ETA - 31/10/13

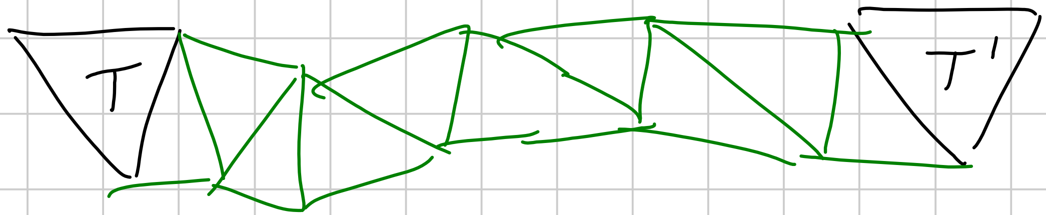
$(\Sigma_1 \setminus \mathring{T}_1) \cup_f (\Sigma_2 \setminus \mathring{T}_2)$ $\quad f: \partial T_1 \to \partial T_2$

$\Sigma_1 \# \Sigma_2 :=$

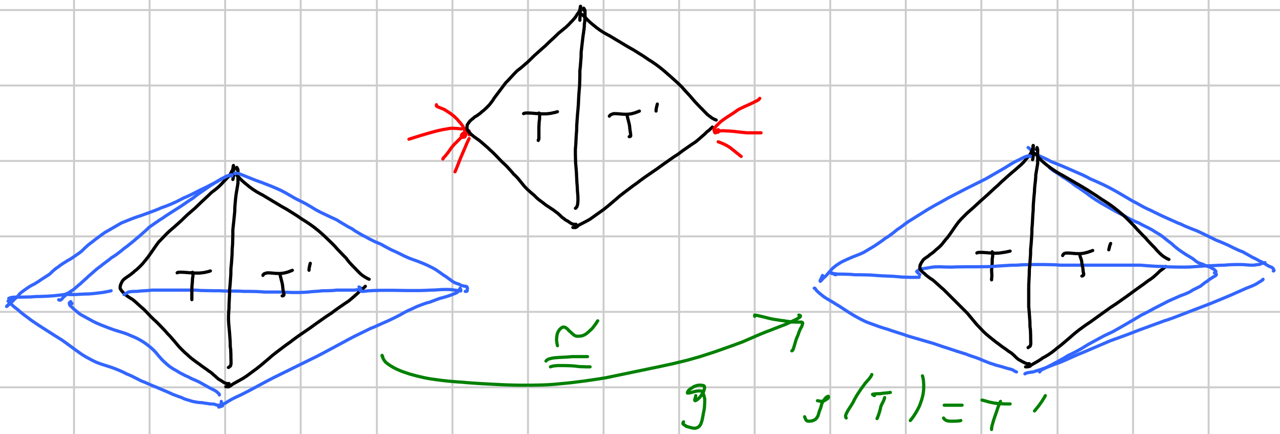
- Indip. da T_1 e T_2: segue da

$T, T' \in \Sigma^{[2]} \Rightarrow \exists g: \Sigma \to \Sigma$ omeo PL t.c. $g(T) = T'$

Basta vederlo per T, T' con lato in comune:



Per T, T' con lato comune :

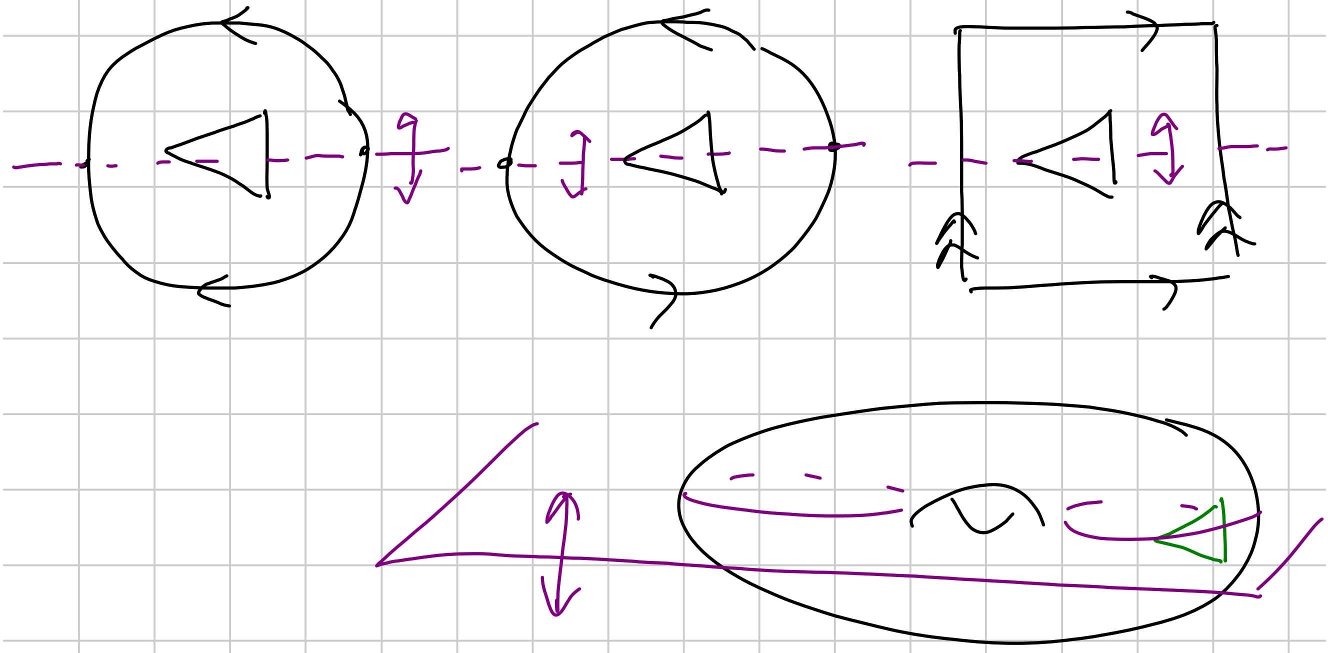


- È indip. da f a meno di perm pari dei vertici di T_1, T_2

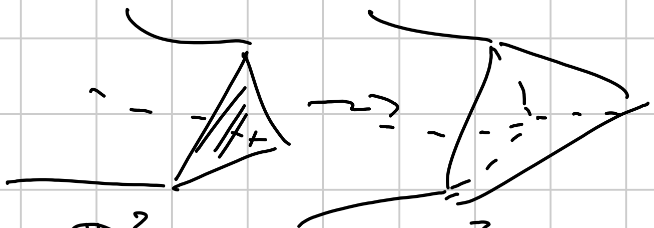
Segue da: ogni permutaz pari di $T^{[0]}$ è indotta da $g: \Sigma \to \Sigma$ omeo
(Dimo locale)

- Se Σ_1 o Σ_2 hanno un autoomorfismo che induce una permutaz. dispari su $T_1^{[0]}$ o $T_2^{[0]}$ allora non dipende da f

Fatto: S^2, $\mathbb{P}^2$, T ce l'hanno:



Oss: $\Sigma \# S^2 = \Sigma$

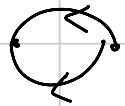
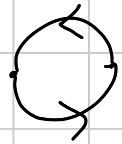


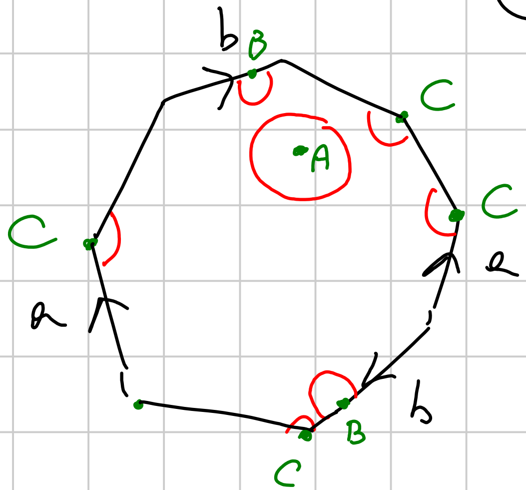
Conseguenze: $\underbrace{n \cdot T}_{= \underbrace{T \# \dots \# T}_{n}}$, $n \cdot \mathbb{P}^2$, $n \cdot T \# m \cdot \mathbb{P}^2$ sono ben def.

Teo: le sup/omeo PL sono

S^2, $n \cdot T$, $n \cdot \mathbb{P}^2$

$m \cdot T \# \mathbb{P}^2$ $\quad$ $m \cdot T \# K$

Dim: (1) Ogni poligono con lati identificati a coppie tramite funzioni affini dà superficie.
(compreso i casi  e )

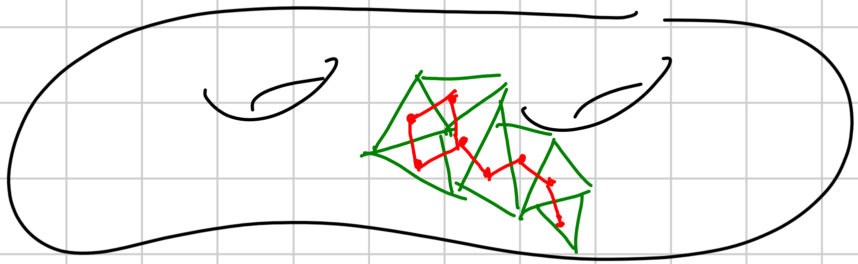


Tutti i link dei punti sono $S^1 \cong \partial\Delta_2$

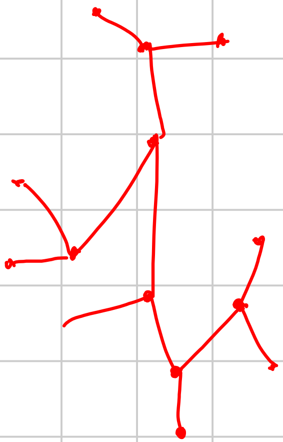
(+ realizzazione in $\mathbb{R}^n$ del c.s. astratto)

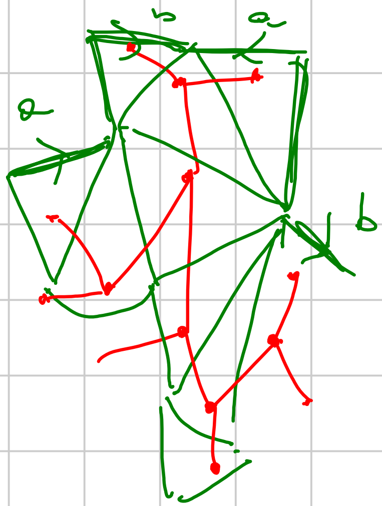
② Ogni superficie emerge così: prendo il "grafo di incollamento di una tria":

- un vertice per ogni triangolo
- un lato per ogni lato comune a due triangoli

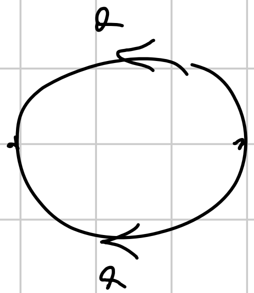


Prendo un albero max (contiene tutti i vertici) e lo realizzo nel piano; reintroduco i triangoli in modo da realizzare gli incollamenti dati dai lati nell'albero:

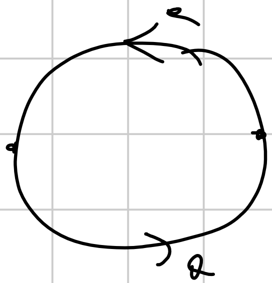




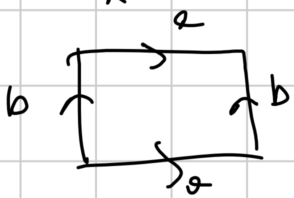
Ora: una superficie è una parola di lunghezza $2n$ in n lettere in cui ogni lettera compare due volte, eventualmente con esponente -1:



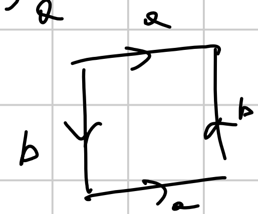
$= aa^{-1}$



$= aa$



$= aba^{-1}b^{-1}$



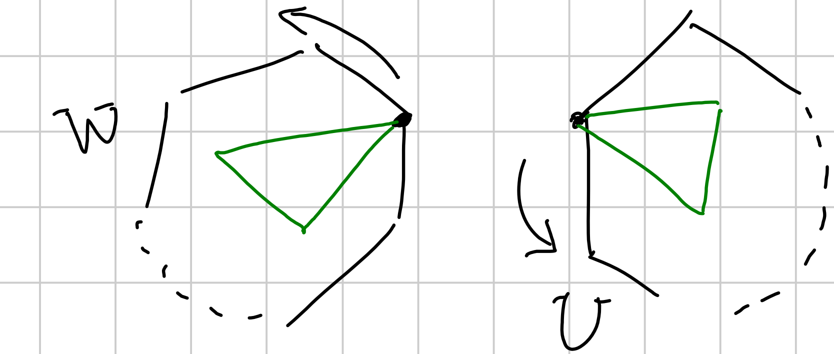
$aba^{-1}b$

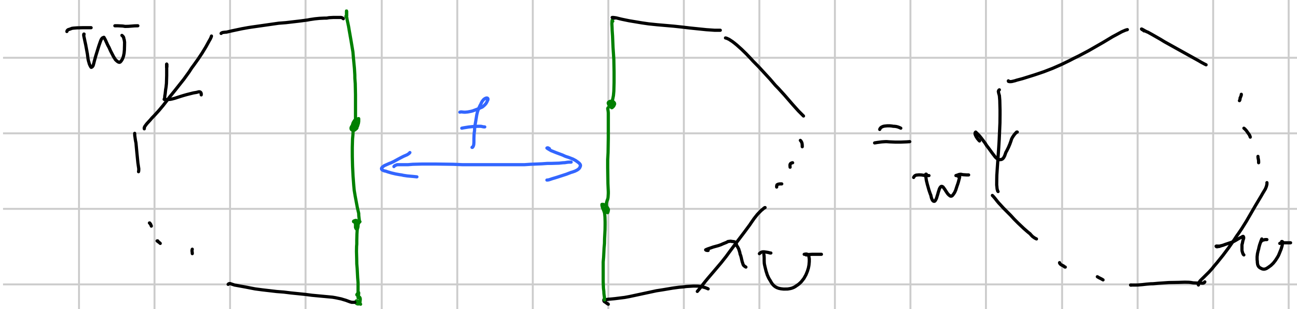
Viste a meno di: • permutazione ciclica

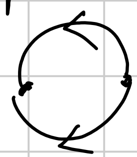
• inversione

• sostituire a con a^{-1} / $(a^{-1})^{-1} = a$

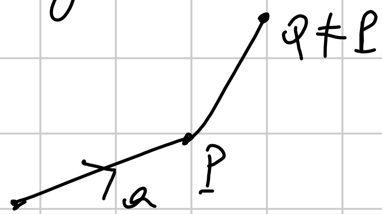
Oss: $W \# U = W \cdot U$

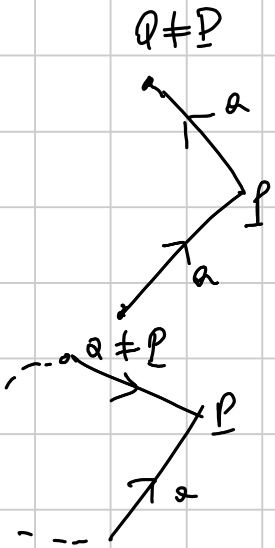




③ Wlog tutti i vertici del poligono si proiettano sullo stesso vertice di Σ (tranne ).

Supponiamo che nel quoziente ci sia un vertice P ma non solo lui. Nel poligono vedo  $Q \neq P$; guardo il compagno di a

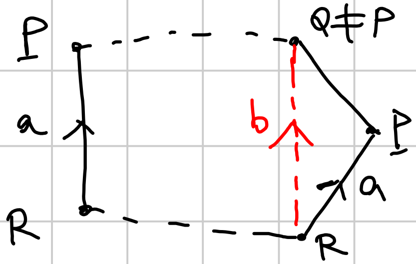


i-1) 

No : ovvi $Q = P$

i-2) 

ii-1)



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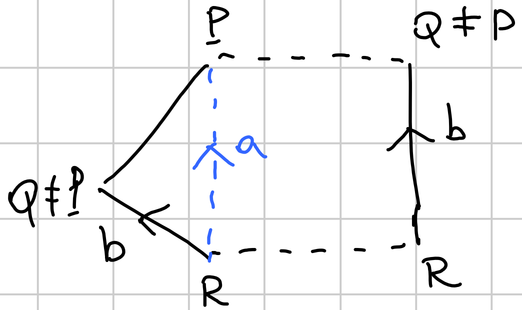
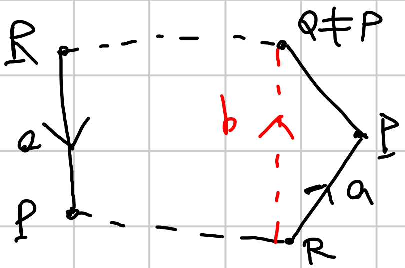


ii-2)



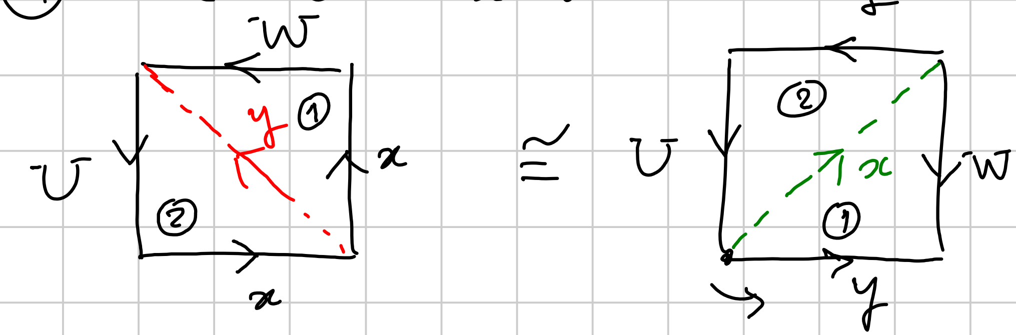
= . . . .

Ho ridotto di 1 il numero di vertici che si proiettano su P_

Ricorsivamente concludo_

(x = lettera W = parola)

④ $xxWU \cong xW^{-1}x\bar{U}$.



$yW^{-1}y\bar{U}$.

Conseguenza: $\mathbb{P}^2 \# \mathbb{P}^2 = K$, $T \# \mathbb{P}^2 = K \# \mathbb{P}^2$

$\mathbb{P}^2 \# \mathbb{P}^2 = aabb \cong ab^{-1}ab = K$

$$T \# \mathbb{P}^2 = a b a^{-1} b^{-1} c c = a b c b a c =$$

$$= a b b c^{-1} a c = c^{-1} a c a b b$$

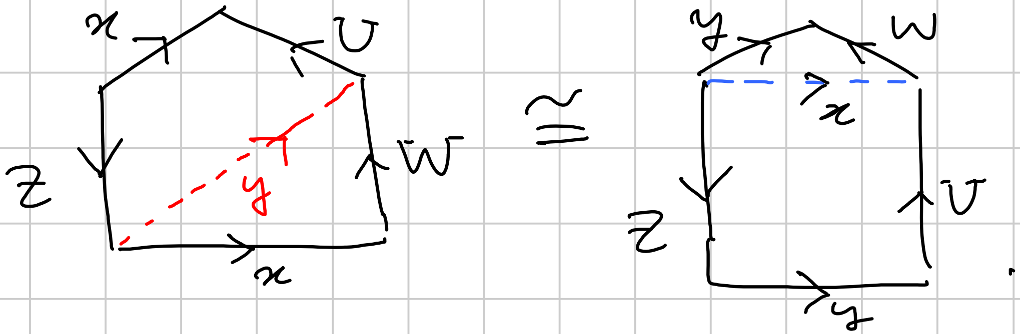
$$= K \# \mathbb{P}^2$$

Oss: ④ preserva "un solo vertice"

- usando ④ posso supporre che ogni x che compare come ... x ... x ... compaia consecutivo ... xx ...

⑤ Preservando "un solo vertice" e "xx consec." ho

$$x W U x^{-1} Z \cong x U W x^{-1} Z$$



⑥ $x W y U x^{-1} V y^{-1} Z = x y x^{-1} y^{-1} T$

$$x W y U x^{-1} V y^{-1} Z = x y U W x^{-1} V y^{-1} Z$$

$$= x y x^{-1} V U W y^{-1} Z = y x^{-1} V U W y^{-1} Z x$$

$$= yx^{-1}y^{-1}ZVUWx = xyx^{-1}y^{-1}ZVUW.$$

Ora abbiamo: un solo vertice

$a \ldots a$ consecutive $aa\ldots$

$a \ldots b \ldots a^{-1} \ldots b^{-1}$ consecutive $aba^{-1}b^{-1}\ldots$

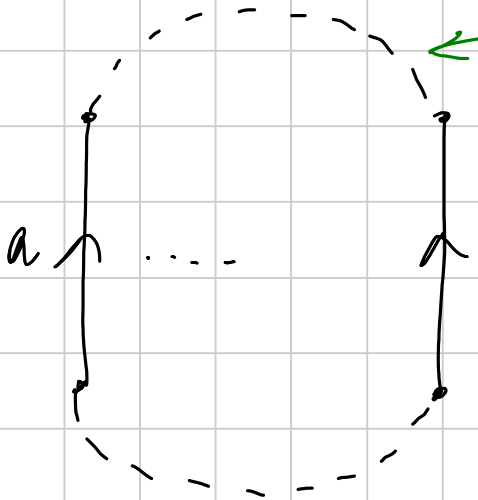
Se tutti i lati sono in di tali config. ok

infatti ho

$$n\cdot T \# m\cdot \mathbb{P}^2 \;\begin{cases} \to\; m=0, & n\cdot T \\ \to\; m>0 & (2n+m)\cdot \mathbb{P}^2 \end{cases}$$

Provo che ho usato tutti i lati: p.a.

sia a non usato; certamente ho $a \dots a^{-1} \dots$



Se b compare qui!

compagno è b

anche compagno sopra

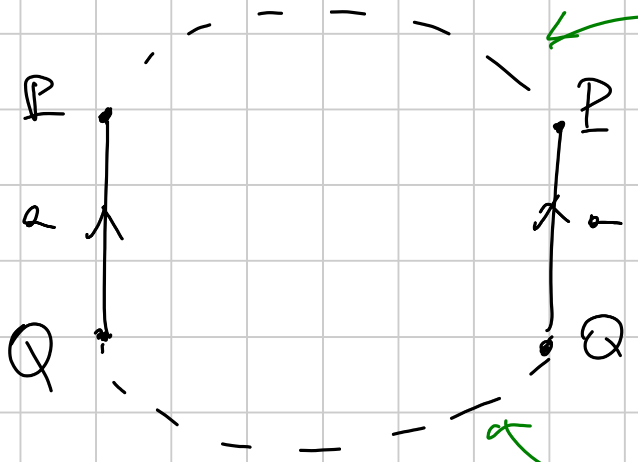
compagno b^{-1}

se b^{-1} fosse sotto avrei

$a \dots b \dots a^{-1} \dots b^{-1}$

<u>NO</u>

Se in config. $bcb^{-1}c^{-1}$, compagno sopra



i lati sopra si identificano solo tra loro

$\Downarrow$

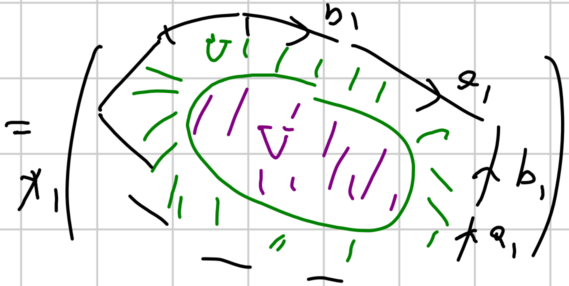
$P \neq Q$

i lati sotto si identificano solo tra loro

Resta da vedere che S^2, $n \cdot T$, $m \cdot \mathbb{P}^2$ sono distinti.

$$\pi_1(S^2) = 0 \Rightarrow H_1(S^2) = 0$$

$$\pi_1(n \cdot T) =$$



$$= \langle a_1, b_1, \dots, a_n, b_n \mid [a_1, b_1] \cdot \ldots \cdot [a_n, b_n] \rangle$$

$$\Rightarrow H_1(n \cdot T) = \mathbb{Z}^{2n}$$

$$\pi_1(m \cdot \mathbb{P}^2) = \langle a_1, \dots, a_m \mid a_1^2 \cdot \ldots \cdot a_m^2 \rangle$$

$$H_1(m\cdot\mathbb{P}^2) = \mathbb{Z}a_1 \oplus \dots \oplus \mathbb{Z}a_m \Big/ \langle 2a_1 + \dots + 2a_m \rangle$$

$$a_1 \dots a_m \longleftrightarrow b_1 = a_1 \ \dots \ b_{m-1} = a_{m-1}\ , \ b_m = a_1 + \dots + a_m$$

$$\begin{pmatrix} 1 & & 1 \\ & \ddots & \vdots \\ & 1 & 1 \end{pmatrix}$$

$$\mathbb{Z}b_1 \oplus \dots \oplus \mathbb{Z}b_m \Big/ 2b_m = \mathbb{Z}^{m-1} \oplus \mathbb{Z}/2 \, .$$

Sono gruppi abeliani distinti.

$\blacksquare$

— o —

Omologia relativa (per complessi simpliciali finiti)

K c.s. $L \subset K$ sottocomplesso.

$C_n(K,L) =$ gruppo abeliano libero generato da $K^{[n]} \setminus L^{[n]}$

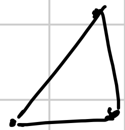
$$\partial_n^{(K,L)} \sigma = \sum_{\substack{\tau \in \partial^K \sigma \\ \tau \notin L^{[n-1]}}} \varepsilon(\sigma,\tau) \cdot \tau$$

Da $\partial_{n-1}^{K} \circ \partial_n^{K} = 0$ segue che $\partial_{n-1}^{(K,L)} \circ \partial_n^{(K,L)} = 0$

$\Longrightarrow$ Ho un complesso di catene $\left\{ \left(C_n(K,L), \partial_n^{(K,L)} \right) \right\}_{n \geq 0}^{+\infty}$

$\Longrightarrow$ ho $H_n(K,L)$.

<u>Oss</u>: $H_n(K,L) \neq H_n(\overline{K \setminus L})$ in gen.

<u>Es</u>: $K =$  $L =$  $\overline{K \setminus L} = K$

$$\Rightarrow H_n(\overline{K}, L) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{altrimenti} \end{cases}$$

$$\begin{array}{ccccccccccc} 0 \to \cdots 0 & \to & C_2(K,L) & \to & C_1(K,L) & \to & C_0(K,L) & \to \\ & & \parallel & & & & & \\ 0 \to \cdots 0 & \to & \mathbb{Z} & \to & 0 & \to & 0 & \to \end{array}$$

$$\Rightarrow H_n(K, L) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{altrimenti} \end{cases}$$

Fatti analoghi a quelli visti per $H_n(K)$:

- Se (A, B) suddivide (K, L) allora c'è $H_n(K, L) \xrightarrow{\cong} H_n(A, B)$ canonico

• Se $f : (K, L) \to (A, B)$ cioè $f : K \to A$, $f(L) \subset B$

ho $f_* : H_n(K, L) \to H_n(A, B)$ e

se $f_0 \simeq_{\text{rel } L} f_1$ cioè $\exists\, F : K \times [0,1] \to A$

$F(\cdot, 0) = f_0$

$F(\cdot, 1) = f_1$

$F(L \times [0,1]) \subset B$

allora $f_{0*} = f_{1*}$

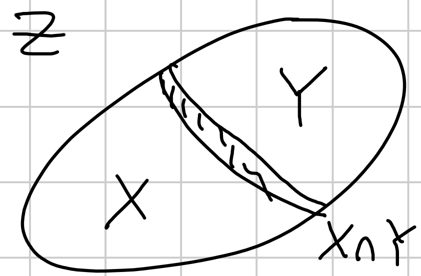
(anche per f_0, f_1 PL ma F continua)

Usare versione rel. del Teo Approx Simpliciale

- $H_n(K,L) = H_n(|K|,|L|)$ / isomorfismo

<u>Proprietà</u> (excision): $Z = X \cup Y$

X, Y <u>sottocompl.</u>



$\Rightarrow H_n(Z,Y) \cong H_n(X, X\cap Y)$

Infatti: $C_n(Z,Y) = \langle \sigma \in Z^{[n]} \setminus Y^{[n]} \rangle$

$= \langle \sigma \in X^{[n]} \setminus (X\cap Y)^{[n]} \rangle$

$$= C_n(X, X \cap Y) -$$