ETA 12/11/13

$j^{(p)} \circ i^{(p)} = \ell$

$C(L) \xrightarrow{\ell} C(K)$

Prop: $K = A_1 \cup A_2$, $L = A_1 \cap A_2$,

$$C(L) \xrightarrow{i^{(p)}} C(A_p) \xrightarrow{j^{(p)}} C(K).$$

$$\Rightarrow 0 \longrightarrow C(L) \xrightarrow[i]{(i^{(1)}, -i^{(2)})} C(A_1) \oplus C(A_2) \xrightarrow[j]{j^{(1)} + j^{(2)}} C(K) \to 0$$

è esatto.

Dim: i iniettiva: $i^{(1)}, i^{(2)}$ iniettive

verifico esattezza $0 \longrightarrow C_k(L) \to \dots \quad \forall k$

$\underline{\mathrm{Ker}}\, j \supset \underline{\mathrm{Im}\ i}$ cioè $j \circ \underline{i = 0}$:

$$(j \circ i)(\sigma) = \left(j^{(1)} + j^{(2)}\right)\left(i^{(1)}(\sigma), -i^{(2)}(\sigma)\right)$$

$$= \left(j^{(1)} \circ i^{(1)}\right)(\sigma) - \left(j^{(2)} \circ i^{(2)}\right)(\sigma)$$

$$= e(\sigma) - e(\sigma) = 0 \ \_$$

$\underline{\mathrm{Ker}}\, j \subset \underline{\mathrm{Im}\ i}$: $x = \left(\sum_{\sigma \in A_1^{[k]}} n_\sigma^{(1)} \cdot \sigma, \sum_{\sigma \in A_2^{[k]}} n_\sigma^{(2)} \cdot \sigma \right) \in \mathrm{Ker}\, j$

cioè $\sum_{\sigma \in A_1} n_\sigma^{(1)} \cdot \sigma + \sum_{\sigma \in A_2} n_\sigma^{(2)} \cdot \sigma = 0$ $\quad$ $\underline{\underline{\text{etc}}}$ (in $C(X)$)

$$\left(\underbrace{\sum_{\sigma\in A_1\setminus L} m^{(1)}_\sigma\cdot\sigma}_{m^{(1)}_\sigma=0\ \forall\sigma\in A_1\setminus L} + \underbrace{\sum_{\sigma\in L} m^{(1)}_\sigma\cdot\sigma\right) + \left(\sum_{\sigma\in L} m^{(2)}_\sigma\cdot\sigma}_{m^{(2)}_\sigma=-m^{(1)}_\sigma\ \forall\sigma\in L} + \underbrace{\sum_{\sigma\in A_2\setminus L} m^{(2)}_\sigma\cdot\sigma}_{m^{(2)}_\sigma=0\ \forall\sigma\in A_2\setminus L}\right)$$

$$\Rightarrow x = \left(\sum_{\sigma\in L} m^{(1)}_\sigma\cdot\sigma,\ -\sum_{\sigma\in L} m^{(1)}_\sigma\cdot\sigma\right)$$

$$\Rightarrow \quad x = \left(i^{(1)}, -i^{(2)} \right) \left(\sum_{\sigma \in L} n_\sigma^{(1)} \cdot \sigma \right)$$

j surjective :

$$C_k(K) \ni \sum_{\sigma \in K^{[k]}} n_\sigma \cdot \sigma$$

$$= \underbrace{\sum_{\sigma \in A_1^{[k]}} n_\sigma \cdot \sigma}_{= j^{(1)}(\ldots)} + \underbrace{\sum_{\sigma \in A_2^{[k]} \setminus A_1^{[k]}} n_\sigma \cdot \sigma}_{= j^{(2)}(\ldots)} . \qquad \square$$

Mayer-Vietoris: $\quad (H(C \oplus C') \cong H(C) \oplus H(C'))$

$$\cdots \longrightarrow H_n(L) \xrightarrow{i_*} H_n(A_1) \oplus H_n(A_2) \xrightarrow{j_*} H_n(K) \xrightarrow{d} H_{n-1}(L) \cdots$$

$$\begin{array}{ccccc} & & \underset{w}{C_n(A_1) \oplus C_n(A_2)} & \xrightarrow{j = j_n^{(1)} + j_n^{(2)}} & \underset{z}{C_n(K)} \longrightarrow 0 \\ & & \Big\downarrow (\partial_n^{A_1}, \partial_n^{A_2}) & & \\ 0 \longrightarrow \underset{u}{C_{n-1}(L)} & \xrightarrow{i = (i_{n-1}^{(1)}, -i_{n-1}^{(2)})} & C_{n-1}(A_1) \oplus C_{n-1}(A_2) & & \end{array}$$

$$z = \sum_{\sigma \in K^{[n]}}{}' m_\sigma \cdot \sigma$$

$$w = \Bigg(\underbrace{\sum_{\sigma \in A_1^{[n]} \setminus L^{[n]}}{}' m_\sigma \cdot \sigma + \sum_{\sigma \in L^{[n]}}{}' p_\sigma \cdot \sigma}_{w_1},$$

$$\underbrace{\sum_{\sigma \in A_2^{[n]} \setminus L^{[n]}}{}' m_\sigma \cdot \sigma + \Big(\sum_{\sigma \in L^{[n]}}{}' (m_\sigma - p_\sigma) \cdot \sigma \Big)}_{w_2} \Bigg)$$

Poiché z è un ciclo, calcolando $(\partial_n^{A_1}, \partial_n^{A_2})(w)$...

Se $\tau \in A_1^{[n-1]} \setminus L^{[n-1]}$ ho $\varepsilon(\sigma,\tau) \neq 0$ solo per

$$\sigma \in A_1^{[n]} \setminus L^{[n]}$$

ma allora il coeff di τ in $\partial_n z$

è uguale al coeff. di τ in $\partial_n^{A_1} w_1$.

Stesso! $\tau \in A_2^{[n-1]} \setminus L^{[n-2]} \Rightarrow$

il coeff di τ in $\partial_n z$ è uguale al

coeff. di τ in $\partial_n^{A_2} w_2$.

Morale: siccome $\partial_n z = 0$

ho $\partial_n w_1 \in C_{n-1}(L)$ anzi a $Z_{n-1}(L)$

$\partial_n w_2 \in C_{n-1}(L)$

anzi si ha $(\partial_n w_1, \partial_n w_2) = (i^{(1)}_{n-1}(u), i^{(2)}_{n-1}(u))$

e u non dipende dalla scelta dei pr.

Morale: $d : H_n(K) \to H_{n-1}(L)$

funziona così:

- Si prende $z = \sum n_\sigma \cdot \sigma$ $\quad [z] \in H_n(K)$;
- Si scrive $z = (w_1, w_2)$ $\quad w_1 \in C_n(A_1)$, $w_2 \in C_n(A_2)$
- $d([z]) = [\partial_n w_1]$

(automaticamente $\partial_n w_1 \in Z_{n-1}(L)$

(e non solo $\partial_n w_1 \in Z_{n-1}(A_1)$)).

<u>Esercizio 1</u>: Sia M una 3-varietà (anche ≥ 3)

$N = M \setminus (\mathrm{int}(\Delta_3))$ $\quad$ (Oss: $N \simeq M \setminus \{pt\}$)

Provare che $H_1(N) \cong H_1(M)$.

(Segue anche dal fatto che $N^{(2)} = M^{(2)}$

ma H_1 dipende solo dal 2-scheletro.)

via M-V: $\quad N = A_1 \quad \Delta_3 = A_2$

$L = A_1 \cap A_2 = \partial \Delta_3 = S^2$

$$\underset{0}{\underset{\parallel}{H_1(S^2)}} \longrightarrow H_1(N) \oplus \underset{0}{\underset{\parallel}{H_1(D^3)}} \longrightarrow H_1(M) \longrightarrow \underset{0}{\underset{\parallel}{\tilde{H}_0(S^2)}}$$

$\Longrightarrow$ ok.

Esercizio: M chiusa orientata, $N = M \setminus (\mathrm{int}(\Delta_3))$

$\Rightarrow H_2(N) \cong H_2(M)$. M-V

$$
\begin{tikzcd}
H_3(D^3) \oplus H_3(N) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_3(M) \arrow[r, "d"] \arrow[d, phantom, "\shortparallel"] & H_2(S^2) \arrow[r, "g"] \arrow[d, phantom, "\shortparallel"] & H_2(D^3) \oplus H_2(N) \arrow[r, "f"] \arrow[d, phantom, "\shortparallel"] & H_2(M) \arrow[d] \\
0 \oplus 0 & \mathbb{Z} & \mathbb{Z} & 0 & H_1(S^2) \arrow[d, phantom, "\shortparallel"] \\
 & & & & 0
\end{tikzcd}
$$

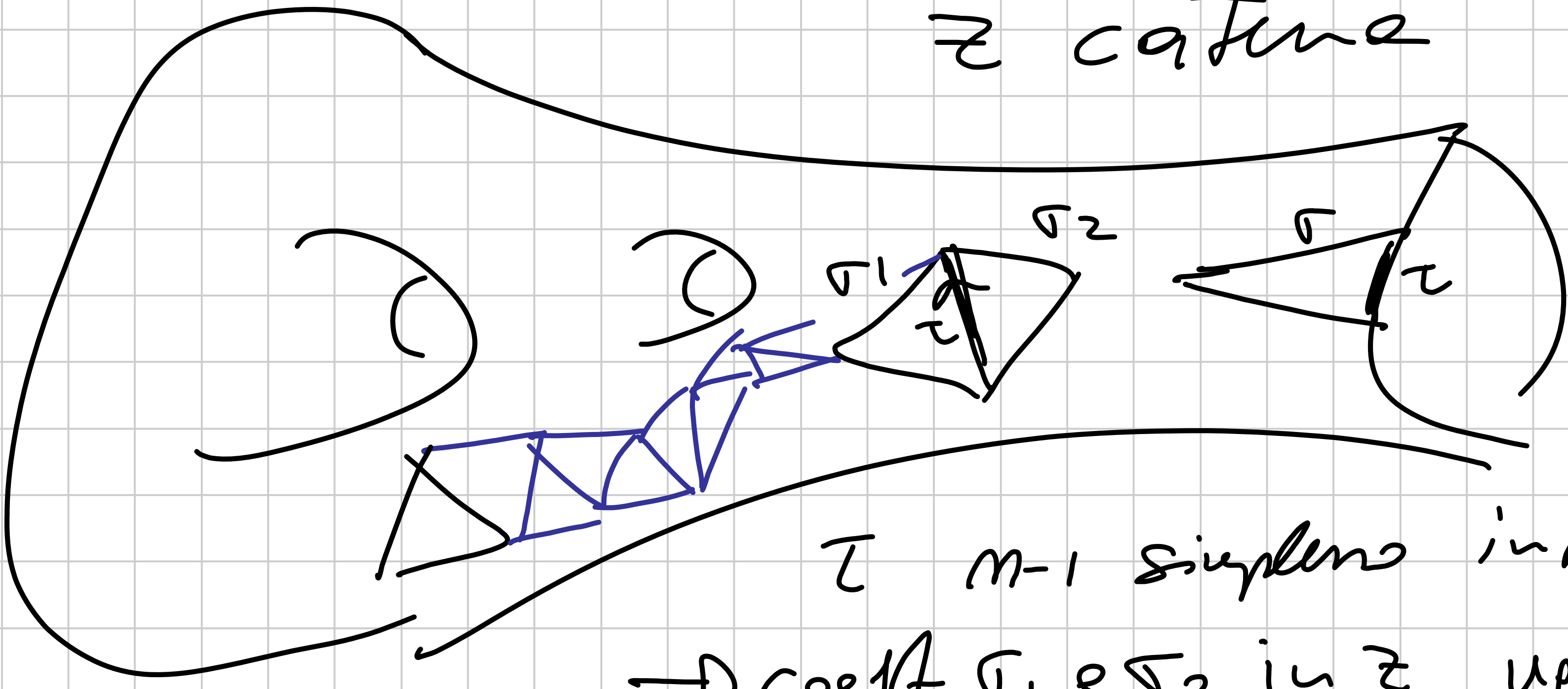
$\mathrm{Ker}\, f = \mathrm{Im}\, g$. Se provo che d è $\cong$

ho d suriettiva $\Rightarrow g = 0 \Rightarrow \mathrm{Ker}\, f = 0$.

(Sotto-es: $N^{(n)}$ compatta connessa, $\partial N \neq \emptyset$

$\Rightarrow H_n(N) = 0$

z catena



τ $n-1$ simplesso interno

$\Rightarrow$ coeff σ_1 e σ_2 in z uguali

τ $n-1$ simplesso di bordo

$\Rightarrow$ coeff σ in z è 0

$\Rightarrow z = 0$)

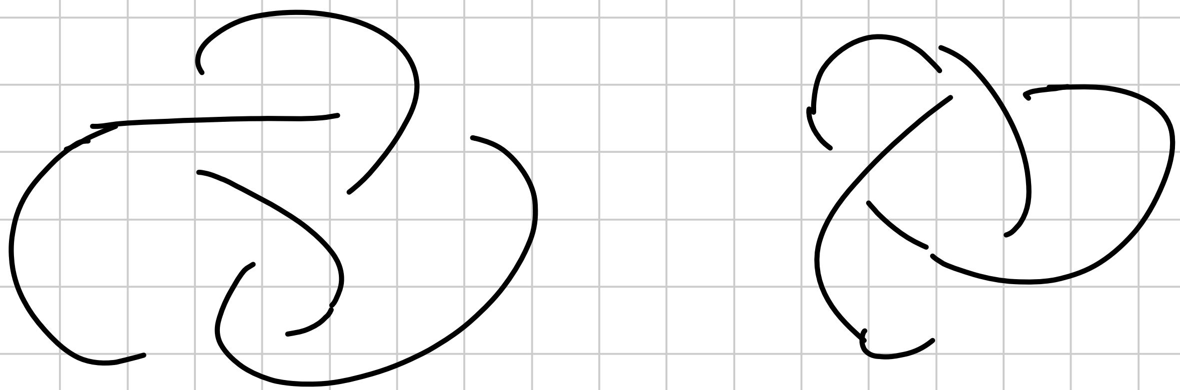
d isomorfismo: $d([M]) = \ldots$

$$\sum_{\sigma \in M} 1\,\sigma = \Delta_3 + \sum_{\sigma \neq \Delta_3} 1\,\sigma$$

e $d([M]) = [\partial \Delta_3]$ cioè

$$\begin{aligned} \mathbb{Z} &\xrightarrow{d} \mathbb{Z} \\ 1 &\longmapsto 1. \end{aligned}$$

Esercizio: Sia $K \subset S^3$ un nodo PL
(sottovarietà PL omeo a S^1) .



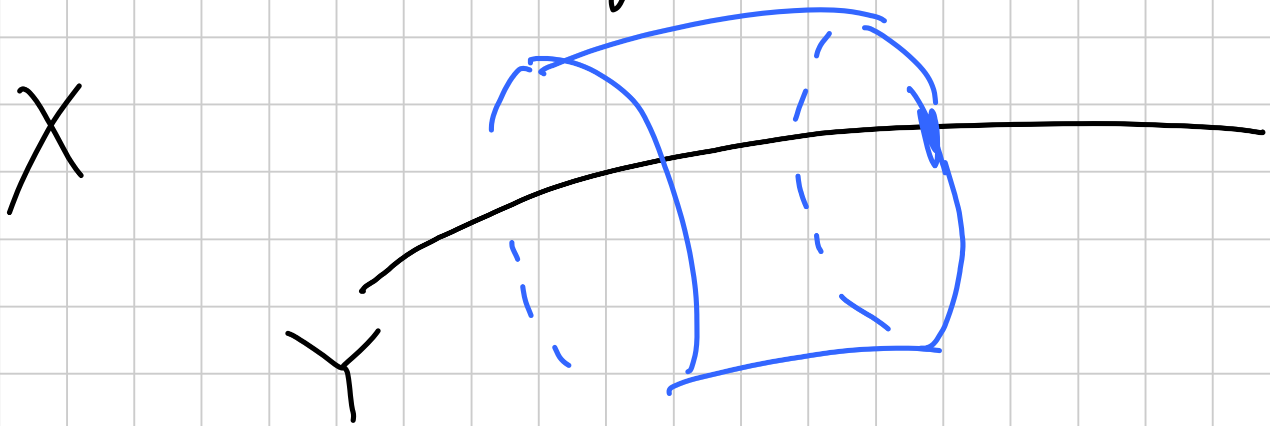
Fatto: esiste un $\underbrace{\text{intorno}}_{}^{U}$ di K in S^3
omeomorfo a $K \times D^2$ (U intorno regolare)

(In generale $Y^{(k)} \hookrightarrow X^{(n)}$ sottovarietà

cpt liscia o PL

$\Longrightarrow \exists$ intorno di Y in X che è

un fibrato in D^{n-k} su Y

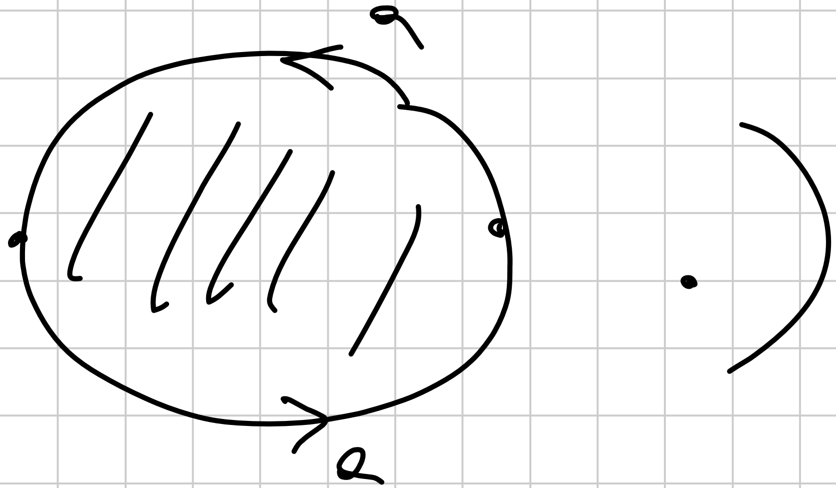


solo localmente

$Y \times D^{n-k}$.

Esempio: l'intorno regolare in $\mathbb{P}^2(\mathbb{R})$ di γ dove $[\gamma] \neq 0 \in \pi_1(\mathbb{P}^2\mathbb{R}) = \mathbb{Z}/2$

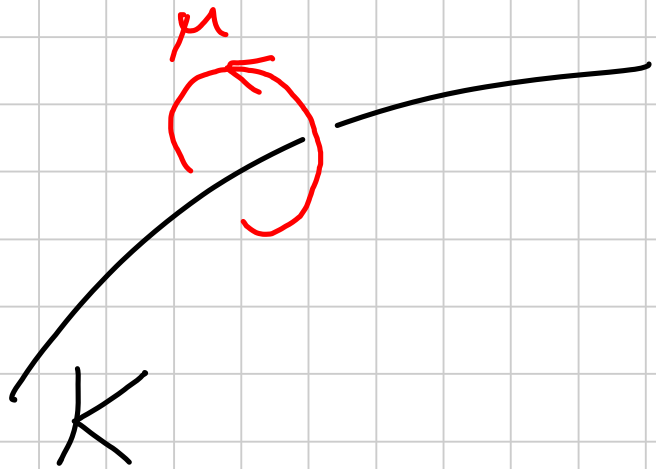
$\Rightarrow U$ è nastro di Möbius, non $S^1 \times D^1$



$K \subset S^3$ nodo ; $U \cong K \times D^2$ intorno regolare

$N = S^3 \setminus \mathring{U}$ (morally $N \simeq S^3 \setminus K$)

$\Rightarrow H_1(N) \cong \mathbb{Z}_\mu = \langle \mu \mid - \rangle$

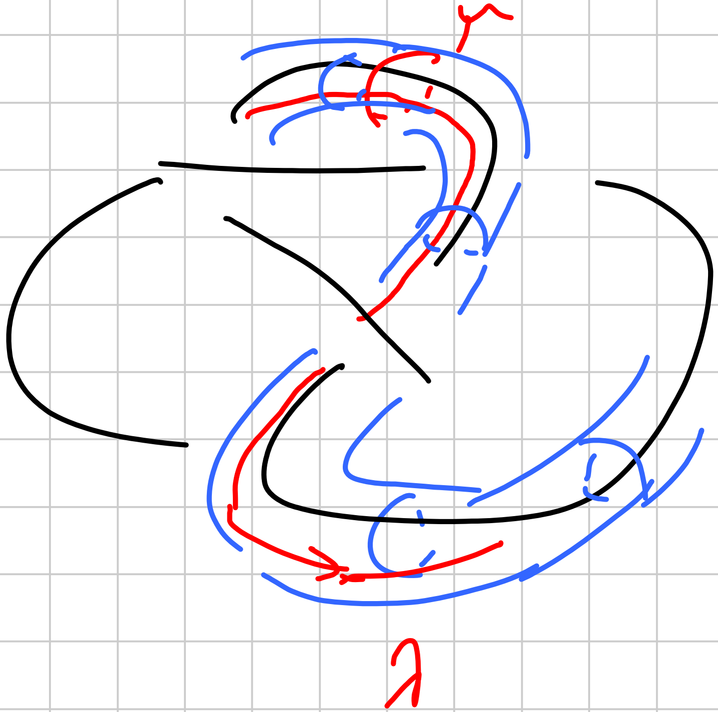


Uso M.-V.:

$A_1 = U$, $A_2 = N$, $K = S^3$, $L = \partial U \cong T$

$T \cong S^1 \times S^1$

$\Rightarrow H_1(T) = \mathbb{Z}_\mu \oplus \mathbb{Z}_\lambda$



<u>Oss</u>: λ definita in modo ambiguo: posso sostituirla con $\lambda + p \cdot \mu \quad \forall p \in \mathbb{Z}$

$$\begin{array}{ccccccc}
H_2(S^3) & \longrightarrow & H_1(\partial U) & \longrightarrow & H_1(U)\oplus H_1(N) & \longrightarrow & H_1(S^3) \\
\| & & \| & & \| & & \| \\
0 & & \mathbb{Z}_\mu \oplus \mathbb{Z}_\lambda & & \mathbb{Z}_\lambda & & 0
\end{array}$$

$$\Longrightarrow H_1(N) \simeq \mathbb{Z}_\mu .$$

Oss:

$$\begin{array}{ccc}
H_1(\partial U) & \longrightarrow & H_1(N) \\
\| & & \\
\mathbb{Z}_\mu \oplus \mathbb{Z}_\lambda & \longrightarrow & \mathbb{Z}_\mu
\end{array}$$

Allora esiste un ben preciso $p \in \mathbb{Z}$ t.c.

$p \cdot \mu + \lambda \in \mathrm{Ker}(i)$: chiamo tale longitudine quella privilegiata.

<u>Esercizio</u>: $N = S^3 \setminus \mathring{U}(K) \Longrightarrow H_2(N) = 0$.

$$\underset{\substack{\| \\ 0}}{H_3(U)} \oplus \underset{\substack{\| \\ 0}}{H_3(N)} \to \underset{\substack{\| \\ \mathbb{Z}}}{H_3(S^3)} \xrightarrow{\partial} \underset{\substack{\| \\ \mathbb{Z}}}{H_2(\partial U)} \to H_2(U) \oplus H_2(N)$$

$$H_2(U) \oplus H_2(N) \to H_2(S^3) = 0$$

Dico che d è isomorfismo ($\Rightarrow H_2(N) = 0$) –

(Come segue: $[S^3] = \left[\sum_\sigma \sigma\right] \xmapsto{d} \left[\partial\left(\sum_{\sigma \in U} \sigma\right)\right]$

$$= [\partial U]$$

dunque $\mathbb{Z} \xrightarrow{d} \mathbb{Z}$

$1 \longmapsto 1$.)

— o —

Proprietà (assiomatiche) dell'omologia (simpliciale).

E' un funtore covariante

$$\left(\begin{array}{l}\text{Coppie } (K,L) \\ (\text{c.s.f}, \text{sottocomplesso}); \\ \text{mappe di coppie}\end{array}\right) \rightsquigarrow \left(\begin{array}{l}\text{Successioni di} \\ \text{gruppi abeliani}; \\ \text{successioni di} \\ \text{omomorfismi}\end{array}\right)$$

Cioè $\forall (K,L)$ ho $\left(H_n(K,L) \right)_{n=0}^{+\infty}$

e $\forall f : (K, L) \to (A, B)$ cioè $f : K \to A$ e $f(L) \subset B$

ho $f_{*_n} : H_n(K, L) \to H_n(A, B)$

funtoriali _

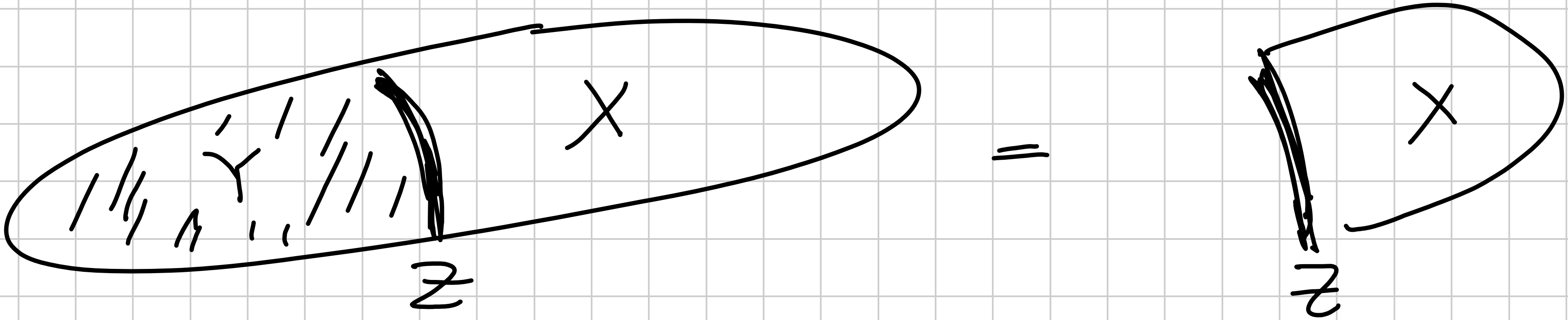
I. Omotopia : $f_0, f_1 : (K, L) \to (A, B)$

$f_0 \simeq_L f_1$ ($\exists f_t$ con $f_t(L) \subset B$ $\forall t$)

$$\Longrightarrow f_{0*} = f_{1*}$$

II. Escissione: $K = X \cup Y$, $Z = X \cap Y$

$$\Longrightarrow H_n(K, Y) = H_n(X, Z)$$



III. LES (long exact sequence) (K, L)

$$\dots \to H_n(L) \to H_n(K) \to H_n(K, L) \to H_{n-1}(L) \to \dots$$

(funtoriale)

IV. Dimensione $H_n(\{pt\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$

V. 0-omotopia: $H_0(X) = \mathbb{Z}^{\#\text{ comp. connesse di } X}$

(Non c'è Mayer-Vietoris)

<u>Oss</u>: Questi assiomi bastano a calcolare

$$H_n(D^p, S^{p-1}) \cong H_n(S^p)$$

(provare a verificare la dimostrazione fatta)