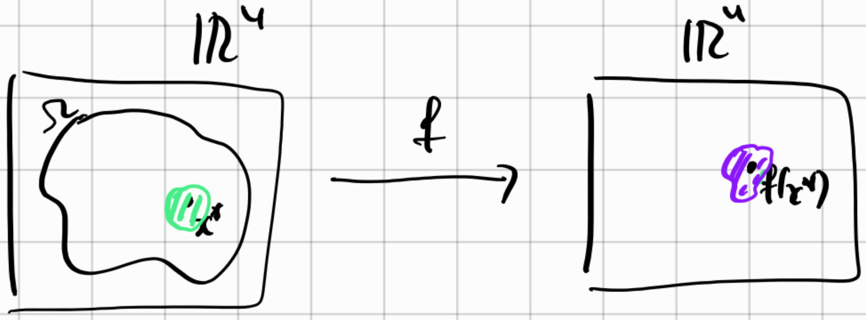
ANALISI MATEMATICA
LEZIONE 8 - 24.3.2023

INVERTIBILITÀ LOCALE

Teo $f:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}^n$ se in $\underline{x}^*\in\Omega$ $\det Df(x^*)\neq 0$
allora c'è un piccolo intorno di x^* in cui
f risulta essere invertibile. Cioè $\exists\,\rho>0$ t.c.

$f:\boxed{B_\rho(x^*)}\to\boxed{f(B_\rho(x^*))}$
è invertibile

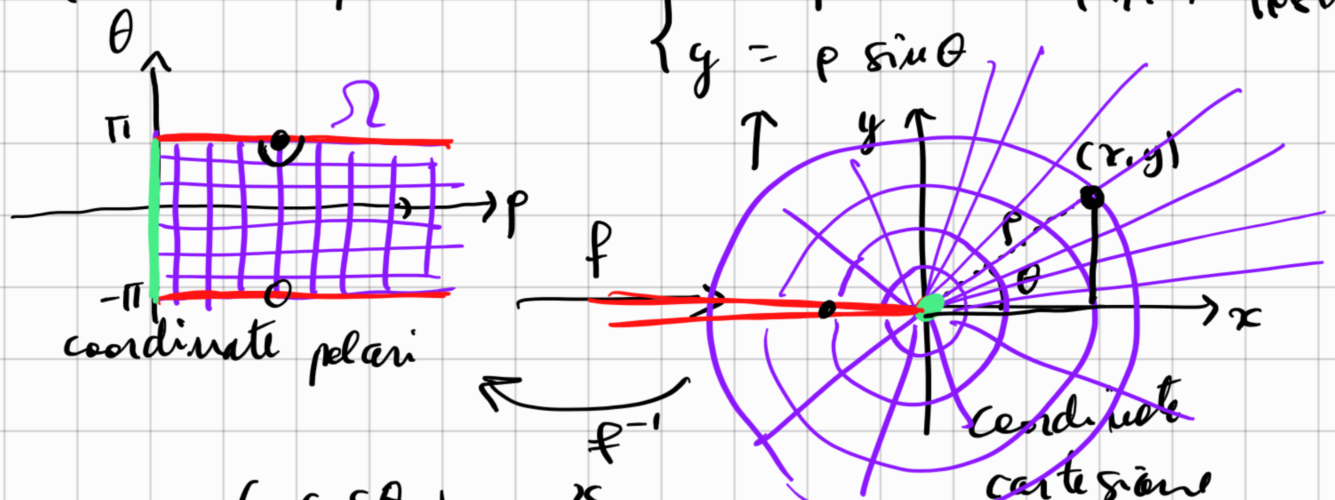
f^{-1}



Es. (coordinate polari)

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\begin{array}{c} \mathbb{R}^2 \\ \downarrow \\ f(\rho,\theta) \end{array} = \begin{array}{c} \mathbb{R}^2 \\ \downarrow \\ \begin{pmatrix}\rho\cos\theta \\ \rho\sin\theta\end{pmatrix} \end{array}$$



$$\begin{pmatrix}x\\y\end{pmatrix} = f(\rho,\theta) = \begin{pmatrix}\rho\cos\theta \\ \rho\sin\theta\end{pmatrix}\begin{array}{c} x \\ y\end{array}$$

$$Jf = Df = \begin{array}{cc} \begin{pmatrix}\cos\theta, & -\rho\sin\theta \\ \sin\theta, & \rho\cos\theta\end{pmatrix} & \begin{array}{c} x \\ y\end{array} \\ \quad\frac{\partial}{\partial\rho}\qquad\frac{\partial}{\partial\theta} & \end{array}$$

$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad-bc$$

$$\det Df = \cos\theta\,\rho\cos\theta - \sin\theta\cdot(-\rho\sin\theta)$$

$$= \rho\left(\cos^2\theta + \sin^2\theta\right) = \rho$$

- Se $\rho > 0$ f è invertibile in (θ, ρ). (LINEA VERDE)

- Se $\theta = \pm\pi$ f non è invertibile globalmente ma localmente sì. (LINEE ROSSE)

Come si inverte f? $\quad \binom{x}{y} = f(\rho, \theta) \quad \binom{\rho}{\theta} = f^{-1}(x, y)$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$ ← devo ricavare ρ e θ dal sistema

se $\cos\theta \neq 0$ $\rightsquigarrow$ $x \neq 0$

$$\begin{cases} \rho = \dfrac{x}{\cos\theta} \\ y = \dfrac{x}{\cos\theta} \cdot \sin\theta \end{cases} \quad \begin{cases} \text{———} \\ y = \operatorname{tg}\theta \cdot x \end{cases} \quad \begin{cases} \text{———} \\ \operatorname{tg}\theta = \dfrac{y}{x} \end{cases}$$

$$\begin{cases} \rho = \dfrac{x}{\cos(\operatorname{arctg}\frac{y}{x})} \\ \theta = \operatorname{arctg}\dfrac{y}{x} \end{cases}$$ ← si semplifica

$$x^2 + y^2 = \rho^2\cos^2\theta + \rho^2\sin^2\theta = \rho^2$$

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \operatorname{arctg}\dfrac{y}{x} \end{cases}$$

Se fosse $x = 0$ posso ricavare y: $\quad \sin\theta \neq 0$

$$\begin{cases} x = \dfrac{y}{\sin\theta} \cdot \cos\theta \\ \rho = \dfrac{y}{\sin\theta} \end{cases} \quad \begin{cases} \dfrac{x}{y} = \dfrac{\cos\theta}{\sin\theta} = \operatorname{cotg}\theta \\ \rho = \sqrt{x^2 + y^2} \end{cases}$$ ← come prima

$$\begin{cases} \theta = \operatorname{arccotg}\dfrac{x}{y} \\ \rho = \sqrt{x^2 + y^2} \end{cases}$$

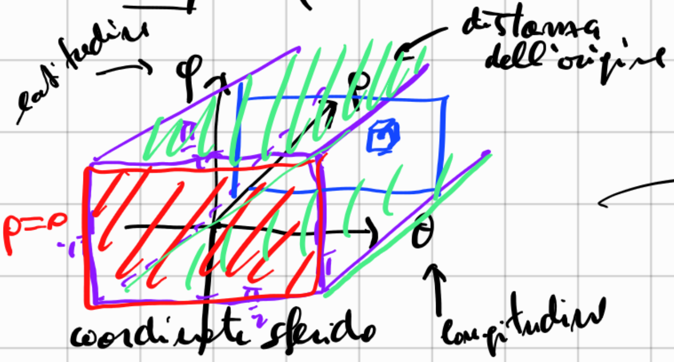
Se $x \neq 0$ ho la prima formula per f^{-1}!

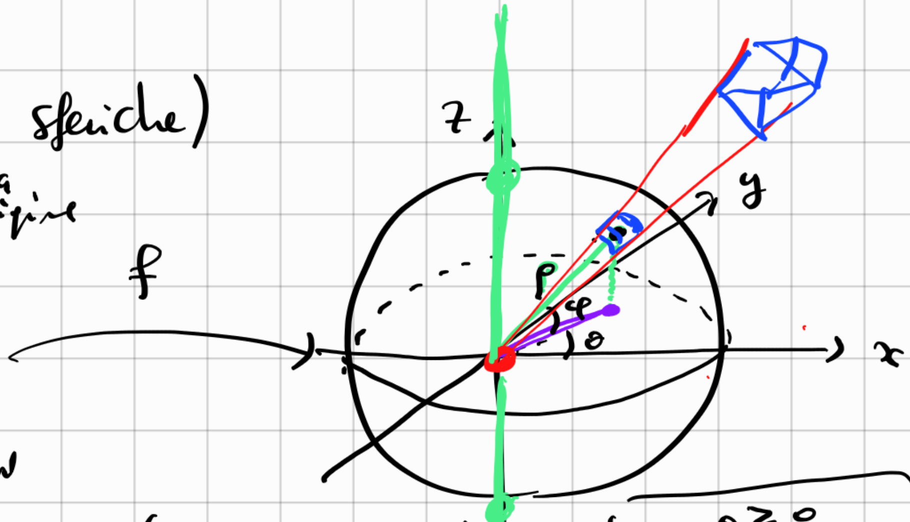
Se $y \neq 0$ ho la seconda formula

Se $x = 0$ e $y = 0$ non posso invertire f

$\hookrightarrow \rho = 0$

Esempio (coordinate sferiche)





$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = f(\rho, \theta, \varphi) = \begin{pmatrix} \rho \cos\theta \cos\varphi \\ \rho \sin\theta \cos\varphi \\ \rho \sin\varphi \end{pmatrix}$$

$$\boxed{\begin{array}{c} \rho \geq 0 \\ -\pi \leq \theta \leq \pi \\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \end{array}}$$

$$\begin{cases} x = \rho \cos\theta \cos\varphi \\ y = \rho \sin\theta \cos\varphi \\ z = \rho \sin\varphi \end{cases}$$

$$Df = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\varphi & -\rho\sin\theta\cos\varphi & -\rho\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\varphi & 0 & \rho\cos\varphi \end{pmatrix}$$

$$\det Df = \cos\theta\cos\varphi \cdot \det\begin{pmatrix} \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ 0 & \rho\cos\varphi \end{pmatrix}$$

$$- \sin\theta\cos\varphi \cdot \det\begin{pmatrix} -\rho\sin\theta\cos\varphi & -\rho\cos\theta\sin\varphi \\ 0 & \rho\cos\varphi \end{pmatrix}$$

$$\sin\varphi \cdot \det\begin{pmatrix} -\rho\sin\theta\cos\varphi & -\rho\cos\theta\sin\varphi \\ \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \end{pmatrix}$$

$$= \cos\theta\cos\varphi \cdot \rho^2\cos\theta\cos^2\varphi + \sin\theta\cos\varphi \cdot \rho^2\sin\theta\cos^2\varphi \; +$$

$$+ \sin\varphi \cdot \left(\rho^2\sin^2\theta\cos\varphi\sin\varphi + \rho^2\cos^2\theta\cos\varphi\sin\varphi\right)$$

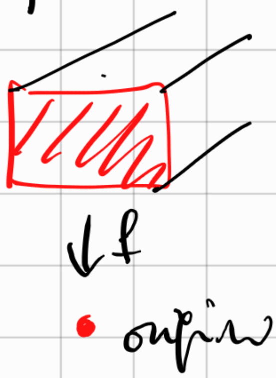
$$= \rho^2\left[\cos^2\theta\cos^3\varphi + \sin^2\theta\cos^3\varphi + \sin^2\theta\sin^2\varphi\cos\varphi + \cos^2\theta\sin^2\varphi\cos\varphi\right]$$

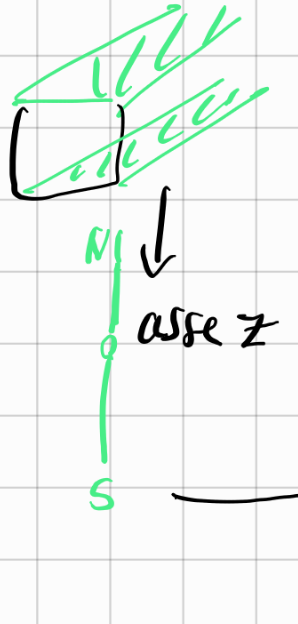
$$= \rho^2\left[\cos^3\varphi + \sin^2\varphi\cos\varphi\right] = \rho^2\cos\varphi\left[\cos^2\varphi + \sin^2\varphi\right]$$

$$= \rho^2 \cdot \cos\varphi.$$

$\det Df = 0$ se $\rho = 0$ o $\cos\varphi = 0$

In questi punti
f non è invertibile
neanche localmente





SUPERFICI [di dimensione 2 in uno spazio di dimensione 3]

Una superficie può essere rappresentata in diversi modi:

(1) forma implicita: $f : \mathbb{R}^3 \to \mathbb{R}$

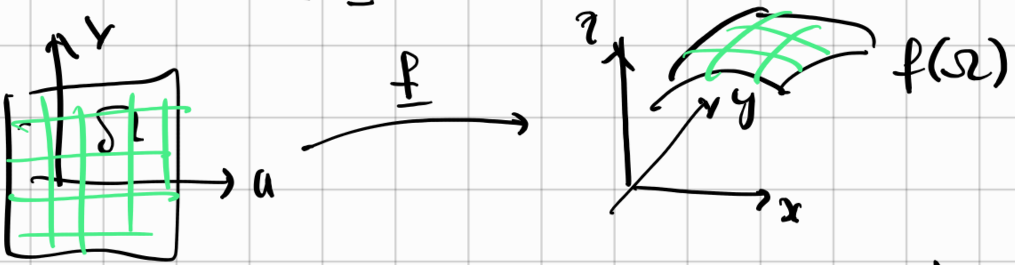
$f(x,y,z)=0$ ovvero $S=\{(x,y,z)\in\mathbb{R}^3 : f(x,y,z)=0\} = f^{-1}(\{0\})$

ES: $x^2+y^2+z^2-1=0$ è la superficie di una sfera di raggio $R=1$ centrata nell'origine.

insieme di livello 0

(2) forma parametrica: $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, $(u,v) \mapsto (x,y,z)$

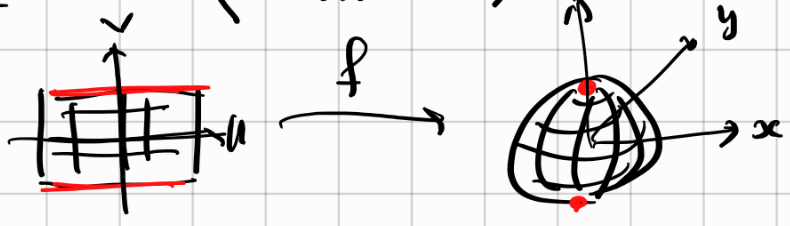
$$S=\{ \underline{f}(u,v) : (u,v)\in\Omega \} = \mathrm{Im}\, f = f(\Omega).$$



ES $u=\theta$

$v=\varphi$

$$\underline{f}(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix}$$

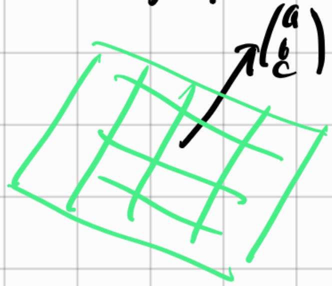


Nel caso lineare:

$$f(x,y,z) = a\cdot x + b\cdot y + c\cdot z - d = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} - d$$

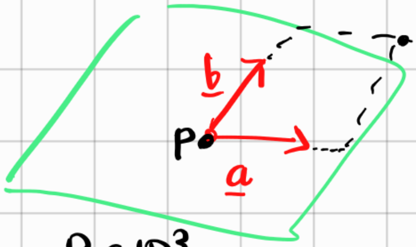
è l'equazione di un piano perpendicolare al vettore $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

FORMA IMPLICITA



FORMA PARAMETRICA

$$\{\underline{P} + u\cdot\underline{a} + v\cdot\underline{b} \ : \ u\in\mathbb{R},\ v\in\mathbb{R}\}$$



$\underline{P}\in\mathbb{R}^3$

$\underline{a},\underline{b}\in\mathbb{R}^3$

$$f(u,v) = \underline{P} + u\cdot\underline{a} + v\cdot\underline{b}$$

$$= \begin{pmatrix} P_x + u\cdot a_x + v\cdot b_x \\ P_y + u\cdot a_y + v\cdot b_y \\ P_z + u\cdot a_z + v\cdot b_z \end{pmatrix}$$

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

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(3) forma di grafico: $\qquad f : \Omega\subseteq\mathbb{R}^2 \longrightarrow \mathbb{R}$

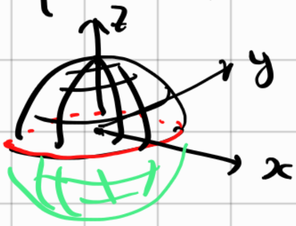
$$z = f(x,y) \quad \text{ovvero} \quad S = \{(x,y,z) : z = f(x,y)\}$$

Es :

$$z = \sqrt{1-x^2-y^2}$$

$$\Downarrow$$

$$(x^2+y^2+z^2=1)$$



$$z = -\sqrt{1-x^2-y^2}$$

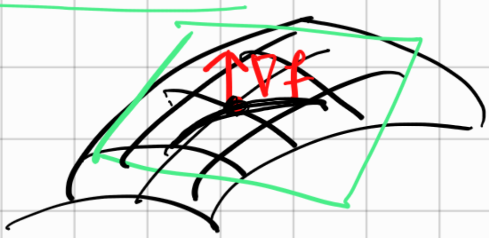
è un caso particolare di (2). $\qquad \underline{f}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$

$$S = \left\{(x,y,z) : \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases} \right\}$$

PIANO TANGENTE

① Forma implicita

$$S = \{f(x,y,z) = 0\}$$



∇f è perpendicolare agli insiemi di livello infatti se $\underline{\gamma}(t)$ è una qualunque curva lungo S

$$f(\underline{\gamma}(t)) = 0 \quad \Rightarrow \quad \frac{d}{dt} f(\underline{\gamma}(t)) = 0$$

$$\nabla f \cdot \underline{\gamma}' = 0$$

∇f è perpendicolare a S. $\Leftarrow$ ∇f è perpendicolare alla tangente $\underline{\gamma}'$ alla curva.

Il piano tangente a S nel punto (x^*, y^*, z^*)

è

$$\nabla f(x^*, y^*, z^*) \cdot \begin{pmatrix} x - x^* \\ y - y^* \\ z - z^* \end{pmatrix} = f(x^*, y^*, z^*) = 0$$

$$\nabla f(p^*) \cdot (p - p^*) = f(p^*) = 0$$

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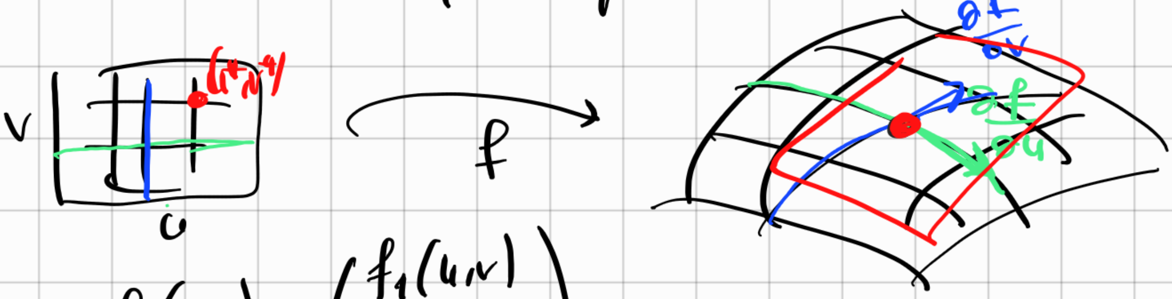
(2) Se S è in forma parametrica



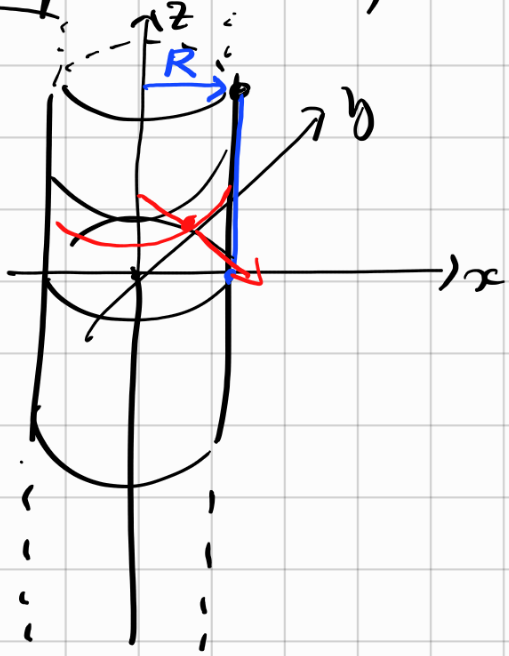
$$f(u,v) = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \\ f_3(u,v) \end{pmatrix}$$

$\dfrac{\partial f}{\partial u}$ e $\dfrac{\partial f}{\partial v}$ sono due vettori tangenti al piano S

se sono indipendenti ottengo il piano tangente prendendo tutte le loro combinazioni lineari:

$$(u,v) \longmapsto f(u^*, v^*) + u \cdot \frac{\partial f}{\partial u}(u^*, v^*) + v \cdot \frac{\partial f}{\partial v}(u^*, v^*)$$

Esempi (Cilindro)



① Forma implicita:

$$\sqrt{x^2+y^2} = R$$

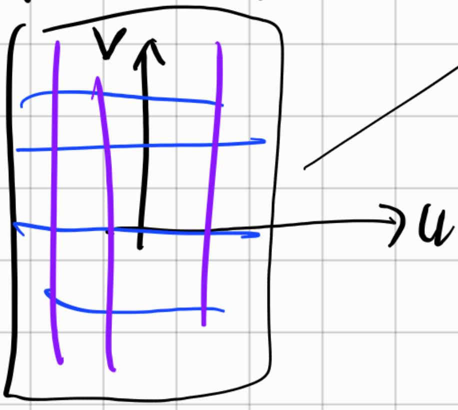
$$x^2+y^2-R^2=0$$

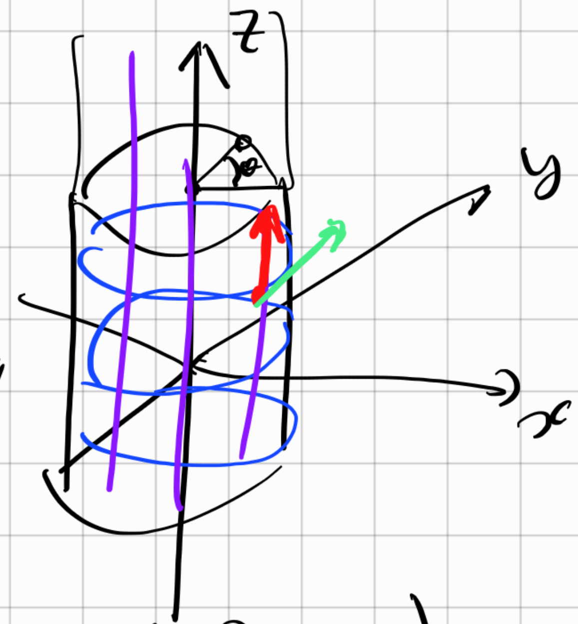
$$f(x,y,z) = x^2+y^2-R^2 = 0$$

$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$$ è perpendicolare al cilindro.

② Forma parametrica

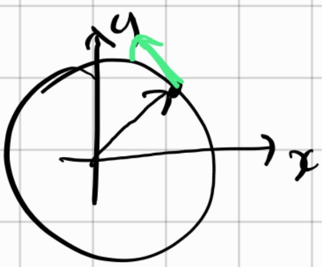
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\cos u \\ R\sin u \\ v \end{pmatrix}$$





$$\underline{f}(u,v) = \begin{pmatrix} R\cos u \\ R\sin u \\ v \end{pmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{pmatrix} -R\sin u \\ R\cos u \\ 0 \end{pmatrix} \qquad \frac{\partial f}{\partial v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



TEOREMA del DINI (FUNZIONE IMPLICITA)

Sia S una superficie data in forma parametrica

$$S=\{(x,y,z) : F(x,y,z)=0\} \qquad F:\mathbb{R}^3\to\mathbb{R}$$

Se in un punto $p^*=(x^*,y^*,z^*)$

si ha $\frac{\partial F}{\partial z}(p^*)\neq 0$ allora è possibile

trovare una funzione $f:\mathbb{R}^2\to\mathbb{R}$ definita in un intorno di (x^*,y^*) tale che

$$S=\{(x,y,z) : z=f(x,y)\}.$$

<u>Idea</u>: se f esiste si ha:

$$F(x,y,f(x,y))=0$$

Derivo rispetto a x:

$$\boxed{0=\frac{d}{dx}F(x,y,f(x,y))=\frac{\partial F}{\partial x}+\cancel{\frac{\partial F}{\partial y}\cdot 0}+\frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial x}}$$

$$0=\frac{d}{dy}F(x,y,f(x,y))=\cancel{\frac{\partial F}{\partial x}\cdot 0}+\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial y}$$

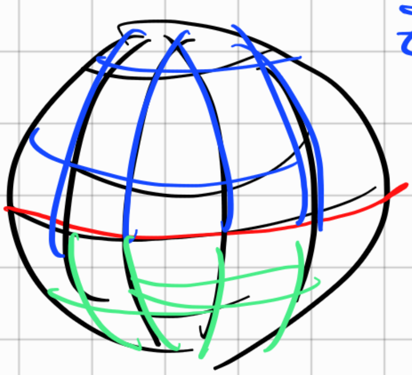
$$\begin{cases} \dfrac{\partial f}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \\[2em] \dfrac{\partial f}{\partial y} = \dfrac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \end{cases}$$

se $\dfrac{\partial F}{\partial z} \neq 0$ è possibile trovare f che soddisfa queste relazioni.

Esempio $S = \text{sfera} = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

$$F(x, y, z) = x^2 + y^2 + z^2 - 1 \leftarrow$$

$$\frac{\partial F}{\partial z} = 2z \neq 0 \qquad \text{se } z \neq 0$$



$$z = \sqrt{1 - x^2 - y^2} = f(x, y)$$

$$z = -\sqrt{1 - x^2 - y^2}$$

Inoltre dovrebbe essere $\dfrac{\partial f}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{-2x}{\sqrt{1 - x^2 - y^2}} = -\frac{x}{\sqrt{1 - x^2 - y^2}}$$

$$\frac{\partial F}{\partial x} = 2x \qquad \frac{\partial F}{\partial z} = 2z$$

$$-\frac{x}{\sqrt{1-x^2-y^2}} \stackrel{?}{=} -\frac{2x}{2z} = -\frac{x}{z}$$

$$x^2+y^2+z^2=1 \qquad x^2+y^2=1-z^2$$

$$\sqrt{1-x^2-y^2} = \sqrt{z^2} = z \quad \text{ok}$$