ETA 19/12/13

R anello commut. con 1

$C_n(X,A;R)$ R-modulo

$C^n(X,A;R)$ R-modulo

$\Longrightarrow$ $H_n(X,A;R)$, $H^n(X,A;R)$ R-moduli.

Introduciamo su $H^*(X,A;R)$ anche struttura di anello (se R campo: algebra su R) via cup product. (Tutti i coeff in R fino a nuovo ordine.)

$$\left(Q_8 \not\cong \mathbb{Z}/4 \rtimes \mathbb{Z}/2 . \right)$$

$$\cup : C^k(X) \times C^l(X) \to C^{k+l}(X)$$

(uso teoria simpliciale o singolare)

$$(\varphi \cup \psi)[v_0, \dots, v_{k+\ell}] = \varphi([v_0, \dots, v_k]) \underset{\text{in } R}{\cdot} \psi([v_k, \dots, v_{k+\ell}])$$

<u>Prop</u>: $\delta_{k+\ell}(\varphi \cup \psi) = (\delta_k \varphi) \cup \psi + (-1)^k \varphi \cup (\delta_\ell \psi)$

<u>Dim</u>: $\delta_{k+\ell}(\varphi \cup \psi)([v_0, \dots, v_{k+\ell+1}])$

$$= (\varphi \cup \psi)(\partial_{k+\ell+1}[v_0, \dots, v_{k+\ell+1}])$$

$$= (\varphi \cup \psi)\left(\sum_{i=0}^{k+\ell+1} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_{k+\ell+1}] \right)$$

$$= (\varphi \cup \psi)\Big(\sum_{i=0}^{k} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k, \dots, v_{k+\ell+1}] + \sum_{j=1}^{\ell+1} (-1)^{k+j} [v_0, \dots, v_k, \dots, \hat{v}_{k+j}, \dots, v_{k+\ell+1}] \Big)$$

$$= \sum_{i=0}^{k} (-1)^i \varphi([v_0, \ldots, \hat{v}_i, \ldots, v_k, v_{k+1}]) \cdot \psi([v_{k+1} \ldots v_{k+\ell+1}])$$

$$+ (-1)^{k+1} \varphi([v_0, \ldots, v_k]) \cdot \psi([v_{k+1}, \ldots, v_{k+\ell+1}])$$

$$+ (-1)^{k} \varphi([v_0, \ldots, v_k]) \cdot \psi([v_{k+1}, \ldots, v_{k+\ell+1}])$$

$$+ \sum_{j=1}^{\ell+1} (-1)^{k+j} \varphi([v_0, \ldots, v_k]) \psi([v_k, \ldots, \hat{v}_{k+j}, \ldots, v_{k+\ell+1}])$$

$$= \varphi\left(\sum_{i=0}^{k+1} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}]\right) \cdot \psi([v_{k+1}, \ldots, v_{k+\ell+1}])$$

$$+ (-1)^k \varphi([v_0, \ldots, v_k]) \cdot \psi\left(\sum_{j=0}^{\ell+1} (-1)^j [v_k \ldots \hat{v}_{k+j} \ldots v_{k+\ell+1}]\right)$$

$$= \varphi(\partial_{k+1} [v_0, \ldots, v_{k+1}]) \cdot \psi([v_{k+1}, \ldots, v_{k+\ell+1}])$$

$$+ (-1)^k \cdot \varphi([v_0, \ldots, v_k]) \cdot \psi(\partial_{\ell+1} [v_{k+1} \ldots v_{k+\ell+1}])$$

$$= (\delta_k \varphi)([v_0, \ldots, v_{k+1}] \cdot \psi[v_{k+1}, \ldots, v_{k+\ell+1}]$$

$$+ (-1)^k \varphi([v_0, \ldots, v_k]) \cdot (\delta_\ell \psi)([v_k, \ldots, v_{k+\ell+1}])$$

$$= \left((\delta_k \varphi) \cup \psi + (-1)^k \varphi \cup (\delta_\ell \psi) \right) [v_0, \dots, v_{k+\ell+1}] .$$

$\square$

<u>Cor</u>: $Z^k \cup Z^\ell \subset Z^{k+\ell}$

$Z^k \cup B^\ell \subset B^{k+\ell}$ (*)

$B^k \cup Z^\ell \subset B^{k+\ell}$

(*): $\varphi \in C^k$, $\delta_k \varphi = 0$ $\quad \psi \in C^\ell$, $\psi = \delta_{\ell-1} \eta$

$$\varphi \cap \psi = \varphi \cap (\delta_{\ell-1} \eta) =$$

$$= \underbrace{(\delta_k \varphi)}_{=0} \cup \left((-1)^k \eta\right) + (-1)^k \varphi \cap \left(\delta_{\ell-1} \left((-1)^k \eta \right) \right)$$

$$= \delta_{k+\ell-1} \left(\varphi \cap (-1)^k \cdot \eta \right) . \quad \square$$

Dunque $\cup$ sulle cocatene induce:

$$\cup : H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X) .$$

Sia $H^*(X) = \bigoplus_{k=0}^{+\infty} H^k(X)$ ho

$\cup : H^*(X) \times H^*(X) \to H^*(X)$

dà struttura di anello graduato commutativo

$1 \in H^0(X;R)$: la funzione che vale $1 \in R$ su ogni 0-simplesso.

Esempio 1: Σ_g = sup. orientabile di genere g.

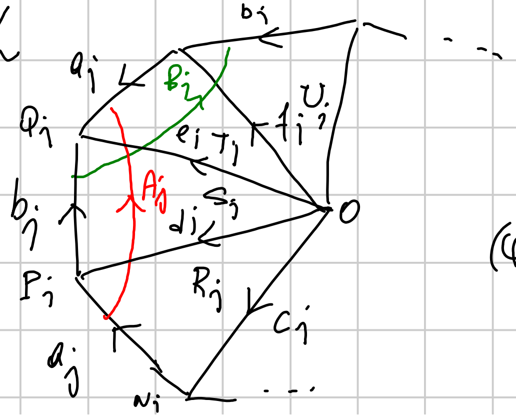
$R = \mathbb{Z}$

$H_0 = \mathbb{Z}$ $\quad$ $H_1 = \mathbb{Z}^{2g} = \langle a_1, b_1, \dots, a_g, b_g \rangle$ $\quad$ $H_2 = \mathbb{Z} = \langle D \rangle$

$H^0 = \mathbb{Z}$ $\quad$ $H^1 = \mathbb{Z}^{2g} = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \rangle$ $\quad$ $H^2 = \mathbb{Z} = \langle \hat{D} \rangle$

Devo calcolare $\quad \cup : H^1 \times H^1 \to H^2 = \mathbb{Z}$.

$\Sigma_g = D/\sim$



$R_j = [O, P_j, N_j]$

così che

$(\varphi \cup \psi)(R_j) =$

$= \varphi(d_j) \cdot \psi(-a_j)$

In coomologia α_j è duale algebrico di a_j, ma come cocatena non posso definire che sia nulla su c_j, d_j, e_j, f_j perché non sarebbe cobordo: $\delta\alpha_j(R_j) = \alpha_j(\partial R_j) = 1$. Definisco invece $\alpha_j(\text{lato}) = \bigcap_{alg} (A_j \cap \text{lato})$, dunque

$$\alpha_j : \begin{cases} a_j \mapsto 1 \\ d_j, e_j \mapsto 1 \\ \text{altri} \mapsto 0 \end{cases} ; \quad \text{Analog. } \beta_j(\text{lato}) = \bigcap_{alg} (B_j, \text{lato}) : \begin{cases} b_j \mapsto 1 \\ e_j, f_j \mapsto 1 \\ \text{altri} \mapsto 0 \end{cases}$$

devo calcolare $\varphi \cup \psi$ su ogni R_j, S_j, T_j, U_j, e $\varphi \cup \psi$ come el. di $\Sigma = H^2$ sarà $(\varphi\cup\psi)\left(\sum_{j=1} R_j + S_j + T_j + U_j\right)$.

$$\varphi \cup \psi : \begin{cases} R_j \mapsto \varphi(d_j) \cdot \psi(-a_j) \\ S_j \mapsto \varphi(e_j) \cdot \psi(-b_j) \\ T_j \mapsto \varphi(f_j) \cdot \psi(a_j) \\ U_j \mapsto \varphi(c_{j+1}) \cdot \psi(b_j) \end{cases}$$

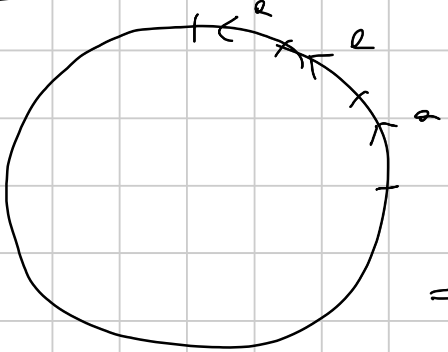
Combinando i calcoli trovo:

$\alpha_j \cup \beta_j = 1 \qquad \beta_j \cup \alpha_j = -1 \qquad \alpha_i \cup \beta_j = \beta_j \cup \alpha_i = 0, \ i \neq j$

$\Rightarrow$ $\cup$ su $\mathbb{Z}^{2g}$ è la forma simplettica

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & & \ddots \end{pmatrix}$$

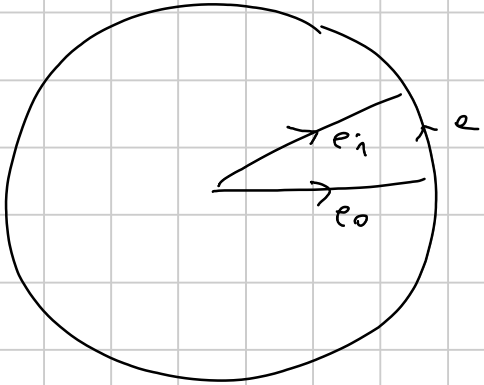
Es: $X = \{ z \in \mathbb{C} : |z| \leq 1 \} /_{\substack{z \sim z^m \\ \text{se } |z| = 1}}$ $\quad$ ($m = 2$ $X = \mathbb{P}^2(\mathbb{R})$)



$H_0 = \mathbb{Z} \quad H_1 = \mathbb{Z}/_m \quad H_2 = 0$

$\Rightarrow$ Se $R = \mathbb{Z}/_m$

$H^0 = \mathbb{Z}/_m \quad H^1 = \mathbb{Z}/_m \quad H^2 = \mathbb{Z}/_m$



$H^1 = \langle \hat{a} \rangle$

come cocatena

$\hat{a}(e_j) = j$

$\Rightarrow \hat{a} \cup \hat{a} = \sum_{j=0}^{m-1} j = \frac{m(m-1)}{2}$

$\Rightarrow H^*$ è banale come anello se m è dispari, non banale per m pari.

$m=2$ $(X = \mathbb{P}^2)$: $\hat{a} \cup \hat{a} = 1 \in \mathbb{Z}/2$

$$\Rightarrow H^*(\mathbb{P}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2[\hat{a}] \big/ \hat{a}^3$$

$1 \in H^0$, $\hat{a} \in H^1$, $\hat{a} \cup \hat{a} \in H^2 \cong \mathbb{Z}/2$

Prodotto $\cup$ è funtoriale: $f^*(\varphi \cup \psi) = f^*(\varphi) \cup f^*(\psi)$.

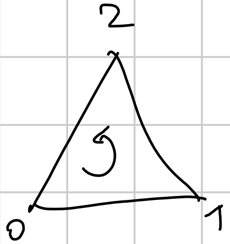
Prop: $\varphi \in H^k, \psi \in H^l \Rightarrow \psi \cup \varphi = (-1)^{k \cdot l} \varphi \cup \psi$

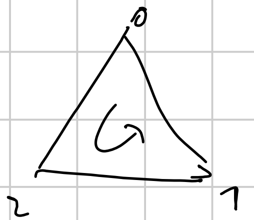
(commutazione per anelli graduati).

Dim: $\rho_n : C_n(X) \circlearrowleft \quad \rho_n([v_0, \dots, v_n]) = \varepsilon_n \cdot [v_n, \dots, v_0]$

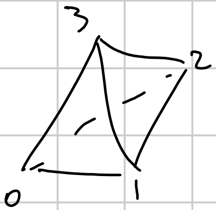
dove $\varepsilon_n = (-1)^{n(n+1)/2}$

Siccome $[v_n, \dots, v_0]$ è il simplesso $[v_0, \dots, v_n]$ con orientazione ε_n

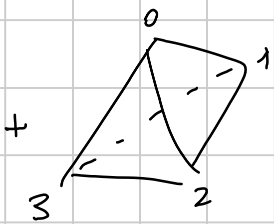




$$(-1)^{2 \cdot 3/2} = -1$$



$\cong_+$



$\Rightarrow \rho_M$ è omotopa all'identità come mappa tra complessi di catene

$\Rightarrow$ induce id su H_* e H^*

$$(\psi\cup\varphi)([v_0,\dots,v_{k+\ell}]) = \varepsilon_{k+\ell}\cdot\big((\psi\cup\varphi)[v_{k+\ell},\dots,v_0]\big)$$

$$= \underbrace{\varepsilon_{k+\ell}\cdot\varepsilon_k\cdot\varepsilon_\ell}_{(-1)^{k\cdot\ell}}\cdot(\varphi\cup\psi)([v_0,\dots,v_{k+\ell}])\,.$$

Dualità di Poincaré:

<u>Teo</u>: Se $M^{(m)}$ è chiusa allora $H_k(M;\mathbb{Z}/2)\cong H^{m-k}(M;\mathbb{Z}/2)$

Se M è orientabile allora
$$H_k(M;\mathbb{Z})\cong H^{m-k}(M;\mathbb{Z})\,.$$

<u>Cor</u>: 1. $H_k(M;\mathbb{Z}/2)\cong H_{m-k}(M;\mathbb{Z}/2)$.

2. M orientabile $\Rightarrow H_k(M;\mathbb{Z}) \cong \dfrac{H_{n-k}(M;\mathbb{Z})}{\mathrm{Tor}(H_{n-k})} \oplus \mathrm{Tor}(H_{n-k-1})$

In particolare $H_{n-1}(M;\mathbb{Z})$ è privo di torsione.
Inoltre $\mathrm{rank}(H_k) = \mathrm{rank}(H_{n-k})$, da cui
$\chi(M^{2m+1}) = 0$.

Dim: Realizzo M come un complesso politopale X. Definisco il complesso duale $\hat{X}$ così:

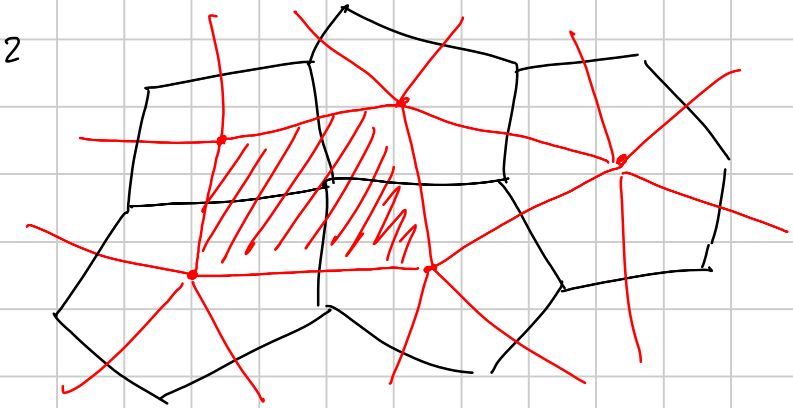
$\hat{X}^{[0]}$ = un punto interno $\hat{c}$ per ogni $c \in X^{[n]}$

$\hat{c}_1, \dots, \hat{c}_p$ sono i vertici di un politopo in $\hat{X}^{[p]}$ se $c_1 \cap \dots \cap c_p \neq \emptyset$. Cioè: per ogni

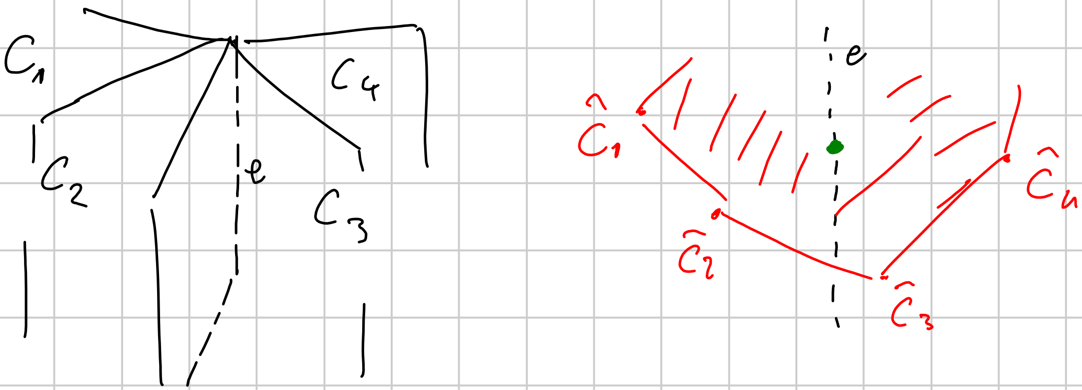
$d \in X^{[m]}$ ho un $\hat{d} \in \hat{X}^{[n-m]}$ che ha vertici $\hat{c}_1, \dots, \hat{c}_p$ se $c_1, \dots, c_p$ sono tutte le $c \in X^{[n]}$ che contengono d.

Fatto: $\hat{X}$ è un complesso politopale e $|\hat{X}| = M$.

$n=2$



$n=3$



Inoltre d e $\hat{d}$ si intersecano trasversalmente in un punto interno ad entrambi.

Definisco: $\varphi_k : C_k(\widehat{X}) \longrightarrow C^{n-k}(X)$

$\hat{d} \longmapsto \bar{d}$

duale topologica

duale algebrico

E' isomorfismo. Affermo che

$$\delta_{n-k} \circ \varphi_k = \varphi_{k-1} \circ \partial_k$$

da cui segue che $H_*(M) = H^{n-*}(M)$.

Se M è orientabile, ed è fissata una orientazione per ogni $d \in X^{[m]}$, oriento $\hat{d}$ in modo che $d \cap \hat{d}$ sia intersezione positiva (in M)

se $m=0$, $d>0 \to$ oriento $\hat{d}$ come M

se $m=n$, oriento d come M $\to$ oriento $d>0$

altrimenti: (base pos di $T d$), (base pos di $T\hat{d}$) =

= base pos di TM

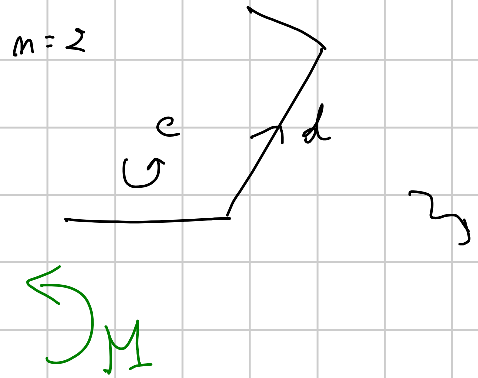
$$\varphi_k : C_k(\hat{X}) \to C^{n-k}(X)$$
$$\hat{d} \longmapsto \bar{d}$$

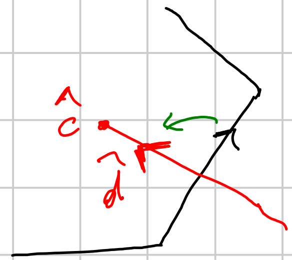
$$\delta_{n-k}(\varphi_k(\hat{d}))(c) = \bar{d}(\partial_{n-k+1} c) = \begin{cases} \pm 1 & \text{se } \pm d \subset \partial c \\ 0 & \text{altrimenti} \end{cases}$$

$$(\varphi_{k-1} \partial_k(\hat{d}))(c) = \qquad = \begin{cases} \pm 1 & \text{se } \hat{c} \subset \partial \hat{d} \\ 0 & \text{altrimenti} \end{cases}$$

Ora la dualità inverte le relazione di inclusione e lo fa in modo coerente con l'orientazione

$m = 2$

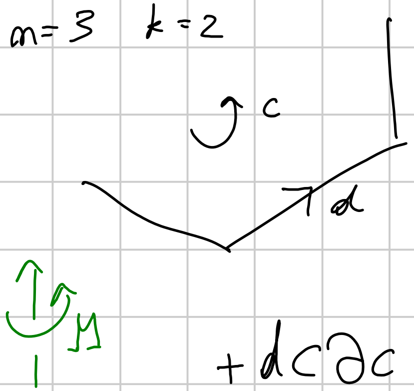




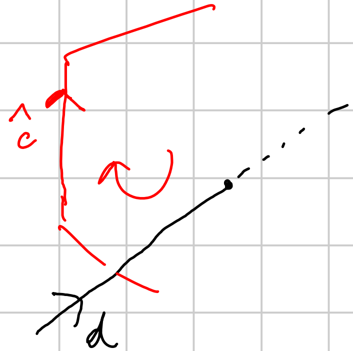
$+\ d \subset \partial c$

$+\ \hat{c} \subset \partial \hat{d}$

$m = 3 \quad k = 2$



$+\ d \subset \partial c$



$+\ \hat{c} \subset \partial \hat{d}$

$\blacksquare$

Prodotto cap : $\quad (k \geq \ell)$

$$\cap : C_k(X) \times C^\ell(X) \to C_{k-\ell}(X)$$

$$[v_0, \dots, v_k] \cap \varphi = \varphi([v_0, \dots, v_\ell]) \cdot [v_\ell, \dots, v_k]$$

Prop: $\partial_{k-\ell}(\sigma \cap \varphi) = (-1)^\ell \left((\partial_k \sigma) \cap \varphi - \sigma \cap (\delta_\ell \varphi) \right)$

Dim: Idea;

Scrivere $(\partial_k [v_0, \dots, v_k]) \cap \varphi$

$$= \sum \dots + \sum$$

aggiungere e togliere

$$(-1)^{\ell+1} \varphi([v_0, \dots, v_\ell]) \cdot [v_{\ell+1} \dots v_k]$$

Come sopra ne segue che è ben def

$\cap : H_k(X) \times H^\ell(X) \to H_{k-\ell}(X)$

$\cap : H_k(X,A) \times H^\ell(X) \to H_{k-\ell}(X,A)$

$\cap : H_k(X,A) \times H^\ell(X,A) \to H_{k-\ell}(A)$ _

"Funtorialità:" $f : X \to Y$

$$\begin{tikzcd} H_k(X) \times H^\ell(X) \arrow[r, "\cap"] \arrow[d, "f_*"'] & H_{k-\ell}(X) \arrow[d, "f"] \\ H_k(Y) \times H^\ell(Y) \arrow[u, shift left=10, "f^*"'] \arrow[r, "\cap"] & H_{k-\ell}(Y) \end{tikzcd}$$

$$f_*(\sigma \cap f^*\varphi) = (f_*\sigma) \cap \varphi$$ —

Forma generale della dualità di Poincaré:

__Teo:__ Se $M^{(n)}$ è una varietà chiusa orientata

$$\begin{aligned} H^k(M) &\longrightarrow H_{n-k}(M) \qquad (\text{su } \mathbb{Z}) \\ \varphi &\longmapsto \underset{\substack{\parallel \\ \text{il generatore canonico di } H_n(M)}}{[M]} \cap \varphi \end{aligned}$$

è un isomorfismo —