rightarrow \underline{z_1 = 2+4i}$; $\underline{z_2 = -4+4i}$

b) $z^4 - 81 = 0$; $(z^2-9)(z^2+9) = 0$

$z^2 - 9 = 0$ $\qquad$ $z^2+9=0$

$z^2 = 9$ $\qquad$ $z^2 = -9$

$z_{1,2} = \pm\sqrt{9}$ $\qquad$ $z^2 = 9i^2$

$\underline{z_{1,2} = \pm 3}$ $\qquad$ $\underline{z_{3,4} = \pm 3i}$

Fila C a) Sia $z = x+iy$; $x,y \in \mathbb{R}$

$\Rightarrow \bar{z} = x - iy$; $z \cdot \bar{z} = (x+iy)(x-yi) = x^2 - y^2i^2 = x^2+y^2$

$(1-i)^6 = \left[(1-i)^2\right]^3 = (1-2i+i^2)^3 = (1-2i-1)^3 = (-2i)^3 = -8i^3 = 8i$

$\Rightarrow x^2+y^2+2(x-iy) = 8i+24$

$x^2+y^2+2x-2yi = 24+8i \Rightarrow \begin{cases} x^2+y^2+2x = 24 \\ -2y = 8 \Rightarrow \boxed{y=-4} \end{cases}$

$x^2+16+2x=24$; $x^2+2x-8=0$;

$x_{1,2} = -1 \pm 3 \rightarrow x_1 = 2 \quad y_1 = -4$; $\Rightarrow \underline{z_1 = 2-4i}$; $\underline{z_2 = -4-4i}$

$\rightarrow x_2 = -4 \quad y_2 = -4$

b) $z^3-27=0$; $(z-3)(z^2+3z+9)=0$ $=11z$

$z-3=0$; $z^2+3z+9=0$

$\boxed{z_1=3}$; $\Delta=9-36=-27$

$z_{2,3}=\dfrac{-3\pm\sqrt{27}i}{2}=\dfrac{-3\pm3\sqrt{3}i}{2}$

Fila 8 a) Sia $z=x+iy$; $x,y\in\mathbb{R}$

$|z|=\sqrt{x^2+y^2} \Rightarrow |z|^2=x^2+y^2$; $\bar{z}=x-iy$

$(1+i)^8=(1+2i+i^2)^4=(1+2i-1)^4=(2i)^4=16i^4=16$

$\Rightarrow x^2+y^2+4(x-iy)=16-4i-4$

$x^2+y^2+4x-4yi=12-4i \Rightarrow \begin{cases} x^2+y^2+4x=12 \\ -4y=-4 \end{cases} \Rightarrow \boxed{y=1}$

$x^2+1+4x=12$

$x^2+4x-11=0$

$x_{1,2}=-2\pm\sqrt{15}$ $\rightarrow x_1=-2+\sqrt{15}$ $y_1=1$

$\rightarrow x_2=-2-\sqrt{15}$ $y_2=1$

$\Rightarrow z_1=-2+\sqrt{15}+i$; $z_2=-2-\sqrt{15}+i$

b) $z^4-16=0$; $(z^2-4)(z^2+4)=0$

$z^2-4=0$; $z^2+4=0$

$z^2=4$; $z^2=-4$

$z_{1,2}=\pm\sqrt{4}=\pm2$; $z^2=4i^2$

$z_{1,2}=\pm2$; $z_{3,4}=\pm2i$