Geometria 15/5/18

$\alpha : [a,b] \to \mathbb{R}^2$

$$\underbrace{\kappa(t)}_{\text{curvatura di } \alpha \text{ nel pto } \alpha(t)} = \underbrace{\pm}_{} \frac{1}{\text{raggio circonf con triplice contatto}} = \underbrace{\pm}_{} \underbrace{\|\alpha''(t)\|}_{\text{se } \|\alpha'\| = 1}$$

se α è orientata,

$+$ = curva a sin

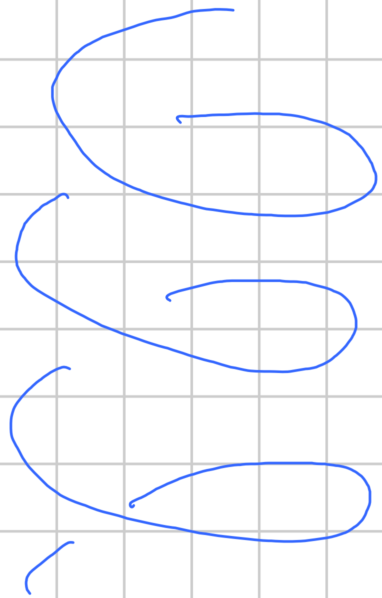
$-$ = curva a dx

Se $\|\alpha'\| \neq 1$

$$\kappa(t) = \frac{\det(\alpha'(t), \alpha''(t))}{\|\alpha'(t)\|}$$

Versione 3D : $\alpha : [a,b] \to \mathbb{R}^3$ curva orientata

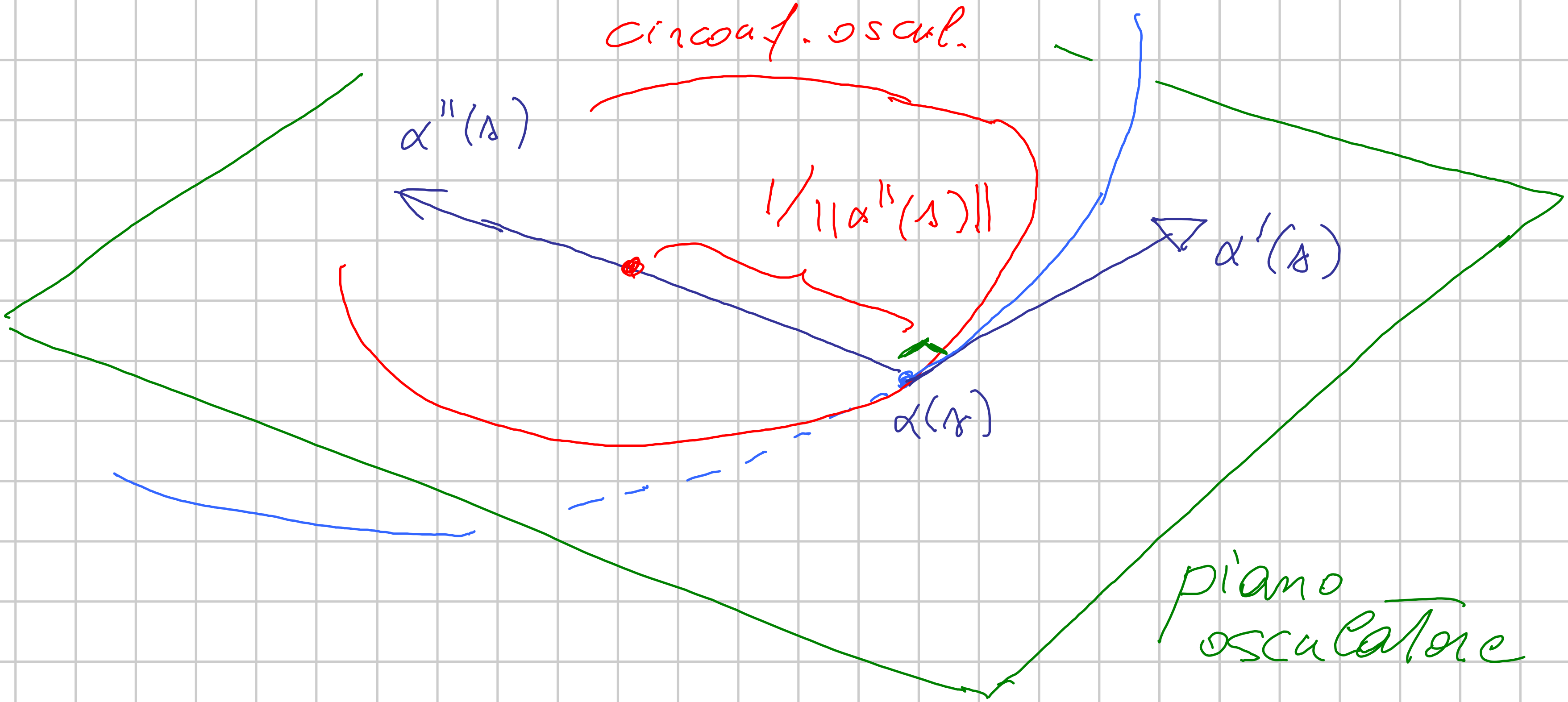
(uso $\alpha(s)$)



Curvatura ($\|\alpha'\| \equiv 1$)

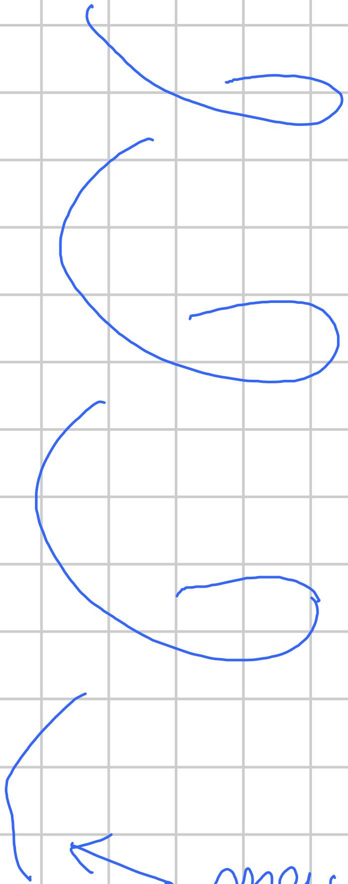
se $\alpha''(s) \neq 0$ il piano che ha triplice contatto con α in $\alpha(s)$ è quello che passa per $\alpha(s)$ e ha giacitura

Span$(\alpha'(s), \alpha''(s))$. Inoltre la circonferenza che ha triplice contatto con α in $\alpha(s)$ è quella che passa per $\alpha(s)$ con centro $\alpha(s) + \dfrac{\alpha''(s)}{\|\alpha''(s)\|^2}$ ($\Rightarrow$ tale circonf. sta sul piano prec.):

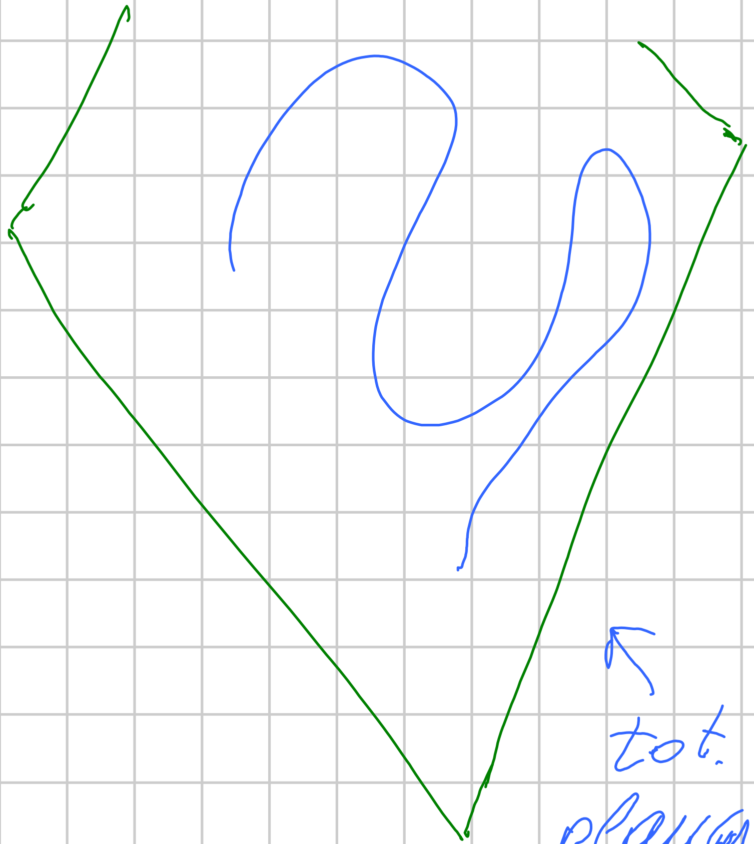


<u>Def</u> : $\kappa(s) = \|\alpha''(s)\|$ (non ha senso segno)

più planare



meno planare



tot. planare

Misura della non-planarità = variazione del piano osculatore

<u>Def</u>: chiamo riferimento di Frénet di α la terna di vettori $(t \; , \; n \; , \; b)$

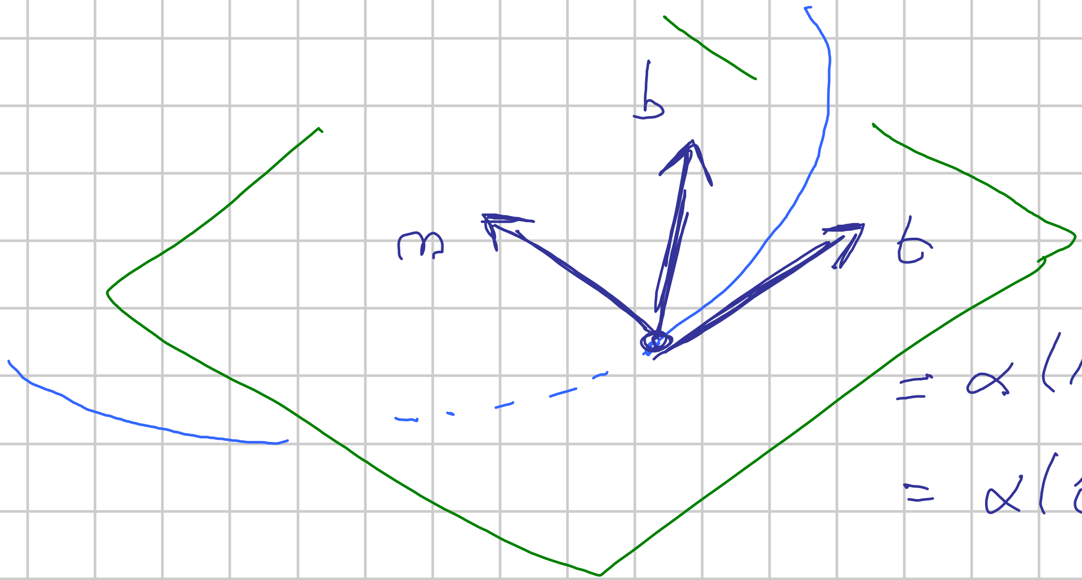
tangente normale binormale

$t(s) = \alpha'(s)$ $\quad$ ($\|\alpha'\| \equiv 1 \Rightarrow t$ unitario)

$n(s) = \alpha''(s) / \|\alpha''(s)\|$ $\quad$ (n unitario e $\perp$ a t)

$$b(s) = t(s) \wedge n(s)$$

$\Longrightarrow (t, n, b)$ base ortonormale positiva di $\mathbb{R}^3$ $(\det = +1)$



$= \alpha(s) + \operatorname{Span}(t(s), n(s))$

$= \alpha(s) + b(s)^{\perp}$

$\Longrightarrow$ misura della variazione di $\mathrm{Span}(t, n)$

è variazione di b.

Prop: Se $\|\alpha'\| = 1$ allora $\exists\, \tau(s)$ t.c.

$$b'(s) = -\tau(s) \cdot n(s)$$

Inoltre

$$n'(s) = -k(s) \cdot t(s) + \tau(s) \cdot b(s)$$

e

$$t'(s) = k(s) \cdot n(s)$$

$$\Longrightarrow \quad (t, n, b)' = (t, n, b) \cdot \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

Dim: $\|b(s)\| \equiv 1 \Longrightarrow b'(s) \perp b(s)$

$$t'(s) = \alpha''(s) = \underbrace{\|\alpha''(s)\|}_{k} \cdot \underbrace{\frac{\alpha''}{\|\alpha''(s)\|}}_{n}$$

$$b(s) = t(s) \times n(s)$$

$$\Longrightarrow b'(s) = t'(s) \times n(s) + t(s) \times n'(s)$$

$$= \kappa(s)\cdot \underbrace{n(s) \wedge n(s)}_{0} + t(s) \wedge n'(s)$$

$$\Longrightarrow b'(s) \perp t(s)$$

$$\Longrightarrow b'(s) = -\tau(s)\cdot n(s)$$

(t, n, b) ortonormale pos

$$\Longrightarrow n = b \times t$$

$$\Longrightarrow n' = b' \wedge t + b \wedge t'$$

$$= -\tau \cdot n \wedge t + b \wedge k \cdot n$$

$$= -k \cdot t + \tau \cdot b .$$ $\blacksquare$

Oss: τ ha un segno (torsione delle viti).

Scoperto che $(t, n, b)' = (t, n, b) \cdot A$

ortogonale

antisimm

Prop: se $M : [a,b] \longrightarrow M_{n\times n}(\mathbb{R})$ e $M(s)$ è ortog. $\forall s$ allora $M'(s) = M(s)\cdot A(s)$ con $A(s)$ antisimmetrica.

Dim: $M^{-1}(s) = {}^tM(s)$

$\Rightarrow {}^tM(s)\cdot M(s) = I_n$

$\Rightarrow {}^tM'(s)\cdot M(s) + {}^tM(s)\cdot M'(s) = 0$.

Se pongo $A(s) = M^{-1}(s)\cdot M'(s) = {}^tM(s)\cdot M'(s)$ ho

$$\underbrace{{}^tM' \cdot M}_{{}^tA} + \underbrace{{}^tM \cdot M'}_{A} = 0$$

$\boxed{\checkmark}$

<u>Fatto</u>: se $M : [0, T] \to M_{n \times n}(\mathbb{R})$

$M'(s) = M(s) \cdot A(s)$ $\quad$ A antisimm

$M(0) = I_n \implies M(s)$ ortog. $\forall s$.

Come calcolare (t, n, b), κ, τ per α qualsiasi?

Difficile esplicitare la riparametrizzazione in p.d'a.

Formule generali per α qualsiasi:

$$\kappa = \frac{\| \alpha' \wedge \alpha'' \|}{\| \alpha' \|^3}$$

$$\tau = \frac{\| \langle \alpha' \wedge \alpha'' \mid \alpha''' \rangle \|}{\| \alpha' \wedge \alpha'' \|^2}$$

<u>Oss</u>: se α è piana, $\beta = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$

$$\frac{\| \beta' \times \beta'' \|}{\| \beta' \|^3} = \frac{\left\| \begin{pmatrix} \alpha' \\ 0 \end{pmatrix} \times \begin{pmatrix} \alpha'' \\ 0 \end{pmatrix} \right\|}{\| \alpha' \|^3} = \frac{|\det(\alpha', \alpha'')|}{\| \alpha' \|^3}$$

$\Rightarrow$ è coerente.

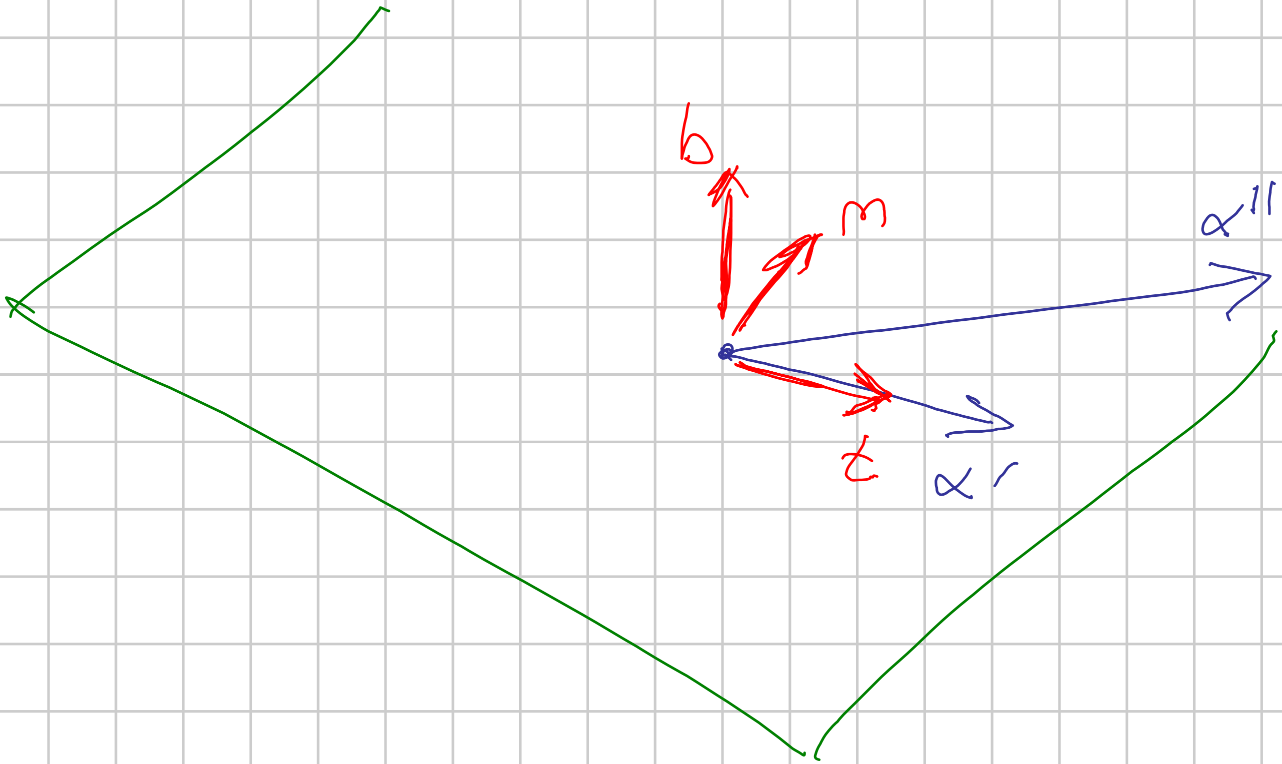
(t, n, b)

I. $t = \dfrac{\alpha'}{\|\alpha'\|}$

n secondo vettore ottenuto ortonormalizzando (α', α'')

$$\left(n = \dfrac{\alpha'' - \dfrac{\langle \alpha' \mid \alpha'' \rangle}{\|\alpha'\|^2} \cdot \alpha'}{\| \quad '' \quad \|} \right)$$

$$b = t \wedge m$$



II. $$t = \frac{\alpha'}{\|\alpha'\|}$$

$$b = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}$$

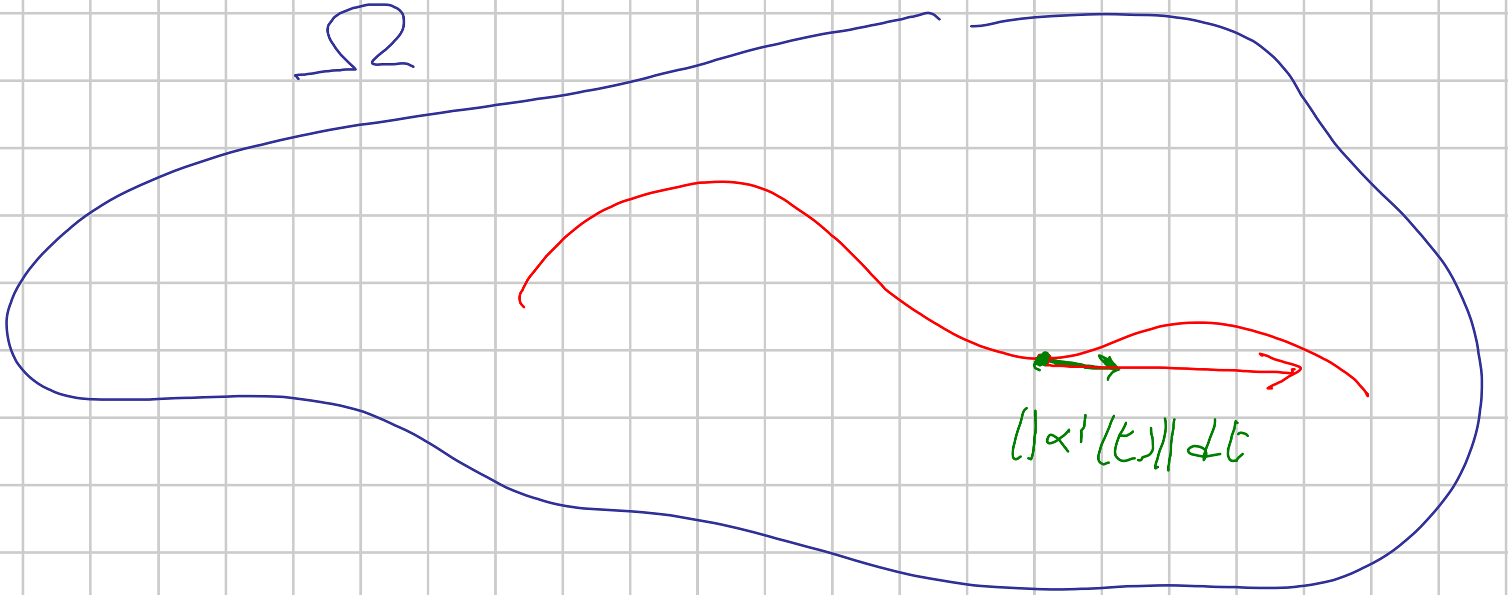
$$m = b \wedge t .$$

Integrazione di forme su curve

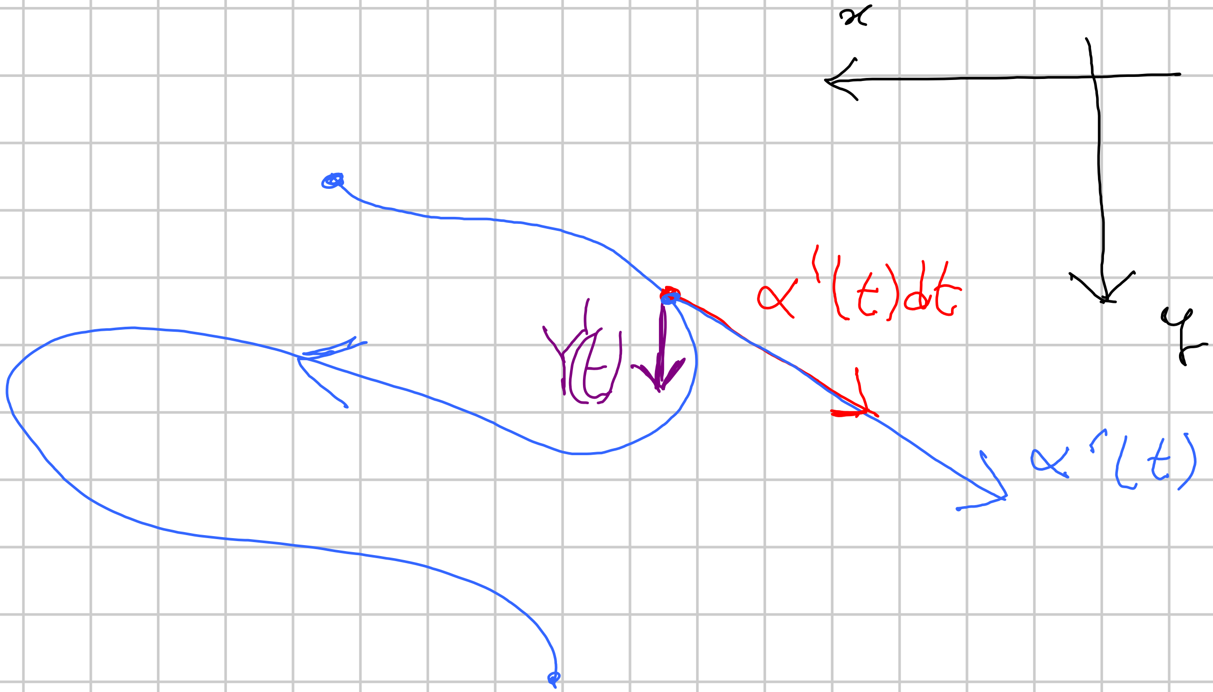
Per funzioni $f : \Omega \longrightarrow \mathbb{R} \qquad \Omega \subset \mathbb{R}^n$

$$\int_{\alpha} f = \int_a^b \underbrace{f(\alpha(t)) \cdot \|\alpha'(t)\| \, dt}$$

non "vede" direzione e verso dello spostamento su α ma solo intensità.

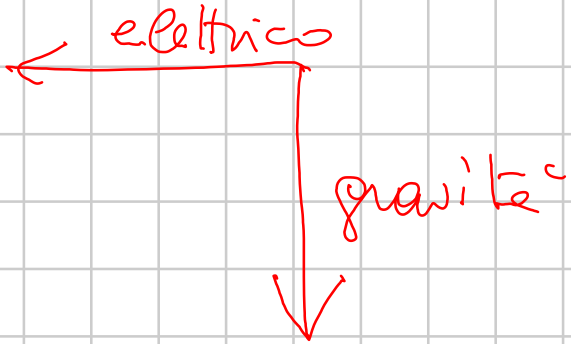

Ω
||α'(t)|| dt

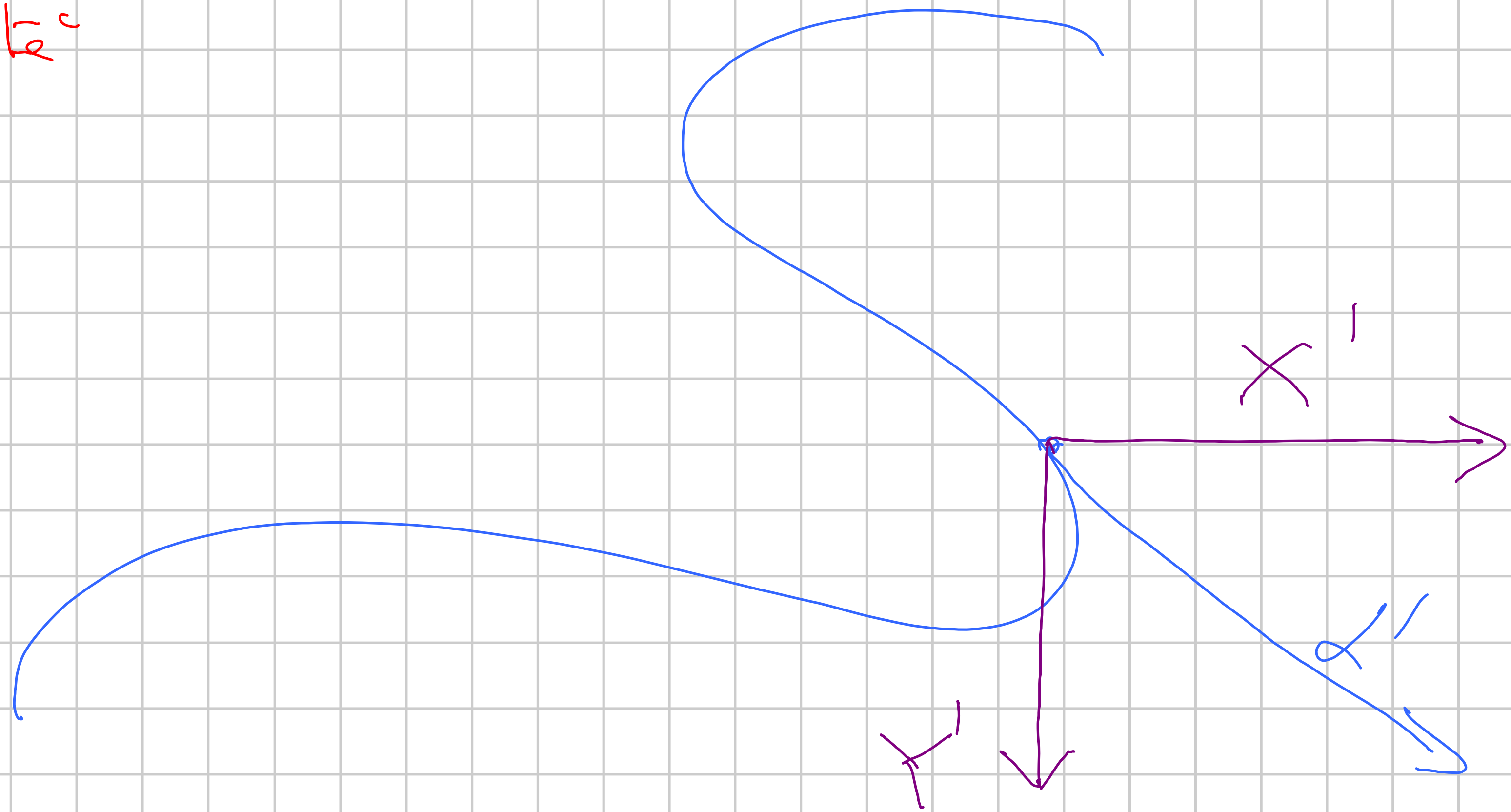
gravità



Lavoro compiuto da forza di gravità è

$$\int_a^b \underbrace{g(\alpha(t))}_{\text{costo in direz } y} \cdot \underbrace{\gamma'(t)\, dt}_{\text{spostamento infinitesimo in direz. } y}$$


elettrico
gravità


x'
y'

Consideriamo: $\displaystyle\int_a^b \Big(g(\alpha(t))\cdot X'(t) + h(\alpha(t))\cdot Y'(t)\Big)dt$

Def: se $\Omega \subset \mathbb{R}^n$ e $g_1, \dots, g_n : \Omega \to \mathbb{R}$ chiamo $\omega = g_1 dx_1 + \dots + g_n dx_n$ 1-forme su Ω. Se $\alpha : [a,b] \to \Omega$ è curva pongo

$$\int_\alpha \omega = \int_a^b \Big(g_1(\alpha(t))\cdot X_1'(t) + \dots + g_n(\alpha(t))\cdot X_n'(t)\Big)dt$$

Esempi: (1) $\Omega = \mathbb{R}^2$

$$\omega(x,y) = e^{x+3y^2}\,dx - \cos(x^2+5y)\,dy$$

$$\alpha : [0,1] \to \mathbb{R}^2 \qquad \alpha(t) = \begin{pmatrix} 1+9t^3 \\ 7t - t^2 \end{pmatrix}$$

$$\int_\alpha \omega = \int_0^1 \left(e^{1+9t^3+3(7t-t^2)^2} \cdot 27t^2 - \cos\left((1+9t^3)^2+5(7t-t^2)\right)(7-2t) \right) dt$$

(2) $\omega(x,y,z) = e^{x-z}\,dx + \sin(x^2+y)\,dy - \ln\left(\frac{x-y}{x-z}\right)dz$

$\alpha = \begin{pmatrix} t^3 \\ t^4 \\ t^5 \end{pmatrix} \quad t \in [1/3, 2/3] \qquad \Omega = \begin{cases} x > y \\ x > z \end{cases}$

$$\int_\alpha \omega = \int_{1/3}^{2/3} \left(e^{t^3-t^5} \cdot 3t^2 + \sin(t^6+t^4) \cdot 4t^3 - \ln\left(\frac{t^3-t^4}{t^3-t^5}\right) 5t^4 \right) dt$$

Prop: Se β è ottenuto da α per cambio parametro ho $\int_\beta \omega = \int_\alpha \omega$ se stessa orientaz.,

$\int_\beta \omega = -\int_\alpha \omega$ se orientaz. opposto.

Dim:

$$\begin{tikzcd}
{[a,b]} \arrow[rr, "\alpha"] & & \mathbb{R}^n \\
{[c,d]} \arrow[u, "\tau"] \arrow[urr, "\beta"'] & &
\end{tikzcd}$$

$\beta = \alpha \circ \tau$

$$\omega = g\,dx + h\,dy \qquad \alpha = \begin{pmatrix} X_\alpha \\ Y_\alpha \end{pmatrix} \qquad \beta = \begin{pmatrix} X_\beta \\ Y_\beta \end{pmatrix}$$

$$\int_\beta \omega = \int_c^d \Big(g(\beta(s)) \cdot X_\beta'(s) + h(\beta(s)) \cdot Y_\beta'(s) \Big)\, ds$$

$$= \int_c^d \Big(g(\alpha(\tau(s))) \cdot X_\alpha'(\tau(s)) \cdot \tau'(s) + h(\alpha(\tau(s))) \cdot Y_\alpha'(\tau(s)) \cdot \tau'(s) \Big)\, ds$$

$$= \operatorname{sgn}(\tau') \int_c^d \underbrace{\Big(g\big(\alpha(\tau(s))\big)\cdot X_\alpha'(\tau(s)) + h\big(\alpha(\tau(s))\big)\cdot Y_\alpha'(\tau(s))\Big)}_{g(\alpha(t))X'(t)+h(\alpha(t))\cdot Y'(t)} \cdot \underbrace{|\tau'(s)|\,ds}_{dt}$$

$$t = \tau(s)$$

$$= \operatorname{sgn}(\tau') \cdot \int_a^b \Big(g(\alpha(t))X'(t) + h(\alpha(t))Y'(t)\Big)\,dt$$

$$= \pm \int_\alpha \omega . \qquad \blacksquare$$

Se $U : \Omega \to \mathbb{R}$ le associo la 1-forma

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \dots + \frac{\partial U}{\partial x_n} dx_n \quad (\text{differenziale di } U)$$

Esempio: $U(x,y) = \cos(x^2 y^3) \cdot e^{x^5 - y^7}$

$$dU = \left(-2x y^3 \sin(x^2 y^3) \cdot e^{x^5 - y^7} + \cos(x^2 y^3) \cdot 5x^4 e^{x^5 - y^7} \right) dx$$
$$+ \left(-3x^2 y^2 \sin(\dots) \cdot e^{\dots} + \cos(\dots)(-7y^6) e^{\dots} \right) dy$$

Prop: $\displaystyle\int_\alpha dU = U(\alpha(b)) - U(\alpha(a))$

Dim: $\displaystyle\int_\alpha dU = \int_a^b \underbrace{\left(\frac{\partial U}{\partial x}\begin{pmatrix} X(t) \\ Y(t)\end{pmatrix} X'(t) + \frac{\partial U}{\partial y}\begin{pmatrix} X(t) \\ Y(t)\end{pmatrix} Y'(t)\right)}_{\frac{d}{dt}\underbrace{U\begin{pmatrix} X(t) \\ Y(t)\end{pmatrix}}_{U(\alpha(t))}} dt$

$= U(\alpha(b)) - U(\alpha(a))$. $\boxed{}$

Def: data $\omega = g\,dx + h\,dy$ 1-forma
dico che è esatta se $\exists\, U$ t.c. $\omega = dU$.

In fisica: un campo di forze $\begin{pmatrix} g \\ h \end{pmatrix}$ è conservativo se ammette un potenziale U cioè una funz. U t.c. il lavoro del campo lungo traiettoria = differenza di potenziale tra gli estremi.

$\begin{pmatrix} g \\ h \end{pmatrix}$ conservativo $\Longleftrightarrow$ $\omega = g\,dx + h\,dy$ esatta

Q: quali sono le forme esatte?