ETA 11/12/14

UCT: (C_n, ∂_n) complesso di gruppi <u>liberi</u>, R gruppo ab (anello),

$\Rightarrow \exists\ 0 \to H_n \otimes R \to H_n^R \to \mathrm{Tor}(H_{n-1}, R) \to 0$ esatte canonica che splitta non canonicamente.

<u>Dimo</u> (leggermente informale):

$C_n^R = C_n \otimes R \qquad \partial_n^R = \partial_n \otimes id_R$ — Fatti

•) $u \in Z_n,\ \lambda \in R \Rightarrow \partial_n^R(u \otimes \lambda) = (\partial_n u) \otimes \lambda = 0 \otimes \lambda = 0$

$\Rightarrow Z_n \otimes R \subset Z_n^R$

•) $u \in B_n,\ \lambda \in R \Rightarrow u = \partial_{n+1} w,\quad u \otimes \lambda = \partial_{n+1}^R (w) \otimes \lambda$

$\Rightarrow B_n \otimes R \subset B_n^R$.

Risoluzione libera di H_{n-1} : $\quad 0 \longrightarrow B_{n-1} \xrightarrow{j_{n-1}} Z_{n-1} \xrightarrow{p_{n-1}} H_{n-1} \longrightarrow 0$

($j_*^R,\ p_*^R$ analoghe per coeff. in R) .

Vogliamo: $0 \to H_n \otimes R \xrightarrow{g_n} H_n^R \xrightarrow{f_n} \mathrm{Tor}(H_{n-1}, R) \to 0$

$$\mathrm{Tor}(H_{n-1}, R) = \mathrm{Ker}\left(B_{n-1} \otimes R \xrightarrow{j_{n-1} \otimes id_R} Z_{n-1} \otimes R\right)$$

Es: $\mathbb{P}^2(\mathbb{R})$

$C_0 = \mathbb{Z} \cdot \sigma_0$, $C_1 = \mathbb{Z} \cdot \sigma_1$, $C_2 = \mathbb{Z} \cdot \sigma_2$

$\partial_0 \sigma_0 = 0$ $\quad \partial_1 \sigma_1 = 0$ $\quad \partial_2 \sigma_2 = 2\sigma_1$

$R = \mathbb{Z}/2$

$C_0^{\mathbb{Z}/2} = \mathbb{Z}/2 \cdot \sigma_0$, $C_1^{\mathbb{Z}/2} = \mathbb{Z}/2 \cdot \sigma_1$, $C_2^{\mathbb{Z}/2} = \mathbb{Z}/2 \cdot \sigma_2$

$\partial_0^{\mathbb{Z}/2} \sigma_0 = 0$ $\quad \partial_1^{\mathbb{Z}/2} \sigma_1 = 0$, $\quad \partial_2^{\mathbb{Z}/2} \sigma_2 = 0$

$Z_1 = \mathbb{Z} \cdot \sigma_1 \qquad B_1 = \mathbb{Z} \cdot (\underline{2\sigma_1})$

il generatore di B_1;
in B_1 non è il doppio
di nessuno

Invece: $j_1 \otimes id_{\mathbb{Z}/2} : B_1 \otimes \mathbb{Z}/2 \longrightarrow Z_1 \otimes \mathbb{Z}/2$

$$\begin{array}{ccc} B_1 \otimes \mathbb{Z}/2 & \longrightarrow & Z_1 \otimes \mathbb{Z}/2 \\ \psi & & \psi \\ (2\sigma_1) \otimes 1 & \longmapsto & (2\sigma_1) \otimes 1 = (2 \cdot \sigma_1) \otimes 1 = 2(\sigma_1 \otimes 1) = 0 \end{array}$$

Per formalizzare dovrei dare nomi a tutte le mappe

$C_n \otimes R \to C_n^R \qquad Z_n \otimes R \to Z_n^R \qquad B_n \otimes R \to B_n^R$

$Z_n \to C_n \qquad B_n \to C_n \qquad Z_n^R \to C_n^R \qquad B_n^R \to Z_n^R$

Punto fondam: la $Z_n \otimes R \to Z_m^R$ è iniettiva.

Infatti tutti C_n sono liberi $\Rightarrow B_{n-1} \subset C_{n-1}$ è libero

$\Rightarrow \quad 0 \to Z_n \hookrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \to 0$ split

$\Rightarrow C_n \cong Z_n \oplus B_{n-1}$

$\Rightarrow (Z_n \otimes R) \oplus (B_{n-1} \otimes R) \cong C_n \otimes R = C_n^R$

$\Rightarrow Z_n \otimes R \hookrightarrow C_n^R$ e ha immagine in Z_n^R.

Definiamo: $0 \longrightarrow H_n \otimes R \xrightarrow{g_n} H_n^R \xrightarrow{f_n} \operatorname{Ker}(j_{n-1} \otimes id_R) \to 0$

$g_n(p_n(z) \otimes \lambda) = p_n^R(z \otimes \lambda)$ per ogni $z \in Z_n$; $f_n\left(p_n^R\left(\sum_i a_i \otimes \lambda_i\right)\right) = \sum (\partial_n a_i) \otimes \lambda_i$ per ogni $\sum_i a_i \otimes \lambda_i \in Z_n^R$

$\boxed{g_n \text{ ben def}}$ $z \in B_n \Rightarrow z \otimes \lambda \in B_n^R \Rightarrow p_n^R(z \otimes \lambda) = 0$

$\boxed{f_n \text{ ben def}}$ Devo provare che $\sum (\partial_n a_i) \otimes \lambda_i \in \operatorname{Ker}(j_{n-1} \otimes id_R)$ se $\sum a_i \otimes \lambda_i \in Z_n^R$:

$(j_{n-1} \otimes id_R)\left(\sum (\partial_n a_i) \otimes \lambda_i\right) = \sum_i (j_{n-1} \partial_n a_i) \otimes \lambda_i$

$= \underbrace{\sum_i (\partial_n a_i) \otimes \lambda_i}_{\text{ora visto in } Z_{n-1} \otimes R} = \partial_n^R \left(\sum_i a_i \otimes \lambda_i \right)$ $\Longrightarrow$ nullo perché $\sum_i a_i \otimes \lambda_i \in Z_n^R$.

(indip. dal rappresentante ...)

$\boxed{g_n \text{ iniettiva}}$ Sia $g_n \left(\sum_i p_n(z_i) \otimes \lambda_i \right) = 0$

cioè $p_n^R \left(\sum_i z_i \otimes \lambda_i \right) = 0$ cioè

$$\sum_i z_i \otimes \lambda_i = \partial_{n+1}^R \left(\sum_i w_k \otimes \mu_k \right) = (\partial_{n+1} \otimes id_R) \left(\sum_i w_k \otimes \mu_k \right)$$

$$= \sum_i (\partial_{n+1} w_k) \otimes \mu_k$$

$$\Longrightarrow \sum_i p_n(z_i) \otimes \lambda_i = (p_n \otimes id_R)\left(\sum_i z_i \otimes \lambda_i\right)$$

$$= (p_n \otimes id_R)\left(\sum (\partial_{n+1} w_k) \otimes \mu_k\right)$$

$$= \sum \underbrace{p_n(\partial_{n+1} w_k)}_{0} \otimes \mu_k \Longrightarrow \text{è nullo}$$

$\boxed{f_n \text{ surgettive}}$ Visto (buona def. di f_n) che $\forall u = \sum_i (\partial_n a_i) \otimes \lambda_i$ (generico el. di $B_{n-1} \otimes R$)

$$(j_{n-1} \otimes id_R)(x) = \partial_n^R \left(\sum a_i \otimes \lambda_i\right)$$

Ora se $x \in \mathrm{Tor}(H_{n-1}, R) = \mathrm{Ker}(j_{n-1} \otimes id_R)$

$$\Rightarrow \sum a_i \otimes \lambda_i \in Z_n^R \Rightarrow x = f_n\left(p_n^R\left(\sum a_i \otimes \lambda_i\right)\right)$$

$\boxed{\mathrm{Im}\, g_n \subset \mathrm{Ker}\, f_n}$

$$f_n(g_n(p_n(z) \otimes \lambda)) = f_n(p_n^R(z \otimes \lambda)) = (\partial z) \otimes \lambda = 0$$

$\boxed{\operatorname{Ker} f_n \subset \operatorname{Im} g_n}$

Esibisco un isomorfismo

$$\varphi_n : \operatorname{Tor}(H_{n-1}, R) \longrightarrow H_n^R / \operatorname{Im}(g_n)$$

il cui inverso è indotto da f_n.

(Basta: so che $H_n^R \xrightarrow{f_n} \operatorname{Tor}(H_{n-1}, R) \to 0$

$\Rightarrow \operatorname{Tor}(H_{n-1}, R) \cong H_n^R / \operatorname{Ker} f_n$; mettendo insieme concludo che $\operatorname{Ker} f_n = \operatorname{Im} g_n$).

Sia $u = \sum (\partial_n a_i) \otimes \lambda_i \in \operatorname{Ker}(j_{n-1} \otimes id_R)$;

pongo $\varphi_n(x) = \left[\sum_i a_i \otimes \lambda_i\right] + \mathrm{Im}(g_n)$;

ben def: $\partial_n^R\left(\sum_i a_i \otimes \lambda_i\right) = \sum_i (\partial_n a_i) \otimes \lambda_i$

$= (j_{n-1} \otimes id_R)\left(\sum_i (\partial_n a_i) \otimes \lambda_i\right) = 0$.

(Altre verifiche su φ_n: esercizio.)

Le succ. splitta

$$0 \longrightarrow H_n \otimes R \xrightarrow{g_n} H_n^R \xrightarrow{f_n} \mathrm{Tor}(H_{n-1}, R) \to 0$$

$$g_n(p_n(z) \otimes \lambda) = p_n^R(z \otimes \lambda)$$

Cerco $g_n^R : H_n^R \to H_n \otimes R$ t.c. $g_n^R \circ g_n = id_{H_n \otimes R}$.

So che $0 \to Z_n \hookrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \to 0$ splitta perché B_{n-1} è libero; quindi esiste $g_n : C_n \to Z_n$ proiezione ($g_n|_{Z_n} = id_{Z_n}$) non canonica.

Ora poniamo $g_n^R : H_n^R \to H_n \otimes R$

$$P_n^R\left(\sum a_i \otimes \lambda_i\right) \longmapsto \sum p_n(g_n(a_i)) \otimes \lambda_i .$$

Ben def: • $q_n(a_i) \in Z_n \Rightarrow p_n(q_n(a_i))$ ha senso

• applicato a un el. di B_n^R fa 0:

$$0 = p_n^R\left(\partial_{n+1}^R\left(\sum b_j \otimes \lambda_j\right)\right) = p_n^R\left(\sum_j (\partial_{n+1} b_j) \otimes \lambda_j\right)$$

$$\xrightarrow{q_n^R} \sum p_n(q_n(\underbrace{\partial_{n+1} b_j}_{\substack{\cap \\ B_n \\ \cap \\ Z_n}})) \otimes \lambda_j$$

$\underbrace{\qquad\qquad}_{\partial_{n+1} b_j}$

$$\underbrace{\hphantom{xxxxxxxxxxxxxxxx}}_{0}$$

Inoltre $q_n^R(g_n(p_n(z))\otimes\lambda)) = p_n(\underbrace{q_n(\underset{\substack{\uparrow\\ Z_n}}{z})}_{z})\otimes\lambda = p_n(z)\otimes\lambda$.

$\blacksquare$

<u>Oss</u>: $\mathbb{F}$ campo di char $=0 \Longrightarrow \mathrm{Tor}(G,\mathbb{F})=0$

$$\Longrightarrow H_*(X,A;\mathbb{F}) = H_*(X,A)\otimes\mathbb{F} .$$

Oss: ritroviamo via UCT

$$H_0(X;R) = \underbrace{\left(\underbrace{H_0(X)}_{R} \otimes R\right)}_{R} \oplus \operatorname{Tor}(\underbrace{H_{-1}(X)}_{\overset{\shortparallel}{0}}, R)$$

Oss: $H_1(X;R) = \left(H_1(X) \otimes R\right) \oplus \underbrace{\operatorname{Tor}(H_0(X), R)}_{\operatorname{Tor}(\mathbb{Z}, R) = 0}$.

Teo (formula di Künneth): esiste successione esatta funtoriale

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y) \rightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \rightarrow 0$$

che splitta ma non funtorialmente.

Cor: $H_n(X,Y) \cong \left(\bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \right)$.

Dimo: Uso la teoria CW (vale in ogni teoria — qui facile).

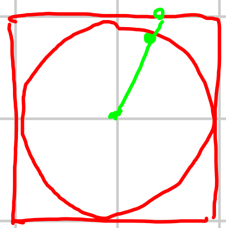
Prendo X, Y CW-complessi. Do struttura di CW-complesso a $X \times Y$:

$a \in X^{[p]}$ cioè $a : \partial D^p \longrightarrow X^{(p-1)} \rightsquigarrow A : D^p \longrightarrow X$

$b \in Y^{[q]}$ cioè $b : \partial D^q \longrightarrow Y^{(q-1)} \rightsquigarrow B : D^q \longrightarrow Y$

definisco $A \cdot B : D^{p+q} \longrightarrow D^p \times D^q \xrightarrow{(A,B)} X \times Y$

proiezione radiale



$$a \cdot b = A \cdot B \big|_{\partial(D^{p+q})}$$

L'insieme di tutte le $a \cdot b$ dà le strutture voluta.

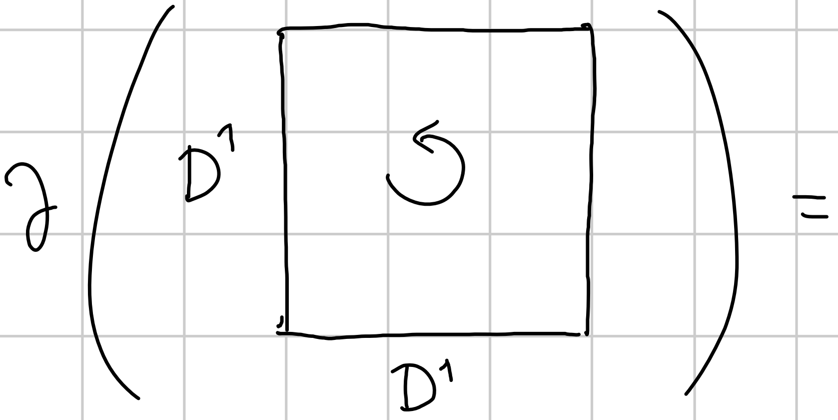
Per costruzione $\left(C_n(X) = \langle X^{[n]} \rangle \right)$

$$C_n(X \times Y) = \bigoplus_{p+q=n} C_p(X) \otimes C_q(Y).$$

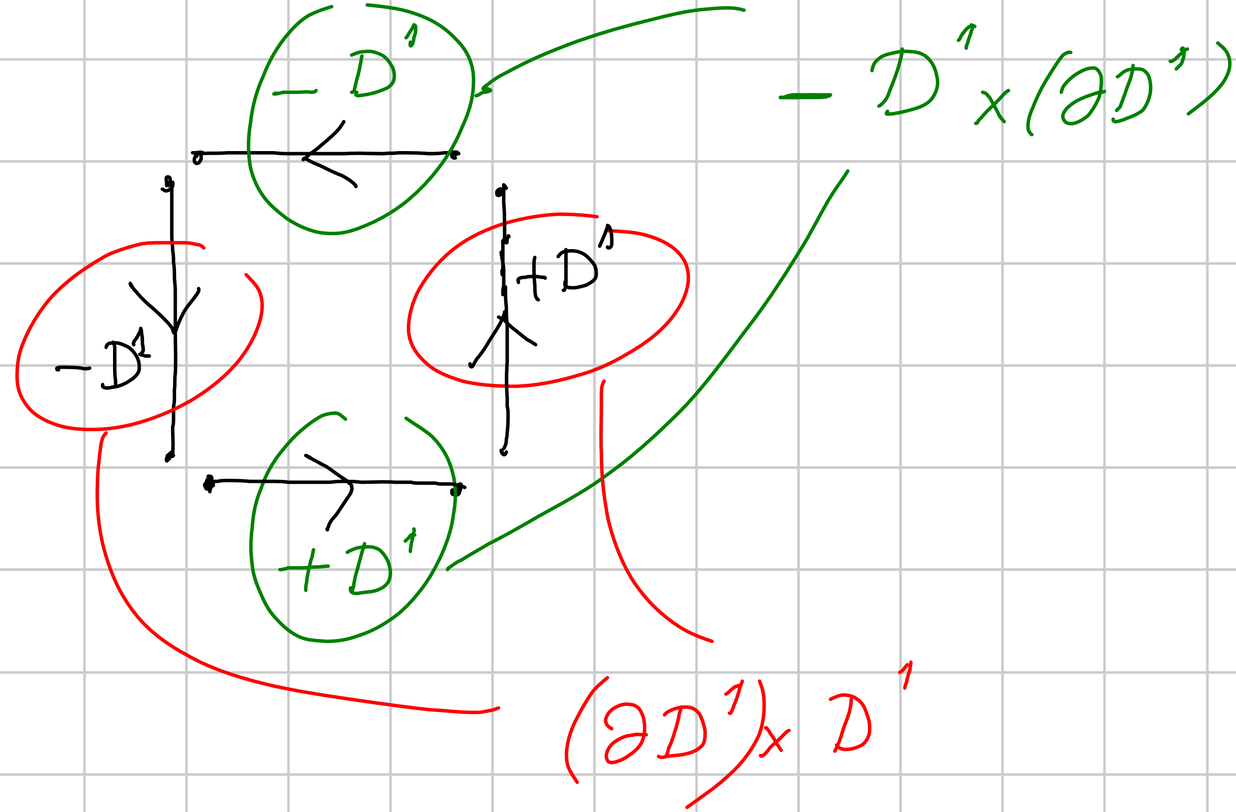
Fatto: $\partial_{p+q}(a_p \otimes b_q) = (\partial_p a) \otimes b + (-1)^p a \otimes (\partial_q b)$.

Ragione: $\partial(D^p \times D^q) = (\partial D^p) \times D^q + (-1)^p D^p \times (\partial D^q)$.

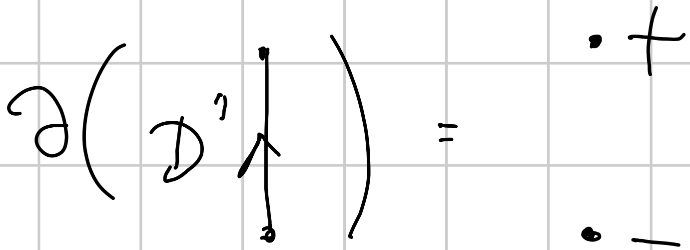
Caso $p=q=1$



∂
D^1
D^1
=



$-D^1$
$-D^1 \times (\partial D^1)$
$-D^1$
$+D^1$
$+D^1$
$(\partial D^1) \times D^1$



∂
D^1
=
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Conclusione segue da fatto algebrico:

<u>Prop</u>: U, W complessi di catene di gruppi abeliani <u>liberi</u>

$$\left(U\otimes W\right)_n = \bigoplus_{p+q=n} U_p \otimes W_q \, , \quad \partial^{U\otimes W}_{p+q}\left(a_p \otimes b_q\right) =$$

$$= \left(\partial^{U} a_p\right) \otimes b_q + (-1)^p a_p \otimes \left(\partial^{W} b_q\right)$$

$$\Rightarrow H_n(U\otimes W) \cong \left(\bigoplus_{p+q=n} H_p(U)\otimes H_q(W)\right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}\left(H_p(U), H_q(W)\right)\right).$$

Dimostriamo la Prop. per via indiretta. (Esercizio: provare UCT per la stessa via.)

Definisco: $(U \oplus W)_n = U_n \oplus W_n$, $\partial_n^{U \oplus W} = (\partial_n^U, \partial_n^W)$

e noto che $H_*(U \oplus W) = H_*(U) \oplus H_*(W)$. Inoltre ogni complesso di gruppi ab. liberi è somma di copie di:

$E(m)$: $E(m)_n = \begin{cases} \mathbb{Z} & n = m \\ 0 & \text{al}\text{—} \end{cases}$ $\partial_n^{E(m)} = 0 \ \forall n$

$$\Longrightarrow H_n(E(m)) = \begin{cases} \mathbb{Z} & n = m \\ 0 & \text{al} \end{cases}$$

$D(m,k)$:

$$D(m,k)_n = \begin{cases} \mathbb{Z} & \text{se } n = m, m+1 \\ 0 & \text{al} \end{cases}$$

$$\partial_{m+1} : \overset{m+1}{\mathbb{Z}} \xrightarrow{\; j \longmapsto k \cdot j \;} \overset{m}{\mathbb{Z}}$$

$\partial_n = 0$ per $n \neq m+1$

$$\Longrightarrow H_n(D(m,k)) = \begin{cases} \mathbb{Z}/k & n = m \\ 0 & \text{al} \end{cases}$$

Ora basta vedere la Prop. quando U e W sono del tipo $E(m)$ oppure $D(m,k)$.

Vediamolo nel caso difficile $D(m,k) \otimes D(r,h)$.

RHS della Prop. è:

$$\bigoplus_{p+q=n} H_p(D(m,k)) \otimes H_q(D(r,h)) = \begin{cases} 0 & n \neq m+r \\ \underbrace{\mathbb{Z}/k \otimes \mathbb{Z}/h}_{\mathbb{Z}/GCD(k,h)} & n = m+r \end{cases}$$

$$\bigoplus_{p+q=n-1} \mathrm{Tor}\Big(H_p(D(m,k)), H_q(D(n,h))\Big) = \begin{cases} 0 & \text{se } n-1 \neq p+q \\ & (n \neq p+q+1 \\ \underbrace{\mathrm{Tor}(\mathbb{Z}/k, \mathbb{Z}/h)}_{\mathbb{Z}/GCD(k,h)} & \text{se } n = p+q+1 \end{cases}$$

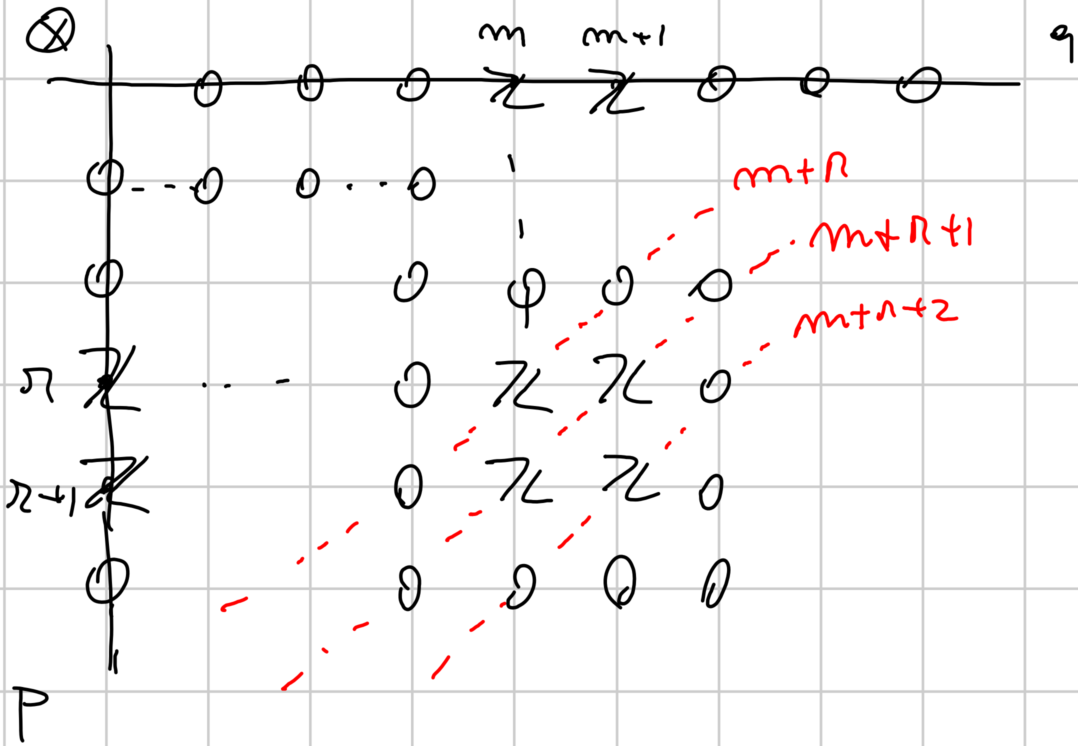
Dunque devo provare:

$$H_n\Big(D(m,k)\otimes D(n,h)\Big) = \begin{cases} \mathbb{Z}/GCD(k,l) \\ 0 \end{cases}$$

$n = m+n$

$n = m+n+1$

d ______ .



$$\left(D(m,k)\otimes D(n,h)\right)_N = \begin{cases} \mathbb{Z} & N = m+n \\ & N = m+n+2 \\ \mathbb{Z}\oplus\mathbb{Z} & N = m+n+1 \\ 0 & \text{al} \end{cases}$$

$N = m+n+1$: gen : $1_{m+1}\otimes 1_{n},\ 1_{m}\otimes 1_{n+1}$:

$$\partial_{m+n+1}(1_{m+1}\otimes 1_n) = (\underbrace{\partial_{m+1} 1_{m+1}}_{k\cdot 1_m})\otimes 1_n + (-1)^{m+1} 1_{m+1}\otimes\underbrace{(\partial_n 1_n)}_{0}$$

$$= k\cdot(1_m\otimes 1_n)$$

$$\partial_{m+n+1}(1_m\otimes 1_{n+1}) = \dots = (-1)^m\cdot h\cdot(1_m\otimes 1_n)$$

$$\partial_{m+n+2}(1_{m+1}\otimes 1_{n+1}) = k\cdot 1_m\otimes 1_{n+1} + (-1)^{m+1} h\cdot 1_{m+1}\otimes 1_n$$

Tutti gli altri bordi nulli. Ora $H_n(\cdot\otimes\cdot\cdot)=0$ per $n<m+n$
$n>m+n+1$.

$$H_{m+n} = \mathbb{Z}\langle 1_m \otimes 1_n \rangle \Big/ \begin{array}{c}\langle k \cdot 1_m \otimes 1_n \rangle \\ \langle h \cdot 1_m \otimes 1_n \rangle\end{array} = \mathbb{Z}\Big/_{GCD(k,h)}$$

$$H_{m+n+1} = \frac{\text{cicli : generati da } \left(h \cdot 1_{m+1} \otimes 1_n - (-1)^m \cdot k \cdot 1_m \otimes 1_{n+1}\right) \Big/ GCD(k,h)}{\text{bordi : generati da } \; k \cdot 1_m \otimes 1_{n+1} + (-1)^{m+1} h \cdot 1_{m+1} \otimes 1_n}$$

$$= a \cdot \mathbb{Z} \Big/_{a \cdot \underset{\substack{\shortparallel \\ GCD(k,h)}}{b} \cdot \mathbb{Z}} = \mathbb{Z}\Big/_{GCD(k,h)} .$$

$a \cdot b = k \cdot h$

$\boxtimes$

Coomologia.

(C_n, ∂_n) complesso di gruppi ab. liberi ; G gruppo abeliano.
(Applico funtore $\mathrm{Hom}(\_, G)$)

$$C^n(G) = \mathrm{Hom}(C_n, G)$$ cocatena a coeff. in G.

$$\delta_n^G : C^n(G) \longrightarrow C^{n+1}(G)$$ come ∂_{n+1}^* : (cobordo)

$$\varphi \longmapsto \left(C_{n+1} \ni a \longmapsto \varphi(\partial_{n+1} a) \in G \right)$$

Chiaro: $\delta^G_{n+1} \circ \delta^G_n = 0$

$\Rightarrow Z^n(G) = \mathrm{Ker}(\delta^G_n)$ cocicli

$B^n(G) = \mathrm{Im}(\delta^G_{n-1})$ cobordi

$H^n(G) = Z^n(G) / B^n(G)$ coomologia a coeff. in G.

Fatto: Non è vero che $H^n(G) \cong \mathrm{Hom}(H_n, G)$.

<u>Es</u>: Con $G = \mathbb{Z}$ per $E(n)$, $D(n,k)$

$E(n)$

$$\begin{array}{lccccccccc}
 & & & n+1 & & n & & n-1 & & \\
C_* & \longrightarrow & & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \;\cdots \\
H_* & \cdots & & 0 & & \mathbb{Z} & & 0 & & 0 \;\cdots
\end{array}$$

$E(n)^{\mathbb{Z}}$

$$\begin{array}{lccccccc}
 & & & & n & & & \\
C^* & \cdots \longleftarrow & 0 & \longleftarrow & \mathbb{Z} & \longleftarrow & 0 & \longleftarrow \;\cdots \\
H^* & \cdots & 0 & & \mathbb{Z} & & 0 & \cdots
\end{array}$$

$D(n,k)$

$$\begin{array}{lcccccccc}
 & & & & n+1 & & n & & \\
C_* & \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{k\cdot} & \mathbb{Z} & \longrightarrow & 0 \longrightarrow \cdots \\
H_* & \cdots & 0 & & 0 & & \mathbb{Z}/k & & 0 \;\cdots
\end{array}$$

$$
D(n,k) \simeq \quad
\begin{array}{lccccccccc}
 & & & & n+1 & & n & & & \\
C^* & \cdots & \leftarrow & 0 & \leftarrow & \mathbb{Z} & \xleftarrow{k\cdot} & \mathbb{Z} & \leftarrow 0 \leftarrow & \cdots \\
H^* & \cdots & & 0 & & \mathbb{Z}/k & & 0 & & \cdots
\end{array}
$$

Dunque: parte libera dell'omologia dà coomologia nello stesso grado
Torsione dell'omologia dà coomologia in grado +1:

$$
\begin{aligned}
H_n &= \mathbb{Z}^r \oplus T \\
H_{n-1} &= \mathbb{Z}^s \oplus U
\end{aligned}
\quad \Longrightarrow \quad H^n \cong \mathbb{Z}^r \oplus U
$$

$\#T < +\infty$, $\#U < +\infty$