ETA 28/11/13

$$C_n^{sing}(X) = \left\{ \sum_{i=1}^{N} p_i \sigma_i \ : \ p_i \in \mathbb{Z} \right\} \qquad \sigma_i : \Delta_n \to X \ \text{cont}$$

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \varphi_i^{(n)} \qquad \varphi_i^{(n)} : \Delta_{n-1} \to \Delta_n$$

$z \in Z_n^{sing}(X)$ corrisponde a $M^{(n)} \longrightarrow X$

n-var. orientate con sing. in codim ≥ 3

Analogamente: $z \in Z_n$ è in B_n

$\Longleftrightarrow$ la mappa $M^{(n)} \longrightarrow X$ si estende a

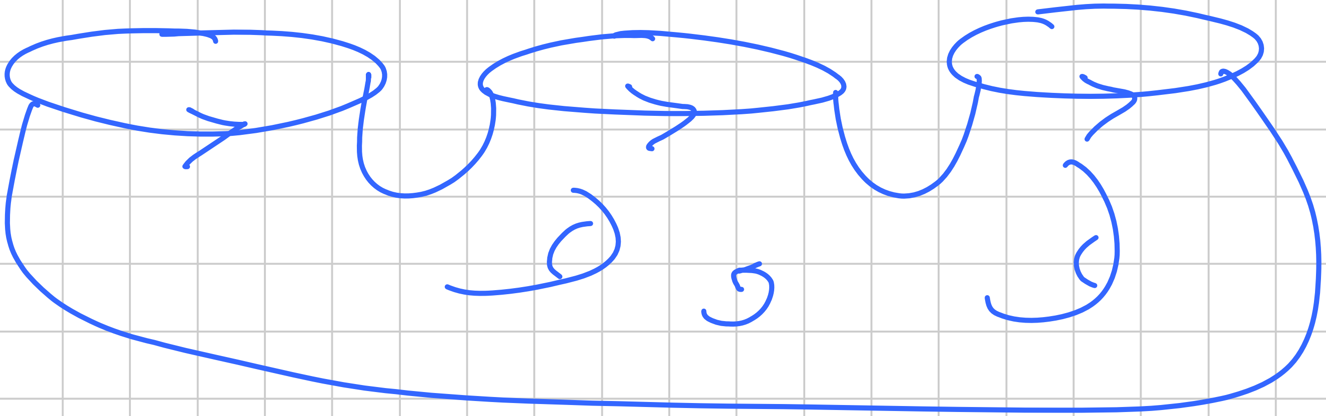
$$W^{(n+1)} \longrightarrow X \quad \text{con } \partial W = M$$

(vero però anche W ha singolarità).

Vero soltanto per $n=2$:

z nulle in $H_1(X)$

$\Updownarrow$



$w|_\partial = z$

Proprietà dell'omologia singolare (assoluta, $A=\emptyset$).

- $H_n(X_1 \sqcup X_2) \cong H_n(X_1) \oplus H_n(X_2)$, ovvero

$H_n(X) \cong \bigoplus H_n(X_\alpha)$ $\quad \{X_\alpha\}$ = c. connesse per archi

- $H_0(X) \cong \mathbb{Z}$ se X è connesso per archi

- $H_n(pt) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{altrimenti} \end{cases}$

$$\begin{array}{cccccccccccc} & & 3 & & 2 & & 1 & & 0 & & -1 & \\ C & \cdots \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow \\ H & & 0 & & 0 & & 0 & & \mathbb{Z} & & & \end{array}$$

- Omologia ridotta:

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & n>0 \\ \operatorname{Ker}\left(H_0(X) \ni [\sum_i n_i x_i] \mapsto \sum n_i \in \mathbb{Z}\right) \end{cases}$$

è l'omologia del complesso di catene aumentato ottenuto da $\{(C_n, \partial_n)\}$ ponendo $C_{-1} = \mathbb{Z}$ e $\partial_0(\sum n_i x_i) = \sum n_i$.

- $f : X \to Y$ continua $\Rightarrow$ $f_\# : C(X) \to C(Y)$

$$f_{\#_n}(\sigma) = f \circ \sigma$$

$$\Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

Immediato: $f_\#$ mappa tra complessi di catene

$$\partial^Y_n \circ f_{\# n} = f_{\# (n-1)} \circ \partial^X_n \Longrightarrow \text{induce } f_* : H(X) \to H(Y)$$

<u>Teorema</u>: $f_0 \simeq f_1$

$$\Longrightarrow f_{0*} = f_{1*}.$$

Def: Siano $\mathcal{C}, \mathcal{D}$ complessi di catene

$\varphi, \psi : \mathcal{C} \to \mathcal{D}$ mappe tra complessi di catene.

Dico che sono omotope, $\varphi \simeq \psi$, se

$\exists\, P_n : C_n \to D_{n+1}$ $\forall n$ t.c. in

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^{\mathcal{C}}_{n+1}"] \arrow[d, "\varphi_{n+1}-\psi_{n+1}"'] & C_n \arrow[r, "\partial^{\mathcal{C}}_{n}"] \arrow[d, "\varphi_n-\psi_n"] \arrow[dl, "P_n"'] & C_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}-\psi_{n-1}"] \arrow[dl, "P_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^{\mathcal{D}}_{n+1}"] & D_n \arrow[r, "\partial^{\mathcal{D}}_{n}"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

vale $\varphi_n - \psi_n = P_{n-1} \circ \partial_n^{\mathcal{C}} + \partial_{n+1}^{\mathcal{D}} \circ P_n$.

Lem: $\varphi \simeq \psi \Longrightarrow \varphi_* = \psi_*$.

Dim: sia $z \in Z_n(\mathcal{C})$

$$\Rightarrow \varphi_{n_*}([z]) - \psi_{n_*}([z]) = [\varphi_n(z)] - [\psi_n(z)]$$

$$= [(\varphi_n - \psi_n)(z)] = [\underbrace{P_{n-1} \circ \underbrace{\partial_n^{\mathcal{C}}(z)}_{0}}_{0} + \underbrace{\partial_{n+1}^{\mathcal{D}} \circ P_n(z)}_{0}]. \quad \blacksquare$$

Per provare il teorema dimostriamo che $f_{0\#} \simeq f_{1\#}$.

Ipotesi: $\exists F : X \times [0,1] \to Y$ t.c.

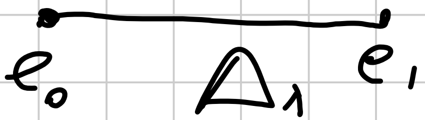
$$F(x,0) = f_0(x) \quad , \quad F(x,1) = f_1(x) .$$

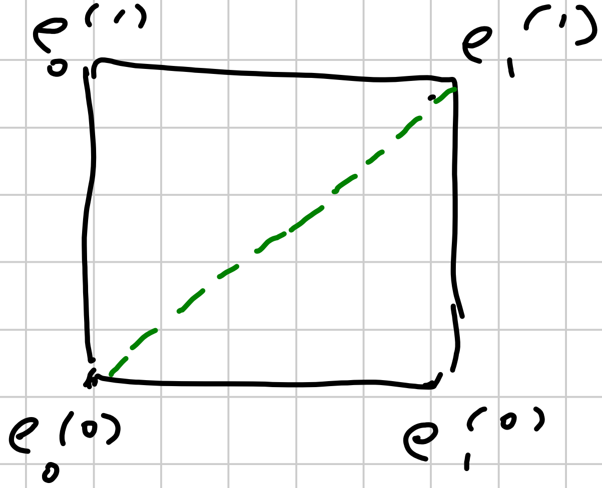
Idee: Cerchiamo $P_n : C_n(X) \to C_{n+1}(Y)$

t.c. $f_{1\#} - f_{0\#} = \partial_{n+1}^Y \circ P_n + P_{n-1} \circ \partial_n^X$;

esprimere $\Delta_n \times [0,1]$ come unione di $n+1$

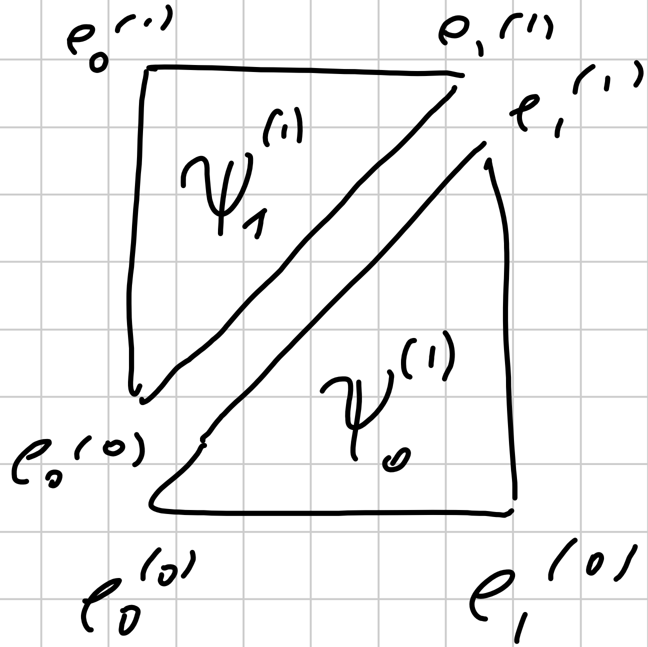
$(n+1)$-simplessi parametrizzati da $\psi_i^{(n)}$ $i = 0 \dots n+1$

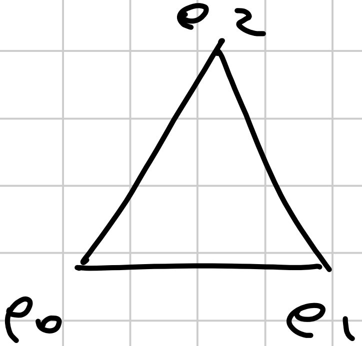




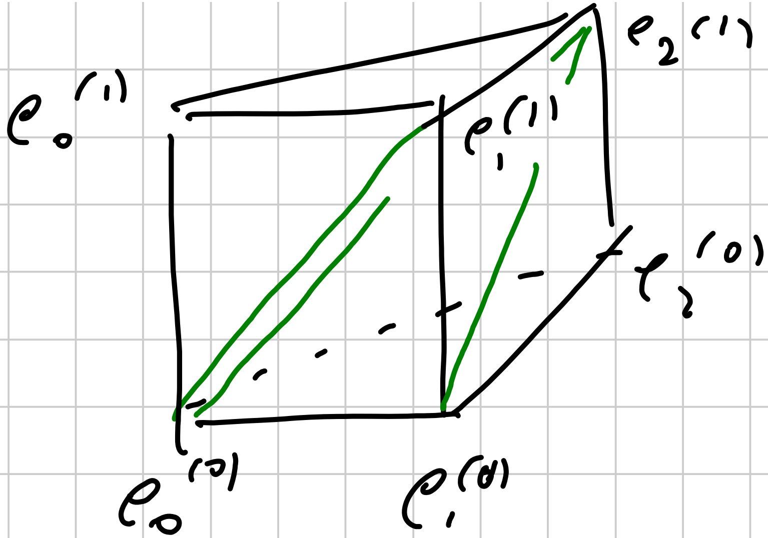
$\Delta_1 \times [0,1]$

=

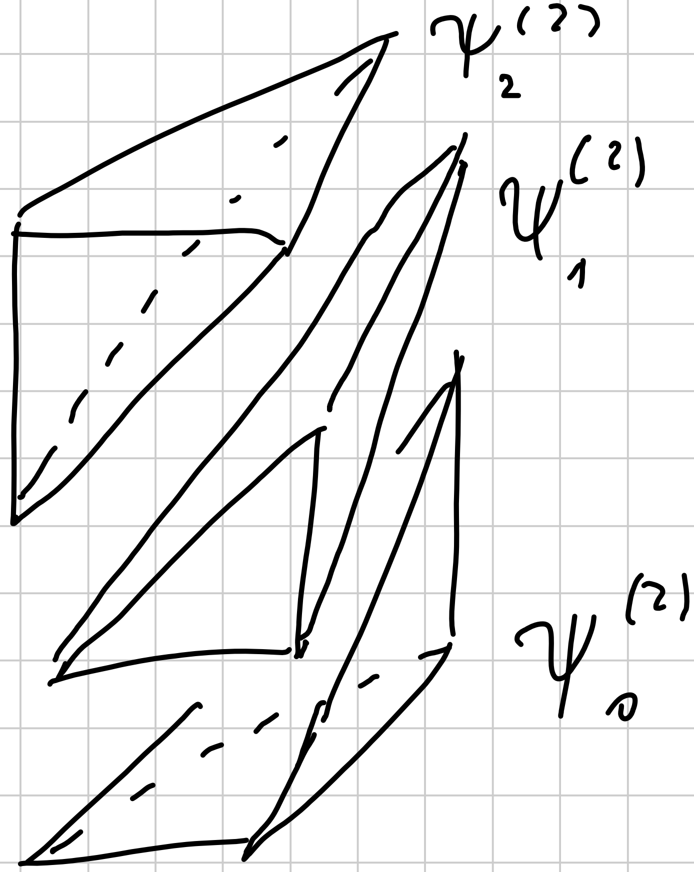




Δ_2



$\Delta_2 \times [0,1]$



Facendo $\underbrace{\sum (-1)^i \psi_i^{(n)}}_{}$ ← il bordo di; le facce interne e $\Delta_n \times [0,1]$

si cancellano: resta $\Delta_n \times \{1\} - \Delta_n \times \{0\} +$
$+ \partial \Delta_n \times [0,1]$

Poniamo:

$$P_n(\sigma) = F \circ (\sigma \times \mathrm{id}_{[0,1]}) \circ \sum (-1)^i \psi_i^{(n)}$$

da cui $\partial_{n+1}^{Y} P_n(\sigma) = f_{1\#}(\sigma) - f_{0\#}(\sigma) + P_{n-1}(\partial_n^X \sigma) -$

<u>Dettagli</u>: $\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \varphi_i^{(n)}$

$$\varphi_i^{(n)} : \Delta_{n-1} \longrightarrow \Delta_n , \quad \varphi_i^{(n)}(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i \end{cases}$$

(parametrizzazione ovvia della faccia di Δ_n opposta ad e_i).

Ora $\psi_i^{(n)} : \Delta_{n+1} \longrightarrow \Delta_n \times [0,1] \qquad i = 0, \dots, n$

$$\psi_i^{(n)}(e_j) = \begin{cases} e_j^{(0)} & j \leq n - i \\ e_{j-1}^{(1)} & j > n - i \end{cases}$$

cioè prendo tutti i vertici ad altezza 0 fino all'$(n-i)$-esimo, poi l'$(n-i)$-esimo ad altezza 1 e continuo ad altezza 1.

Pongo $P_n(\sigma) = \sum_{i=0}^{n} (-1)^{n+i} F \circ (\sigma \times Id_{[0,1]}) \circ \psi_i^{(n)}$.

(Hatcher non lo mette)

Affermo che $\underbrace{P_{n-1} \circ \partial_n}_{X} + \underbrace{\partial_{n+1} \circ P_n}_{Y} = 1_{\#} - f_{0\#}(1)$

Affermo che: $\psi_i^{(n)} \circ \varphi_{n-i}^{(n+1)} = \psi_{i+1}^{(n)} \circ \varphi_{n-i}^{(n+1)}$

(informalmente: le facce di mezzo si cancellano facendo il bordo).

$$
\begin{array}{lccccccc}
 & e_0, & \dots, & e_{n-i-1}, & e_{n-i}, & \dots, & e_n \\
\varphi_{n-i}^{(n+1)} : & \downarrow & & \downarrow & \downarrow & & \downarrow \\
 & e_0, & \dots, & e_{n-i-1}, & e_{n-i+1}, & \dots, & e_{n+1}
\end{array}
$$

$$
\begin{array}{lccccccc}
 & e_0, & \dots, & \boxed{\cancel{e_{n-i}}}, & e_{n-i+1}, & \dots, & e_{n+1} \\
\psi_i^{(n)} : & \downarrow & & \boxed{\cancel{\downarrow}} & & & \\
 & e_0^{(0)}, & \dots, & \boxed{\cancel{e_{n-i}^{(0)}}}, & e_{n-i}^{(1)}, & \dots, & e_n^{(1)}
\end{array}
$$

$$\psi^{(n)}_{i+1} : \begin{array}{cccccccc} e_0, & \dots, & e_{n-i-1}, & \boxed{\cancel{e_{n-i}}}, & \dots, & e_{n+1} \\ \downarrow & & \downarrow & \boxed{\cancel{\downarrow}} & & \downarrow \\ e^{(0)}_0, & \dots, & e^{(0)}_{n-i-1}, & \boxed{\cancel{e^{(1)}_{n-i-1}}}, & \dots, & e^{(1)}_n \end{array}$$

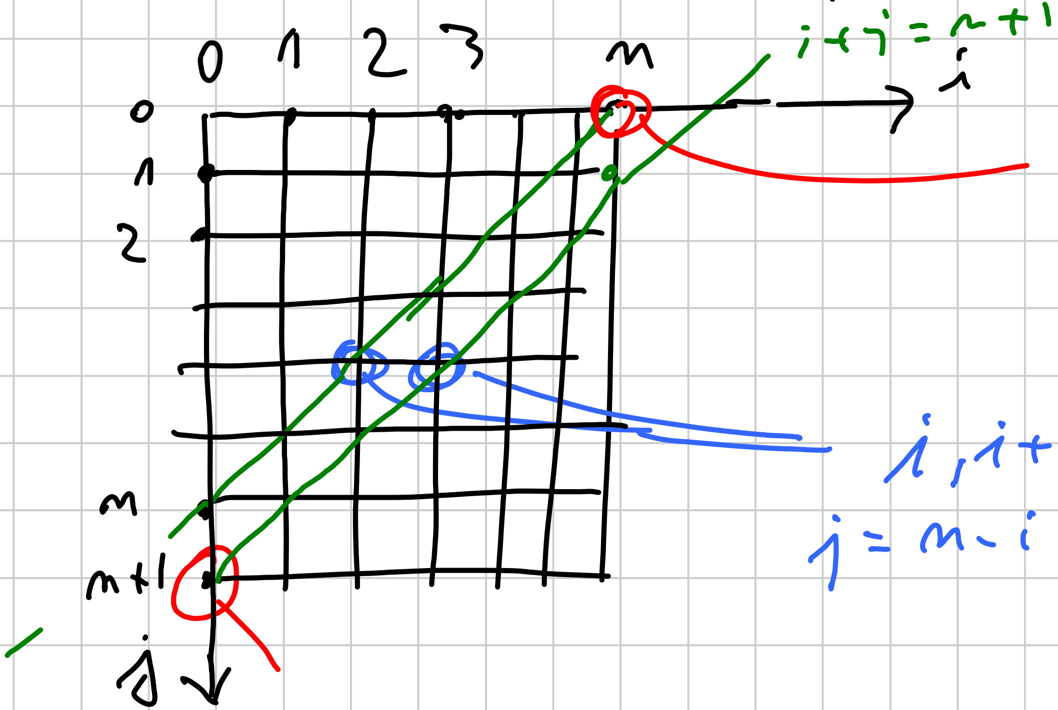
Analogamente $\psi^{(n)}_0 \circ \varphi^{(n+1)}_{n+1} = id_{\Delta_n} \times \{0\}$ (2)

$$\psi^{(n)}_n \circ \varphi^{(n+1)}_0 = id_{\Delta_n} \times \{1\}. \quad (3)$$

(informalmente: facendo il bordo restano

$\Delta_n \times \{1\} - \Delta_n \times \{0\} \Big) -$

$$\partial^{\Upsilon}_{n+1} \left(P_n(\sigma) \right) = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{n+i+j} \cdot F \circ \left(\sigma \times id_{[0,1]} \right) \circ \varphi_i^{(n)} \circ \varphi_j^{(n+1)}$$



$j=0,\ i=n$

$$(-1)^{0+n+n} \; F \circ \left(\sigma \times id \right) \circ \left(id_{\Delta_n} \times \{1\} \right)$$

$$= \sigma_{1\#}(\sigma) \qquad (3)$$

$i,\ i+1$

$j = n-i \qquad (-1)^{\cdots} \ast \; + (-1)^{\cdots} \ast$

upnoi (3)

$i+j=m$

$i=0 \quad j=m+1$

$$(-1)^{0+m+m+1} F\circ(\sigma\times id)\circ\left(id_{\Delta_n}\times\{0\}\right) = \overset{(2)}{} \, \| \, 0$$

$$= f_{0\#}(\sigma)$$

$$\Rightarrow \partial_{m+1}\left(P_n(\sigma)\right) = f_{1\#}(\sigma) - f_{0\#}(\sigma) +$$

$$+ \sum_{\substack{i+j\leq m \\ i+j\geq m+1}} (-1)^{m+i+j} F\circ(\sigma\times id_{[0,1]})\circ\psi_i^{(n)}\circ\varphi_j^{(n+1)}$$

devo vedere che è $P_{n-1}(\partial_n^X \sigma)$.

Affermo che:

$$\psi_i^{(n)} \circ \varphi_j^{(n+1)} = \begin{cases} \left(\varphi_j^{(n)} \times id_{[0,1]}\right) \circ \psi_i^{(n-1)} & \text{se } i+j < n \\ \left(\varphi_{j-1}^{(n)} \times id_{[0,1]}\right) \circ \psi_{i-1}^{(n-1)} & \text{se } i+j > n+1 \end{cases}$$

laborioso ma diretto usando le definizioni.

Dunque l'ultimo addendo è:

$$\sum_{i+j<n} (-1)^{n+i+j} F \circ \underbrace{(\sigma \times id_{[0,1]}) \circ (\varphi_j^{(n)} \times id_{[0,1]})}_{(\sigma \circ \varphi_j^{(n)}) \times id_{[0,1]}} \circ \psi_i^{(n-1)}$$

$$+ \sum_{i+j \geq n+1} (-1)^{n+i+j} F \circ \underbrace{(\sigma \times id_{[0,1]})(\varphi_{j-1}^{(n)} \times id_{[0,1]})}_{(\sigma \circ \varphi_{j-1}^{(n)}) \times id_{[0,1]}} \circ \psi_{i-1}^{(n-1)}$$

pongo $i = h+1$
$j = k+1$

e poi h lo richiamo i
k lo richiamo j

$$= \sum_{i+j<n} (-1)^{n+i+1} F \circ \left(\left(\sigma \circ \varphi_j^{(n)} \right) \times id_{[0,1]} \right) \circ \psi_i^{(n-1)}$$

$$+ \sum_{i+j \geq n} (-1)^{n+i+1} F \circ \left(\left(\sigma \circ \varphi_j^{(n)} \right) \times id_{[0,1]} \right) \circ \psi_i^{(n-1)}$$

$$= P_{n-1} \left(\sum_{j=0}^{n} (-1)^j \sigma \circ \varphi_j^{(n)} \right) = P_{n-1} \left(\partial_n^X (\sigma) \right). \quad \boxed{}$$

(Forse non ci vuole $(-1)^n$ ...)

——— o ———

Omologia singolare relativa.

$$C_n(X,A) = C_n(X) / C_n(A)$$

Poiché $\partial_n \sigma \in C_{n-1}(A)$ per $\sigma \in C_n(A)$

abbiamo $\partial_n^{(X,A)} : C_n(X,A) \longrightarrow C_{n-1}(X,A)$

e $\partial_{n-1}^{(X,A)} \circ \partial_n^{(X,A)} = 0$

$\Longrightarrow H_n(X,A)$ _ per costruzione

$$0 \longrightarrow C(A) \longrightarrow C(X) \longrightarrow C(X,A) \longrightarrow 0$$

$\Longrightarrow$ LES

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \to \cdots$$

$i : A \hookrightarrow X \qquad j : (X, \emptyset) \hookrightarrow (X, A)$

Interpretazione di ∂_n.

<u>Prop</u>: Posto $Z_n'(X,A) = \{ z \in C_n(X) : \partial_n z \in C_{n-1}(A) \}$

$B_n'(X,A) = \{ d + \partial_{n+1} e : d \in C_n(A), e \in C_{n+1}(X) \}$

si ha $H_n(X,A) \cong Z_n'(X,A) / B_n'(X,A)$.

<u>Oss</u>: usando Z'/B' come def. di $H_n(X,A)$

$$\partial_n : H_n(X,A) \to H_{n-1}(A)$$
$$[z] \longmapsto [\partial z].$$

<u>Dim</u> (Prop): Chiamo $q_n : C_n(X) \to C_n(X,A) = \frac{C_n(X)}{C_n(A)}$

$\partial_n^{(X,A)} : C_n(X,A) \to C_{n-1}(X,A)$

$$\partial_n^{(X,A)}\left(q_n(c)\right) = q_{n-1}\left(\partial_n^X(c)\right)$$

Definiamo $\psi_n : H_n'(X,A) \to H_n(X,A)$

$$[c]' \longmapsto [q_n(c)].$$

Affermo che è isomorfismo.

Ben def: • $\partial_n^{(X,A)} q_n(c) = q_{n-1}(\partial_n^X(c))$

$$\underbrace{\hphantom{C_{n-1}(A)}}_{0} \quad C_{n-1}(A) \quad \checkmark$$

- Se $c = d + \partial^X_{n+1} e \qquad d \in C_n(A)\,,\; e \in C_{n+1}(X)$

$$\Rightarrow [q_n(c)] = [q_n(d + \partial^X_{n+1} e)]$$

$$= [\underbrace{q_n(d)}_{\shortparallel \atop 0} + q_n(\partial^X_{n+1}(e))] = \underbrace{[\partial^{(X,A)}_{n+1}(q_{n+1}(e))]}_{\shortparallel \atop 0}.$$

Iniettiva: Se $q_n(c) \in B_n(X,A)$

$\Rightarrow q_n(c) = q_n(\partial^X_{n+1}(e))$

$\Rightarrow q_n(c - \partial^X_{n+1} e) = 0 \Rightarrow c - \partial^X_{n+1} e = d \in C_n(A)$

Surgettiva: Se prendo un n-catena x (X,A)

cioè $\partial^{(X,A)}_{n+1} q_n(c) = 0$ allora

$q_{n-1}(\partial^X_n c) = 0 \Rightarrow \partial^X_n c \in C_{n-1}(A)$

$\Rightarrow c \in Z_n'$ e $[q_n(c)]$ è immagine di $[c]'$.

$\boxtimes$

<u>Oss</u>: Se $f_0, f_1 : (X, A) \to (Y, B)$

e sono omotope tra mappe $(X, A) \to (Y, B)$

allora $P_n^X : C_n(X) \to C_{n+1}(Y)$ manda

$C_n(A)$ in $C_{n+1}(B) \Rightarrow$ induce

$P_n^{(X,A)} : C_n(X,A) \to C_{n+1}(Y,B)$

che prova che $f_{0\#} \simeq f_{1\#}$

$\Rightarrow f_{0*} = f_{1*} : H(X,A) \to H(Y,B)$ ___

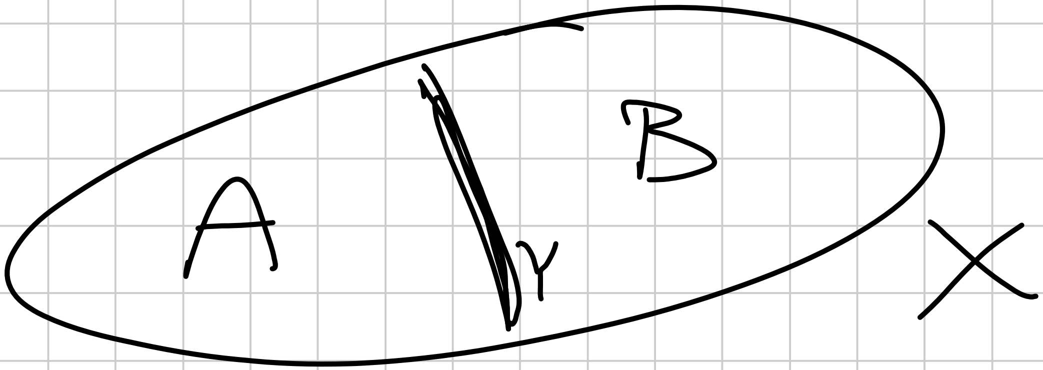
—— o ——

Escissione per complessi simpliciali finiti

$X = A \cup B$, A, B sottocomplessi

$Y = A \cap B$

$$H_n(X,A) = H_n(B,Y)$$

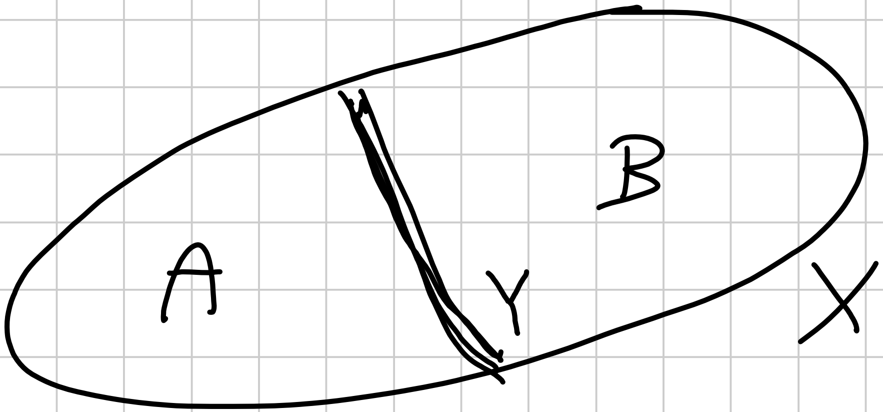


Teo: X sp. top. $Z \subset A \subset X$ con $\overline{Z} \subset \mathrm{int}(A)$

$\Rightarrow (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induce

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A).$$

Oss: dall'escissione topologica segue quella simpliciale



Idea: prendere intorni regolari e usare anche l'omotopia.