ETA 15/12/15

$\mathcal{C} = (C_n, \partial_n)$ $\quad G$ ab $\quad C^n_G = Hom(C_n, G)$, $\delta_n = \partial^*_{n+1}$

$\Longrightarrow \quad H^*(\mathcal{C}; G) = H_*((C^n_G, \delta_n))$

(X, A) coppie PL, Δ, CW, TOP

$\Longrightarrow \quad H^*(X, A; G)$.

Proprietà:

- funtore controvariante da TOP_2 (o ...) in $Ab^{\mathbb{N}}$

$$f : (X,A) \to (Y,B)$$

$$\rightsquigarrow f_{\#_n} : C_n(X,A) \to C_n(Y,B)$$

$$\rightsquigarrow f^n_{\#} : C^n(Y,B;G) \to C^n(X,A;G)$$

$$\partial^{(Y,B)}_{n-1} \circ f_{\#_n} = f_{\#_{n-1}} \circ \partial^{(X,A)}_n$$

$$\Rightarrow \quad f^{\#} \circ \delta^{(Y,B)} = \delta^{(X,A)} \circ f^{\#}$$

$\Longrightarrow$ indotta $f^*: H^n(Y,B;G) \to H^n(X,A;G)$

- omotopia : $f_0 \simeq f_1 \Longrightarrow f_0^* = f_1^*$

  $\Longrightarrow H^*$ inv $/_{\simeq}$

- escissione (con le ipotesi giuste : PL sottocompl
  SING $\overline{Z} \subset \mathring{A}$

  $\Longrightarrow H^n(X,A;G) \cong H^n(X \setminus Z, A \setminus Z; G)$)

- dimensione : $H^n(pt;G) = \begin{cases} G & n=0 \\ 0 & \text{altrimenti} \end{cases}$

• LES : $A \overset{j}{\hookrightarrow} X = (X, \emptyset) \overset{i}{\hookrightarrow} (X, A)$

$$\dots \to H^n(X, A; G) \xrightarrow{i_n^*} H^n(X; G) \xrightarrow{j_n^*} H^n(A; G) \xrightarrow{\Delta_n} H^{n+1}(X, A; G) \to \dots$$

(viene da $0 \to \mathcal{C}' \to \mathcal{C} \to \mathcal{C}'' \to 0$ esatta corta $\Rightarrow$ LES).

Dobbiamo provare che $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$ induce

$$0 \leftarrow C^*(A; G) \leftarrow C^*(X; G) \leftarrow C^*(X, A; G) \leftarrow 0 .$$

Lem: G ab; $\forall T$ sia $T^* = \mathrm{Hom}(T, G)$.

- Se $A \xrightarrow{i} B \xrightarrow{p} C \to 0$ esatta

$\Rightarrow A^* \xleftarrow{i^*} B^* \xleftarrow{p^*} C^* \leftarrow 0$ esatta

- Se inoltre $0 \to A \xrightarrow{i} \dots$ è esatta e splitta

$\Rightarrow 0 \leftarrow A^* \xleftarrow{i^*} \dots$ è esatta

(Falso se non splitta.)

Dim: $\boxed{p^* \text{ iniettiva}}$ $\gamma \in C^*$, $p^*(\gamma) = 0$

$\Longrightarrow \quad p^*(\gamma)(b) = 0 \quad \forall b \in B \quad \Longrightarrow \quad \gamma(p(b)) = 0 \quad \forall b \in B$

p surgettiva $\Longrightarrow \gamma = 0$ _

$$\boxed{\operatorname{Im}(p^*) \subset \operatorname{Ker}(i^*)}$$

$$p \circ i = 0 \Longrightarrow i^* \circ p^* = 0$$

$$\boxed{\operatorname{Ker}(i^*) \subset \operatorname{Im}(p^*)}$$

$$\begin{array}{ccccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ A^* & \xleftarrow{i^*} & B^* & \xleftarrow{p^*} & C^* & \longleftarrow & 0 \end{array}$$

$\beta \in \operatorname{Ker} i^* \Longrightarrow i^*(\beta)(a) = 0 \quad \forall a \in A$

$\Longrightarrow \beta(i(a)) = 0 \quad \forall a \in A$ _

Definisco $\gamma \in C^*$ ponendo $\gamma(p(b)) = \beta(b)$

Ben def perché se $p(b)=0$ h_0 $b = i(a)$

$\Rightarrow \beta(i(a)) = 0$. Chiaro $\beta = p^*(\gamma)$.

Se
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[l, bend left, "q"] \arrow[r] & \dots \end{tikzcd}$$

$q \circ i = id_A \Rightarrow i^* \circ q^* = id_{A^*} \Rightarrow i^*$ suriettiva. $\boxed{\text{///}}$

Per LES:
$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r, "j_\#"] & C_n(X) \arrow[l, bend left, "q_n"] \arrow[r, "i_\#"] & C_n(X,A) \arrow[r] & 0 \end{tikzcd}$$

$$q_n(\sigma) = \begin{cases} \sigma & \text{se } \sigma \subset A \\ 0 & \text{d\_\_\_\_\_} \end{cases}$$

Omomorfismo Δ_n in LES di H^A :

$\varphi \in Z^n(A)$

$$\begin{tikzcd}
 & & C^n(X) \arrow[r, "j_n^\#"] \arrow[d, "\delta_n"] & C^n(A) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & C^{n+1}(X,A) \arrow[r, "i_{n+1}^\#"] & C^{n+1}(X) \arrow[r, dashed] & \cdots &
\end{tikzcd}$$

ψ $\qquad$ φ $\qquad$ η

$\Delta_n([\varphi]) = [\eta]$ $\quad\quad \psi = \varphi$ esteso da A a X

ponendo 0 sui simplessi non in A

$i^{\#}_{n+1}(\eta) = \delta_n(\psi)$

$$\left(i^{\#}_{n+1}(\eta)\right)(u) = (\delta_n \psi)(u) = \psi(\partial_{n+1}(u))$$

$$\| $$

$$\eta(i_{n+1}(u))$$

Ben definito: $C^{n+1}(X,A) = C^{n+1}(X) / C^{n+1}(A)$

se $u \in C^{n+1}(A)$ ho

$$\psi(\partial_{n+1} u) = \varphi(\partial_{n+1} u) - (\delta^A_n \varphi)(u) = 0$$

(sono partito da $z \in Z^n(A)$)

- Teoria ridotta: dualizzo il complesso aumentato

$$C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow \overset{-1}{\mathbb{Z}} \longrightarrow 0$$

$$\sum m_i x_i \longmapsto \sum m_i$$

Per X connesso per archi:

$$Z^0(X;G) = \{ \varphi \in C^0(X;G) : \delta_0 \varphi = 0 \}$$

$$= \{ \varphi : X \to G : \underbrace{(\delta_0 \varphi)(\alpha)}_{\substack{= \varphi(\partial\alpha) \\ = \varphi(\alpha(1)) - \varphi(\alpha(0))}} = 0 \quad \forall \alpha \in C_1(X) \}$$

$$= \{ \varphi : X \to G \text{ costanti} \} \cong G$$

$B^0(X;G) = \{0\}$ $\qquad (C^{-1} = 0)$

$\Longrightarrow \quad H_0(X;G) \cong G$

Aumentando: $\qquad C^{-1}(X;G) = Hom(\mathbb{Z}, G) \cong G$

dove $\quad g(1) = g$

$$(\tilde{\delta}_{-1} g)(x) = g(\tilde{\partial}_0 x) = g(1) = g$$

$\Longrightarrow \quad \tilde{B}^0(X;G) = Z^0(X;G) \cong G$

$\Longrightarrow \quad \tilde{H}^0(X;G) = 0$ _

Cioè: si perde un addendo diretto G_

- Mayer-Vietoris $X = A \cup B$ sottocompl PC
  $X = \mathring{A} \cup \mathring{B}$ SING

$$\dots \to H^n(X;G) \xrightarrow{\Psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\Phi} H^n(A \cap B;G) \xrightarrow{+1} H^{n+1}(X;G) \to \dots$$

$$\begin{tikzcd}
 & A \arrow[dr, "i_A"] & \\
A \cap B \arrow[ur, "j_A"] \arrow[dr, "j_B"'] & & X \\
 & B \arrow[ur, "i_B"'] &
\end{tikzcd}$$

$\Psi = (i_A^*, i_B^*)$

$\Phi = j_A^* - j_B^*$

$$\Theta([\omega]) = [\lambda]$$

$$\lambda(u) = \omega(\partial a - \partial b) \qquad \forall u \in C_{n+1}(X)$$

comunque scritto come $u = a + b$

Relazione tra H_* e H^*_G.

Fisso per sempre G ; $*$ = duale wrt G

Def: se $0 \longrightarrow K \xrightarrow{i} F \xrightarrow{p} A \longrightarrow 0$ è esatta
K, F libere

$$0 \leftarrow \cdots K^* \xleftarrow{i^*} F^* \xleftarrow{p^*} A^* \leftarrow 0$$

$$\mathrm{Ext}(A,G) = \mathrm{Coker}(i^*) = K^* / \mathrm{Im}(i^*)$$

Esempi: •) $A = \mathbb{Z}/m$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/m \longrightarrow 0$$

$$G \xleftarrow{m\cdot} G$$

$$\Rightarrow \operatorname{Ext}(\mathbb{Z}/m, G) = G/m\cdot G$$

$$\left(\operatorname{Ext}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m \right)$$

•) $A = \mathbb{Z}$ $\qquad 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$

$$0 \longleftarrow G$$

$$\Rightarrow \operatorname{Ext}(\mathbb{Z}, G) = 0 .$$

<u>Prop</u>: $\operatorname{Ext}(A, G)$ ben def.

Dim: Siano $0 \longrightarrow K_0 \xrightarrow{i_0} H_0 \xrightarrow{p_0} A \longrightarrow 0$

$0 \longrightarrow K_1 \xrightarrow{i_1} H_1 \xrightarrow{p_1} A \longrightarrow 0$

ris. libere. Cerco isomorfismo $K_0^\alpha / \mathrm{Im}(i_0^\alpha) \cong K_1^\alpha / \mathrm{Im}(i_1^\alpha)$.

Considero:

$$\begin{tikzcd} H_0 \arrow[dr, "p_0"] \arrow[d, dashed, "h_{01}"'] & & \\ H_1 \arrow[r, "p_1"'] & A \arrow[r] & 0 \end{tikzcd}$$

Esiste h_{01} t.c. $p_1 \circ h_{01} = p_0$: scelgo base di H_0 e definisco sulla base come qualsiasi d'x_0 di $p_1^{-1} \circ p_0$.

$$\begin{tikzcd} K_0 \arrow[r, "i_0"] \arrow[d, dashed, "k_{01}"'] & H_0 \arrow[r, "p_0"] \arrow[d, "h_{01}"] & A \arrow[r] & 0 \\ K_1 \arrow[r, "i_1"] & H_1 \arrow[ur, "p_1"'] & & \end{tikzcd}$$

(verifico che k_{01} esiste usando K_0 libero + esattezza)

Dualizzando ho:

$$\begin{tikzcd} K_0^* & H_0^* \arrow[l, "i_0^*"'] & \\ & & 0 \arrow[ul, "p_0^*"'] \arrow[dl, "p_1^*"] \\ K_1^* \arrow[uu, "k_{01}^*"] & H_1^* \arrow[l, "i_1^*"'] \arrow[uu, "h_{01}^*"'] & \end{tikzcd}$$

$$k_{01}^* \circ i_1^* = i_0^* \circ h_{01}^*$$

$$\Longrightarrow \quad k_{01}^*(\mathrm{Im}(i_1^*)) \subset \mathrm{Im}(i_0^*)$$

$$\Longrightarrow k_{01}^* \text{ induce } \overline{k_{01}} : K_1^* / \mathrm{Im}(i_1^*) \to K_0^* / \mathrm{Im}(i_0^*) .$$

Affermo che $\overline{k_{01}}$ non dipende da nessuna scelta.
Se lo faccio concludo;
con la stessa tecnica costruisco $\overline{k_{10}} : K_0^* / \mathrm{Im}(i_0^*) \to \ldots,$

$$\overline{k_{11}} = \overline{k_{10}} \circ \overline{k_{01}} : K_1^* / \mathrm{Im}(i_1^*) \circlearrowleft$$

è unico $\Longrightarrow$ è l'identità perché posso scegliere $h_{11} = id_{H_1}$ $k_{11} = id_{K_1}$ —

Provo affermazione:

$$\begin{tikzcd}
y_0 \; K_0 \arrow[r, "i_0"] \arrow[d, "k_{01}'"] & x_0 \; H_0 \arrow[dr, "p_0"] \arrow[d, "h_{01}'"] & & \\
y_1 \; K_1 \arrow[r, "i_1"] & H_1 \arrow[r, "p_1"'] & A \arrow[r] & 0
\end{tikzcd}$$

Basta provare che $\forall \beta_1 \in K_1^*$ $\quad (k_{01}'^{*} - k_{01}^{*})(\beta_1) \in \operatorname{Im}(i_0^{*})$

$$\begin{tikzcd}
K_0^* & H_0^* \arrow[l, "i_0^*"'] & & \\
 & & A \arrow[ul, "p_0^*"'] \arrow[dl, "p_1^*"] & 0 \arrow[l] \\
K_1^* \arrow[uu, "K_{01}'^* - K_{01}^*"] & H_1^* \arrow[l, "i_1^*"] \arrow[uu, "h_{01}'^* - h_{01}^*"'] & &
\end{tikzcd}$$

α_0 β_1

Voglio definire $\alpha_0 \in H_0^*$; sia $x_0 \in H_0$

$$p_1\left(\left(h_{01}' - h_{01}\right)(x_0)\right) = p_0(x_0) - p_0(x_0) = 0$$

$$\Longrightarrow \left(h_{01}' - h_{01}\right)(x_0) = i_1(y_1)$$

Pongo $\alpha_0(x_0) := \beta_1(y_1)$ (ben def ....)

Voglio provare che $(k_{01}'^{*} - k_{01}^{*})(\beta_1) = i_0^*(\alpha_0)$

Applico a $y_0 \in K_0$:

$$(i_0^*(\alpha_0))(y_0) = \alpha_0(i_0(y_0))$$

come si calcola?

$$i_1 \underbrace{(k_{01}'(y_0) - k_{01}(y_0))}_{\text{lo uso come } y_1} = (h_{01}' - h_{01})(i_0(y_0))$$

$$\Rightarrow \alpha_0(i_0(y_0)) = \beta_1\left(k'_{01}(y_0) - k_{01}(y_0)\right)$$

$$= (k'^{*}_{01} - k^{*}_{01})(\beta_1)(y_0)$$

$$\Rightarrow (k'^{*}_{01} - k^{*}_{01})(\beta_1) \in \mathrm{Im}(i_0^{*}) \qquad \blacksquare$$

$\boxed{\text{UCT per } H^*:}$ esiste una successione esatta canonica

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}, G) \longrightarrow H^n_G \longrightarrow \mathrm{Hom}(H_n, G) \longrightarrow 0$$

che splitta non canonicamente.

<u>Cor</u>: $H^n_G \cong \mathrm{Hom}(H_n, G) \oplus \mathrm{Ext}(H_{n-1}, G)$.

(Dim Cor: per via indiretta provandolo per $E(n)$ $D(n, k)$.)

Come calcolare $H^*(X; G)$.

Versione PL: complicata.

Esercizio: verifica PL diretta che $\tilde{H}^k(S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$

Più pratica: teoria CW.

$$X = \bigcup X^{(n)} \qquad X^{(n)} = X^{(n-1)} \cup_{a_i} \left(\bigcup D_i^n \right)$$

$$X^{(0)} = \{ a_i^0 \}$$

$$a_i^n : S_i^{n-1} \to X^{(n-1)}$$

$$A_i^n : D_i^n \to X^{(n)}$$

$$C_n(X) = \mathbb{Z}\langle X^{[n]} \rangle$$

Intrinsecamente:
$$H_n\left(X^{(n)}, X^{(n-1)}\right) \overset{\text{canonico}}{\cong} H_n\Big(\underbrace{X^{(n)}/X^{(n-1)}}_{\substack{\text{bouquet di } S^n \\ \# X^{[n]} \text{ copie}}}\Big)$$

$=$ una copia di $\mathbb{Z}$ per ogni S^n

$a_i^n \in X^{[n]}$ induce mappa $D^n/S^{n-1} = S^n \to X^{(n)}/X^{(n-1)}$

è uno dei generatori canonici di $H_n\left(X^{(n)}/X^{(n-1)}\right)$.

<u>Morale</u>: $C_n(X) \cong H_n\left(X^{(n)}/X^{(n-1)}\right)$

canonicamente generato da $\overline{a_i^n}$.

Interpretazione di $\partial_n : C_n(X) \to C_{n-1}(X)$

$$
\begin{tikzcd}
C_n(X) \arrow[rr, "\partial_n"] \arrow[d, equal] & & C_{n-1}(X) \arrow[d, equal] \\
H_n(X^{(n)}, X^{(n-1)}) \arrow[rr, dashed] \arrow[dr, "\partial"'] & & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \\
& H_{n-1}(X^{(n-1)}) \arrow[ur, "i"'] &
\end{tikzcd}
$$

LES (X^n, X^{n-1})

LES

X^{n-1}, X^{n-2}

En cohomologie :

$$X^{(n)} / X^{(n-1)} \simeq \bigvee_{X^{[n]}} S^n$$

$$\Rightarrow C^n(X;G) = H^n(X^{(n)}, X^{(n-1)}; G) \cong \bigoplus_{X^{[n]}} G$$

Cobondo

$$\begin{tikzcd}
H^n(X^n, X^{n-1}; G) \arrow[rr, dashed] \arrow[dr, "j^*"] & & H^{n+1}(X^{n+1}, X^n; G) \\
& H^n(X^n) \arrow[ur, "\delta"'] &
\end{tikzcd}$$

LES
X^n, X^{n-1}

LES
X^{n+1}, X^n