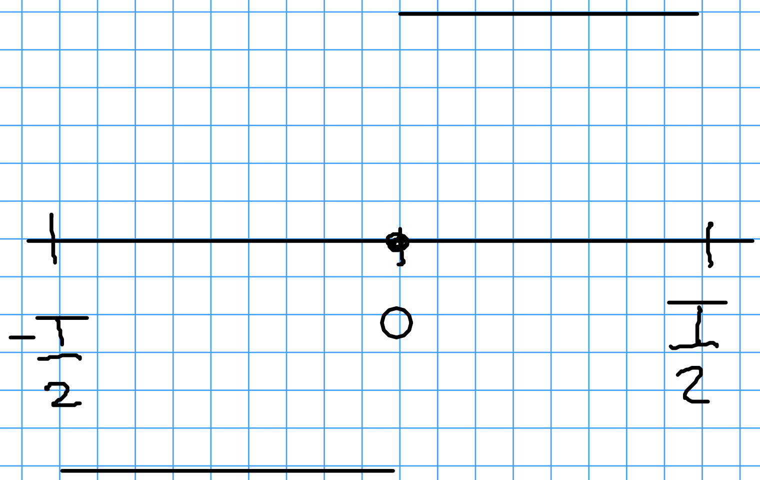
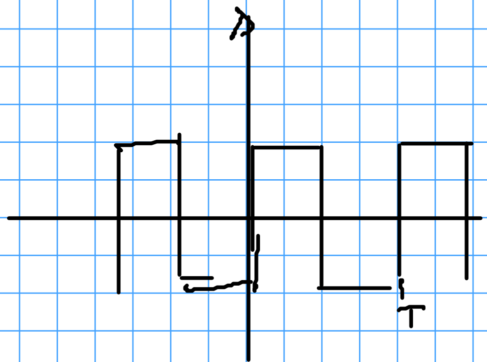
Esempio (onda quadra)



$$f(t) = \begin{cases} 1 & 0 < t < \frac{T}{2} \\ 0 & t = 0 \\ -1 & -\frac{T}{2} < t < 0 \end{cases}$$

$f(t)$ T periodica

$$\left(\omega_0 = \frac{2\pi}{T} \right)$$



Voglio calcolare i coeff di Fourier c_k.

$$c_k = \frac{1}{T} \int_0^T f(t)\, e^{-ik\omega_0 t}\, dt = \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} f(t)\, e^{-i\omega_0 k t}\, dt =$$

$$= \frac{1}{T} \int_{-T/2}^{0} -e^{-i\omega_0 k t}\, dt + \frac{1}{T} \int_0^{T/2} e^{-i\omega_0 k t}\, dt =$$

$$\frac{1}{T}\left[\frac{-e^{-i\omega_0 K t}}{-i\omega_0 K}\right]_{-\frac{T}{2}}^{0} + \frac{1}{T}\left[\frac{e^{-i\omega_0 K t}}{-i\omega_0 K}\right]_{0}^{\frac{T}{2}} =$$

$$\left(\int e^{at} = \frac{e^{at}}{a} + \text{cost}\right)$$

$$\frac{1}{T}\frac{1}{-i\omega_0 K}\left\{-1 + \underset{e^{i\pi K}}{e^{i\omega_0 K \frac{T}{2}}} + \underset{e^{-i\pi K}}{e^{-i\omega_0 K \frac{T}{2}}} - 1\right\} =$$

$$\omega_0 \cdot T = 2\pi$$

$$\frac{i}{\underbrace{T\,\omega_0}_{2\pi} K}\left(2(-1)^K - 2\right) = \qquad e^{\pm i K \pi} = \underset{(-1)^K}{\cos(\pm K\pi)} + i\,\underset{0}{\sin(\pm K\pi)}$$

$$\frac{i}{\pi K}\left((-1)^K - 1\right) = \begin{cases} 0 & K \text{ pari} \\ \dfrac{-2i}{\pi K} & K \text{ dispari} \end{cases}$$

DUNQUE, APPLICANDO IL TEOREMA

$$\circledast = \sum_{K=-\infty}^{+\infty} c_K e^{i\omega_0 K t} \longrightarrow f(t) \quad \forall t \quad \left(\text{perché ho definito } f(0) = 0\right)$$

(CONV. PUNTUALE)

Notiamo che $\circledast = \sum_{k=1}^{\infty} c_k e^{i\omega_0 k t} + \sum_{k=1}^{\infty} c_{-k} e^{-i\omega_0 k t} =$

$$\sum_{k=1}^{\infty} c_k \left(e^{i\omega_0 k t} - e^{-i\omega_0 k t}\right) = \qquad (c_{-k} = -c_k)$$

$$\overset{\|}{\sum_{h=0}^{\infty}} c_{2h+1} \left(e^{i\omega_0 (2h+1)t} - e^{-i\omega_0 (2h+1) t}\right) = \qquad (c_k = 0 \text{ se } k \text{ pari})$$

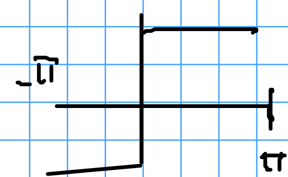
$$\sum_{h=0}^{\infty} \frac{-2i}{\pi(2h+1)} \, 2i \sin((2h+1)\omega_0 t) = \qquad \left(e^{ia} - e^{-ia} = \cancel{\cos(a)} + i \sin(a) - (\cancel{\cos(-a)} + i \sin(-a)) = 2i \sin(a) \right)$$

$$\sum_{h=0}^{\infty} \frac{4}{\pi} \frac{1}{2h+1} \sin((2h+1)\omega_0 t)$$

QUINDI

$$f(t) = \frac{4}{\pi} \sum_{h=0}^{\infty} \frac{1}{2h+1} \sin((2h+1)\omega_0 t) \qquad \text{(puntualmente)}$$

se $T = 2\pi \Rightarrow \omega_0 = 1$

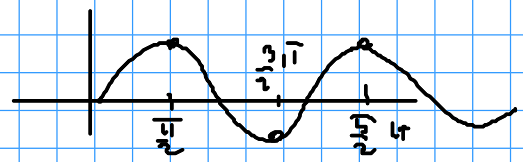


$$= \frac{4}{\pi} \sum_{h=0}^{\infty} \underbrace{\frac{1}{2h+1}} \sin((2h+1)t)$$

Se metto $t = \frac{\pi}{2}$ ho che $f\left(\frac{\pi}{2}\right) = 1 \Rightarrow$

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \underbrace{\sin\left((2k+1)\frac{\pi}{2}\right)}_{(-1)^k} =$$

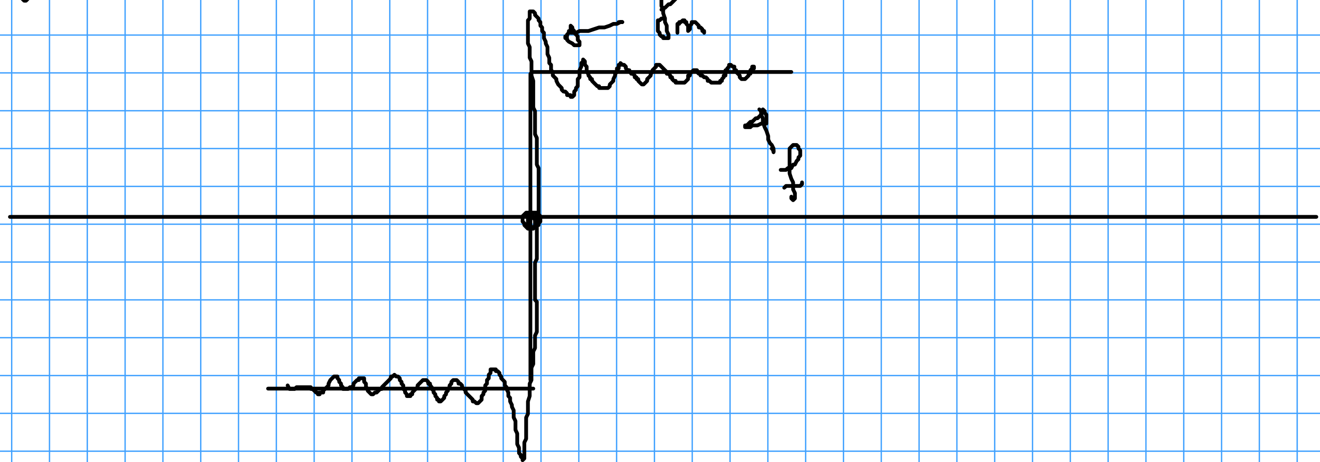
$k\pi + \frac{\pi}{2}$



$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

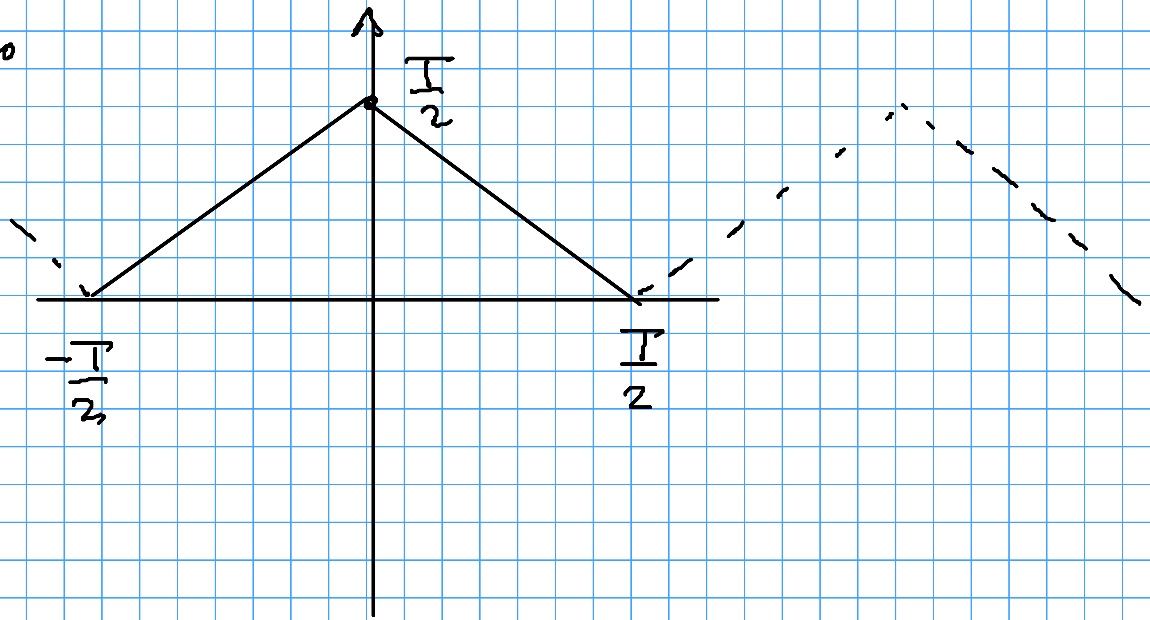
NOTA CHE LA SERIE SOPRA <u>NON</u> È ASSOLUTAM. CONV.

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} = +\infty \quad (\text{serie armonica})$$



VICINO A ZERO LA CONV. DI f_n a f È MENO BUONA (NON È UNIFORME)

Altro esempio
(onda triangolare)



$f(t) = \frac{T}{2} - |t|$. Cerchiamo i c_k

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-i\omega_0 k t}\, dt = \frac{1}{T}\int_{-\frac{T}{2}}^{0} f(t)\, e^{-i\omega_0 k t}\, dt + \frac{1}{T}\int_{0}^{\frac{T}{2}} f(t) e^{-i\omega_0 k t}\, dt$$

(cambio di variabile $t \to -t$ nel primo integrale, $dt \to -dt$)

$$\frac{1}{T}\int_{T/2}^{0} f(-t)\, e^{i\omega_0 k t} (-1)\, dt + \frac{1}{T}\int_{0}^{T/2} f(t)\, e^{-i\omega_0 k t}\, dt =$$ (f è pari)

$$\frac{1}{T}\int_{0}^{T/2} f(t) \left(e^{i\omega_0 k t} + e^{-i\omega_0 k t}\right) dt =$$ (integro per parti)

ATTENZIONE: LO POSSO FARE SE $k \neq 0$

$$\frac{1}{\pi}\left[f(t)\left(\frac{e^{i\omega_0 k t}}{i\omega_0 k} + \frac{e^{-i\omega_0 k t}}{-i\omega_0 k}\right)\right]_0^{T/2} - \frac{1}{\pi}\int_0^{T/2} f'(t)\left(- - - - -\right)dt =$$

$k \neq 0$

$[\dots]_0^{T/2}$

Il primo pezzo è zero: $f(T/2)=0$, in zero $(- - - -)=0 \rightarrow$

$$= -\frac{1}{\pi}\int_0^{T/2} (-1)\,\frac{e^{i\omega_0 k t} - e^{i\omega_0 k t}}{i\omega_0 k}\,dt = \frac{1}{\pi}\int_0^{T/2} \frac{2\sin(\omega_0 k t)}{\omega_0 k}\,dt =$$

$$\frac{1}{\pi}\,\frac{2}{\omega_0 k}\left[\frac{-\cos(\omega_0 k t)}{\omega_0 k}\right]_0^{T/2} = \frac{1}{\pi}\,\frac{2}{(\omega_0 k)^2}\left(1 - \cos\Big(k\,\underbrace{\frac{\omega_0 T}{2}}_{=\pi}\Big)\right) =$$

$$\frac{2}{2\pi\,\omega_0 k^2}\left(1-(-1)^k\right) = \frac{1-(-1)^k}{\pi\,\omega_0\,k^2} = \begin{cases} 0 & k \text{ pari} \\ \underbrace{\dfrac{2}{\pi\,\omega_0 k^2}}_{\text{SOMMABILE}} & k \text{ dispari} \end{cases}$$

$\boxed{k \neq 0}$

FACENDO UN CALCOLO A PARTE $C_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t) = \frac{T}{4}$!!

(NOTIAMO CHE f' = − onda quadra)

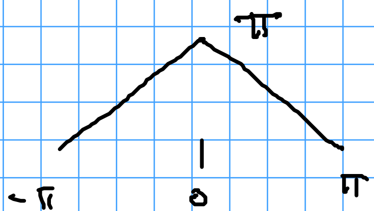
Applicando i teoremi, ragionando come prima:

$$f(t) \underset{\forall t}{\overset{-T/4}{=}} \sum_{k\in\mathbb{Z}} c_k\, e^{ik\omega_0 t} = \sum_{k=1}^{\infty} c_k\, e^{ik\omega_0 t} + \sum_{k=1}^{\infty} c_{-k}\, e^{-ik\omega_0 t}$$

$$\left(c_{-k} = c_k\right) = \sum_{k=1}^{\infty} c_k \left(e^{ik\omega_0 t} + e^{-ik\omega_0 t}\right) = \text{(fare le verifica)}$$

$$\sum_{k=1}^{\infty} c_k \, 2\cos(k\omega_0 t) = \sum_{h=0}^{\infty} c_{2h+1} \, 2\cos((2h+1)\omega_0 t) =$$

$$\frac{4}{\pi\omega_0} \sum_{h=0}^{\infty} \frac{1}{(2h+1)^2} \cos((2h+1)\omega_0 t)$$; se per es. $T = 2\pi$, $\omega_0 = 1$



$$\longrightarrow \frac{4}{\pi} \sum_{h=0}^{\infty} \frac{1}{(2h+1)^2} \cos((2h+1)t) + \frac{\pi}{2}$$

Osserviamo che la serie delle derivate fa

$$\frac{4}{\pi} \sum_{h=0}^{\infty} \frac{1}{(2h+1)^{\cancel{2}}} \left(-\sin((2h+1)t)\right) \cancel{(2h+1)} = \text{serie di } - \text{onda quadra}$$

Calcolando tutto in $t=0$ si trova $\pi = \frac{4}{\pi} \sum_{h=0}^{\infty} \frac{1}{(2h+1)^2} + \frac{\pi}{2} \iff$

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{h=0}^{\infty} \frac{1}{(2h+1)^2} \iff \sum_{h=0}^{\infty} \frac{1}{(2h+1)^2} = \frac{\pi^2}{8} \quad !!$$

Non siamo nelle ipotesi scritte nel esercizio per avere che $f_n \xrightarrow{UNIF} f$

MA IN REALTÀ, IN QUESTO CASO LA CONVERGENZA È UNIFORME

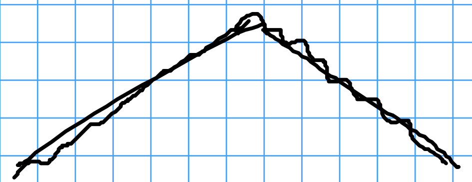
INFATTI STO CONSIDERANDO $\sum_{h=0}^{\infty} \frac{1}{(2h+1)^2} \cos((2h+1)t)$ (ho tolto $\frac{4}{\pi}$)

Questa serie converge TOTALMENTE su $[0, 2\pi]$ infatti:

$$\max_{[0,2\pi]} \left| \frac{1}{(2h+1)^2} \cos((2h+1)t) \right| \leq \frac{1}{(2h+1)^2}$$

e so che $\underline{\sum_{h=0}^{\infty} \frac{1}{(2h+1)^2} < +\infty}$ ($\simeq \frac{1}{h^2}$, serie armonica di esponente 2)

$\Rightarrow$ la serie è unif. conv. Per il teorema (T) la serie converge puntualmente a f $\Rightarrow$ la serie conv unif. a f



la conv. è migliore