Soluzioni

1. Equazione diff. lineare del I ordine
$a(x) = -\frac{2}{x+1}$, $x > -1$; $A(x) = -2\lg(x+1)$; $e^{A(x)} = \frac{1}{(x+1)^2}$

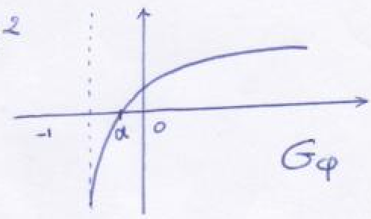
$$\left(\frac{y}{(x+1)^2}\right)' = \frac{4}{(x+1)(x+2)} \Rightarrow \frac{y}{(x+1)^2} = 4\lg\left(\frac{x+1}{x+2}\right) + c \Rightarrow$$

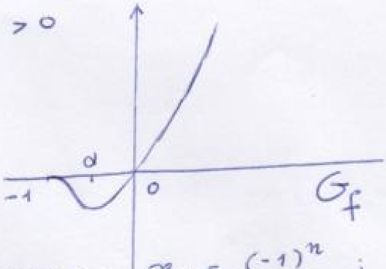
$$y(x) = 4(x+1)^2 \lg\left(\frac{x+1}{x+2}\right) + c(x+1)^2.$$

2. nel dominio $(-1,+\infty)$ assegnato la funzione risulta definita.
$\lim_{x\to -1} f(x) = 0$; $x=-1$ discontinuità eliminabile ($f(-1)=0$)
$\lim_{x\to +\infty} f(x) = \lim_{x\to+\infty} x^2 \lg 2 = +\infty$ senza asintoto
$f(x) \geq 0 \iff \frac{2(x+1)}{x+2} \geq 1 \iff x \geq 0$

$$\underset{-1}{\circ} \; - \; \underset{0}{\circ} \; +$$

$f'(x) = 2(x+1)\lg\frac{2(x+1)}{x+2} + \frac{x+1}{x+2}$
$\lim_{x\to -1} f'(x) = 0$; la fz. prolungata è derivabile in $x_0 = -1$ ($f'(-1)=0$).
Per ricavare il segno di f', studiamo la fz. $\varphi(x) = 2\lg\frac{2(x+1)}{x+2} + \frac{1}{x+2}$.
$\lim_{x\to -1}\varphi(x) = -\infty$, $\varphi(0) = 1/4$, $\lim_{x\to+\infty}\varphi(x) = 2\lg 2$
$\varphi'(x) = \frac{x+3}{(x+1)(x+2)^2} > 0$





3. . Un possibile esempio: $x_n = \frac{(-1)^n}{n}$; inf = min = -1 , sup = max = $\frac{1}{2}$
limite = 0.

. $\sqrt{n^2+n} - n = n\left(\sqrt{1+\frac{1}{n}} - 1\right) \sim \frac{n}{2n} \to \frac{1}{2}$

. $\sqrt{n^2+n} < n + \frac{1}{2} + \varepsilon \Rightarrow \cancel{n^2} + n < \cancel{n^2} + 2\left(\frac{1}{2}+\varepsilon\right)n + \left(\frac{1}{2}+\varepsilon\right)^2$
$\cancel{n} < \cancel{n} + 2\varepsilon n + \left(\frac{1}{2}+\varepsilon\right)^2$ sempre verificato

$\sqrt{n^2+n} > n + \frac{1}{2} - \varepsilon$ (definitiv. >0) $\Rightarrow \cancel{n^2} + n > \cancel{n^2} + 2\left(\frac{1}{2}-\varepsilon\right)n + \left(\frac{1}{2}-\varepsilon\right)^2$
$\cancel{n} > \cancel{n} - 2\varepsilon n + \left(\frac{1}{2}-\varepsilon\right)^2$
$n > \left(\frac{1}{2}-\varepsilon\right)^2/2\varepsilon$, come richiesto.

4. . Partiamo dal fatto che $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ per $-1<x<1$.
$\frac{1}{1-x} = 2$ per $x = \frac{1}{2}$.
Per $\alpha \neq 1$, $x_n \sim n^\alpha\left(\frac{1}{n^\alpha} - \frac{\alpha}{n}\right)$; se $\alpha > 1$ $x_n \sim -\frac{\alpha}{n^{1-\alpha}}$ serie div ;
. se $0<\alpha<1$, $x_n \sim \frac{n^\alpha}{n^\alpha} \to 1$ serie div. Per $\alpha = 1$ $x_n \sim -\frac{1}{2n^2}$ serie conv.

5. $\frac{1-\cos x}{x^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{(2n)!}$, $\int_0^1 \frac{1-\cos x}{x^2}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!\,(2n-1)}$

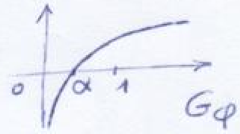
$|E_n| < \frac{1}{(2n+2)!\,(2n+1)} < 10^{-3}$ per $n \geq 2$; $I \sim \frac{1}{2} - \frac{1}{72} \sim 0{,}486$

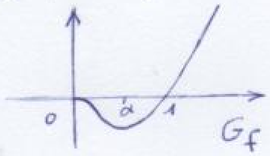
Soluzioni [2]

1. equazione diff. lineare del I ordine.
$a(x) = -\frac{2}{x}$, $x>0$; $A(x) = -2\lg x$; $e^{A(x)} = \frac{1}{x^2}$

$$\left(\frac{y}{x^2}\right)' = \frac{4}{x(x+1)} \Rightarrow \frac{y}{x^2} = 4\lg\left(\frac{x}{x+1}\right)+c \Rightarrow y = 4x^2\lg\left(\frac{x}{x+1}\right)+cx^2.$$

2. nel dominio $(0,+\infty)$ la funzione risulta definita.
$f(x)\geq 0 \iff \frac{2x}{x+1}\geq 1 \iff x\geq 1$ $\quad \underset{0}{\circ}\ -\ \underset{1}{\circ}\ +$
$\lim_{x\to 0} f(x)=0$; $x=0$ disc. eliminabile ; $f(0)=0$.
$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^2\lg 2 = +\infty$ senza asintoto.
$f'(x) = 2x\lg\frac{2x}{x+1} + \frac{x}{x+1}$.
$\lim_{x\to0} f'(x)=0$; la fz. prolungata è derivabile in $x_0=0$ $(f'(0)=0)$
Per ricavare il segno di f', studiamo la fz. $\varphi(x) = 2\lg\frac{2x}{x+1}+\frac{1}{x+1}$.
$\lim_{x\to0}\varphi(x) = -\infty$, $\lim_{x\to+\infty}\varphi(x) = 2\lg 2$, $\varphi(1)=\frac{1}{2}$
$\varphi'(x) = \frac{x+2}{x(x+1)^2}>0$





3. . Vedi [1]
. $\sqrt{n^2-n}-n = n\left(\sqrt{1-\frac{1}{n}}-1\right)\sim \frac{n}{-2n}\to -\frac{1}{2}$
$\sqrt{n^2-n} < n-\frac{1}{2}+\varepsilon \Rightarrow \cancel{n^2}-n < \cancel{n^2}+2\left(-\frac{1}{2}+\varepsilon\right)n+\left(\frac{1}{2}+\varepsilon\right)^2$ sempre verificato
$-n<-n+2\varepsilon n + \left(\frac{1}{2}+\varepsilon\right)^2$
$\sqrt{n^2-n} > n-\frac{1}{2}-\varepsilon \Rightarrow \cancel{n^2}-n > \cancel{n^2}-2n\left(\frac{1}{2}+\varepsilon\right)+\left(\frac{1}{2}+\varepsilon\right)^2$
(definitiv. >0) $\quad -n > -n - 2\varepsilon n + \left(\frac{1}{2}+\varepsilon\right)^2$
$n > \left(\frac{1}{2}+\varepsilon\right)^2/2\varepsilon$ come richiesto

4. . Partiamo dal fatto che $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ per $-1<x<1$.
$\frac{1}{1-x}=4$ per $x=\frac{3}{4}$.
. Per $\alpha\neq1$ $x_n\sim n^\alpha\left(\frac{1}{n^\alpha}-\frac{\alpha}{n}\right)$; se $\alpha>1$, $x_n\sim -\frac{\alpha}{n^{1-\alpha}}$ serie div. ; se $0<\alpha<1$
$x_n \sim \frac{n^\alpha}{n^\alpha}\to1$ serie div. ; se $\alpha=1$ $x_n\sim\frac{1}{2n^2}$ serie converg.

5. $\frac{\operatorname{sen} x}{x} = \sum_{n=0}^{\infty} (-1)^n\frac{x^{2n}}{(2n+1)!}$; $\int_0^1\frac{\operatorname{sen}x}{x}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\,(2n+1)}$.
$|E_n| < \frac{1}{(2n+3)!\,(2n+3)} < 10^{-3}$ per $n\geq1$; $I\sim 0,988$