ANALISI MATEMATICA B
LEZIONE 67 - 22.3.2023

Formula di Stirling: $n! \sim \sqrt{2\pi n}\,\frac{n^n}{e^n}$

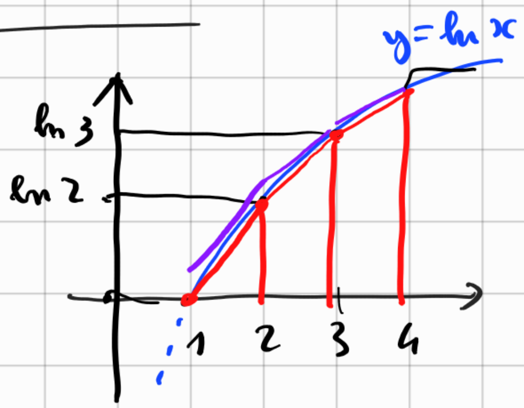
Passo 1: $n! \sim c\cdot\sqrt{n}\cdot\frac{n^n}{e^n}$

Prodotto di Wallis: $\frac{\pi}{2} \sim \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot\ldots\cdot(2n)(2n)}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\ldots(2n-1)(2n-1)(2n+1)}$

Fine: $c=\sqrt{2\pi}$.

Formula di Stirling.

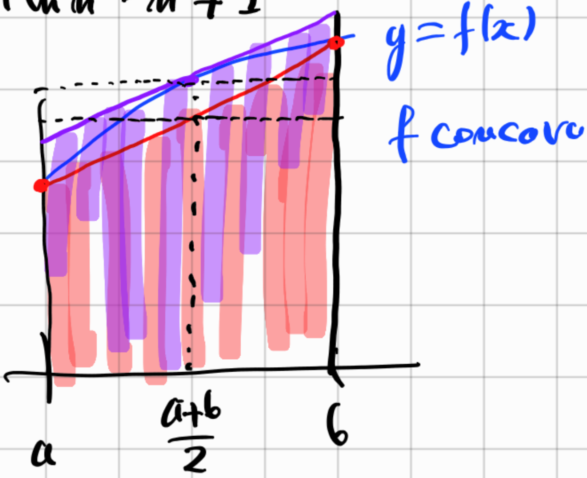
$$\ln(n!) = \sum_{k=1}^{n} \ln k$$



$$\int_1^n \ln x\,dx = \left[x\ln x - x\right]_1^n = n\ln n - n + 1$$

$$\parallel$$

$$\sum_{k=1}^{n-1}\int_k^{k+1}\ln x\,dx$$



In generale se f è concava su $[a,b]$

$$(b-a)\cdot\frac{f(a)+f(b)}{2} \le \int_a^b f(x)\,dx \le (b-a)\cdot f\left(\frac{a+b}{2}\right)$$

$$\underbrace{\frac{\ln(k)+\ln(k+1)}{2} \le \int_k^{k+1}\ln x\,dx \le \ln\left(k+\frac{1}{2}\right)}$$

$$\int_k^{k+1}\ln x\,dx = \frac{\ln(k)+\ln(k+1)}{2} + \varepsilon_k \quad \text{con}$$

$$0 \le \varepsilon_k \le \ln\left(k+\frac{1}{2}\right) - \frac{\ln k+\ln(k+1)}{2}$$

$$\varepsilon_k \le \ln\left(k+\frac{1}{2}\right) - \frac{\ln k + \ln(k+1)}{2} =$$

$$= \ln\left(k\cdot\left(1+\frac{1}{2k}\right)\right) - \frac{\ln k + \ln\left(k\cdot\left(1+\frac{1}{k}\right)\right)}{2}$$

$$= \cancel{\ln k} + \ln\left(1+\frac{1}{2k}\right) - \frac{\cancel{\ln k} + \cancel{\ln k} + \ln\left(1+\frac{1}{k}\right)}{2}$$

$$= \ln\left(1+\frac{1}{2k}\right) - \frac{1}{2}\ln\left(1+\frac{1}{k}\right)$$

$$= \cancel{\frac{1}{2k}} + O\left(\frac{1}{k^2}\right) - \frac{1}{2}\left[\cancel{\frac{1}{k}} + O\left(\frac{1}{k^2}\right)\right]$$

$$\boxed{\begin{aligned}\ln(1+x) &= x - \frac{x^2}{2} + o(x^2)\\ &= x + O(x^2)\end{aligned}}$$

$$= O\left(\frac{1}{k^2}\right)$$

$\sum_{k=1}^{+\infty} \varepsilon_k$ è convergente

Abbiamo visto che:

$$n\ln n - n + 1 = \int_1^n \ln x\,dx = \sum_{k=1}^{n-1}\int_k^{k+1} \ln x\,dx =$$

$$= \sum_{k=1}^{n-1}\left[\frac{\ln k + \ln(k+1)}{2} + \varepsilon_k\right]$$

$$= \frac{1}{2}\sum_{k=1}^{n-1}\ln k + \frac{1}{2}\sum_{k=2}^{n}\ln k + \sum_{k=1}^{n-1}\varepsilon_k$$

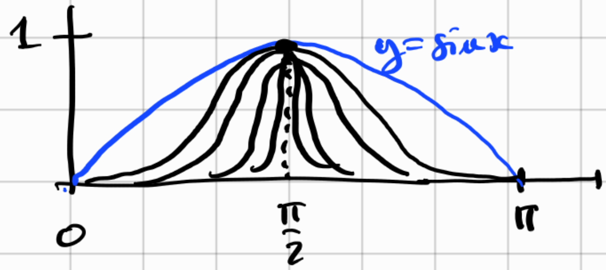
$$= \sum_{k=1}^{n}\ln k - \frac{1}{2}\ln n + \varepsilon_k = \ln n! - \frac{1}{2}\ln n + \sum_{k=1}^{n-1}\varepsilon_k$$

$$\ln n! = n\ln n - n + 1 + \frac{1}{2}\ln n - \sum_{k=1}^{n-1}\varepsilon_k$$

$$n! = \frac{e^{n\ln n} \cdot e^{1} \cdot e^{\frac{1}{2}\ln n}}{e^{n} \cdot e^{\sum_{k=1}^{n}\varepsilon_k}} \sim \frac{n^{n} \cdot e \cdot \sqrt{n}}{e^{n} \; e^{\sum^{\infty}\varepsilon_k}} = c\sqrt{n}\,\frac{n^{n}}{e^{n}}$$

ok!

Prodotto di Wallis



$$I_n = \int_0^{\pi} \underbrace{(\sin x)^n}\, dx$$

$$\sin^{n+1}(x) \leq \sin^{n}(x)$$

$0 \leq I_n$ è decrescente

$$I_0 = \int_0^{\pi} 1\, dx = \pi$$

$$I_1 = \int_0^{\pi} \sin x\, dx = \left[-\cos x\right]_0^{\pi} = 2$$

Cerco una formula ricorsiva:

$$I_{n+2} = \int_0^{\pi} \sin^{n}x \cdot \sin^{2}x\, dx = \int_0^{\pi} \sin^{n}x\,(1-\cos^{2}x)\, dx$$

$$= \int_0^{\pi} \sin^{n}x\, dx - \int_0^{\pi} \underbrace{\sin^{n}x \cdot \cos x} \cdot \cos x\, dx$$

PER PARTI

$$= I_n - \left\{ \cancel{\left[\frac{\sin^{n+1}x}{n+1} \cdot \cos x\right]_0^{\pi}} - \int_0^{\pi} \frac{\sin^{n+1}x}{n+1} \cdot (-\sin x)\, dx \right\}$$

$$= I_n - \frac{1}{n+1} \cdot I_{n+2}$$

$$I_n = I_{n+2} \cdot \left(1 + \frac{1}{n+1}\right) = I_{n+2} \cdot \frac{n+2}{n+1}$$

$$\boxed{I_{n+2} = \frac{n+1}{n+2} \cdot I_n} \quad (*)$$

Idea: $\frac{I_{n+1}}{I_n} \to 1$. ?

$$\frac{2n+1}{2n+2} \overset{(*)}{=} \frac{I_{2n+2}}{I_{2n}} \le \frac{I_{2n+1}}{I_{2n}} \le \frac{I_{2n}}{I_{2n}} = 1$$

(I_n decrescente)

$\downarrow$

1 per $n \to +\infty$ $\qquad \frac{I_{2n+1}}{I_{2n}} \to 1.$

$$I_3 = \frac{2}{3} I_1$$

$$I_5 = \frac{4}{5} I_3$$

$\vdots$

$$1 \leftarrow \frac{I_{2n+1}}{I_{2n}} \overset{(*)}{=} \frac{\frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots\cdots \frac{2}{3} \cdot I_1}{\frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \cdot I_0}$$

$$= \frac{(2n)^2 (2n-2)^2 (2n-4)^2 \cdots\cdots (4)^2 \cdot 2^2}{(2n+1)(2n-1)^2(2n-3)^2 \cdots\cdots 3^2 \cdot 1} \cdot \frac{2}{\pi}$$

$$\boxed{\frac{\pi}{2} \sim \frac{\left((2n)!!\right)^2}{(2n+1)\left((2n-1)!!\right)^2}}$$

WALLIS

$$(2n)!! = \underbrace{(2n)(2n-2)(2n-4)\cdots 2}_{n \text{ fattori}}$$

$$(2n+1)!! = \underbrace{(2n+1)(2n-1)\cdots\cdots 1}_{n \text{ fattori}}$$

METTIAMO INSIEME LE 2 COSE:

$$n! \sim c\sqrt{n} \cdot \frac{n^n}{e^n} \qquad \& \qquad \sqrt{\frac{\pi}{2}} \sim \frac{(2n)!!}{\sqrt{2n+1} \cdot (2n-1)!!}$$

$$(2n)!! = (2n)(2n-2)(2n-4)\cdots\cdots 4 \cdot 2 \overset{(*)}{=} 2^n \cdot n!$$

$$(2n+1)!! \overset{(*)}{=} \frac{(2n+1)!}{(2n)!!} = \frac{(2n+1)!}{2^n \cdot n!}$$

$$\sqrt{\frac{\pi}{2}} \sim \frac{(2n)!! \cdot \sqrt{2n+1}}{(2n+1)!!} \overset{(*)}{=} \frac{\left((2n)!!\right)^2 \cdot \sqrt{2n+1}}{(2n+1)!} \overset{(*)}{=} \frac{\left(2^n \cdot n!\right)^2 \sqrt{2n+1}}{(2n+1)!}$$

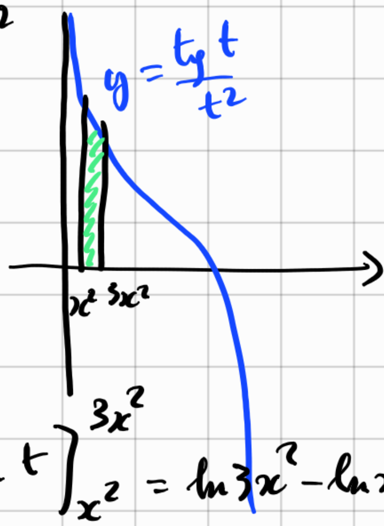
$$= \frac{(2^n \cdot n!)^2}{\sqrt{2n+1}\cdot(2n)!} \overset{\text{Stirling}}{=} \frac{1}{\sqrt{2n+1}} \cdot \frac{\cancel{2^{2n}}\left(c\sqrt{n}\,\frac{\cancel{n^n}}{\cancel{e^n}}\right)^2}{c\sqrt{2n}\,\frac{(\cancel{2}\cancel{n})^{2n}}{\cancel{e^{2n}}}} =$$

$$= \frac{c}{\sqrt{2n+1}}\,\frac{n}{\sqrt{2n}} = c\,\frac{n}{\sqrt{4n^2+2n}} \longrightarrow \frac{c}{2}$$

Dunque $c = 2\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}$.

$$n! = \sqrt{2\pi n}\cdot\frac{n^n}{e^n} \qquad \square$$

<u>Esercizio</u> (volta scorsa) $\displaystyle\lim_{x\to 0}\int_{x^2}^{3x^2}\frac{\operatorname{tg} t}{t^2}\,dt$



$\frac{\operatorname{tg} t}{t^2} \sim \frac{1}{t}$ per $t\to 0$

<u>Idea</u>: per $x\to 0$

$$\int_{x^2}^{3x^2}\frac{\operatorname{tg} t}{t^2}\,dt \sim \int_{x^2}^{3x^2}\frac{1}{t}\,dt = \Big[\ln t\Big]_{x^2}^{3x^2} = \ln 3x^2 - \ln x^2 = \ln 3.$$

<u>Teorema I</u> Se $a(x)\to a$, $b(x)\to a$, $f(x)\sim g(x)$ per $x\to x_0$.
(siamo su un intervallo) f,g continue.

$$\int_{a(x)}^{b(x)} f(t)\,dt \sim \int_{a(x)}^{b(x)} g(t)\,dt \qquad \text{per } x\to x_0$$

$F' = f, \quad G' = g.$

dim

$$\frac{\int_{a(x)}^{b(x)} f}{\int_{a(x)}^{b(x)} g} = \frac{[F]_{a(x)}^{b(x)}}{[G]_{a(x)}^{b(x)}} = \frac{F(b(x)) - F(a(x))}{G(b(x)) - G(a(x))}$$

(CAUCHY)

$$= \frac{F'(c(x))}{G'(c(x))} = \frac{f(c(x))}{g(c(x))} \xrightarrow{f \sim g} 1 \quad \text{per } x \to x_0$$

$\square$

$$c(x) \in [a(x), b(x)]$$
$$\downarrow \quad \swarrow$$
$$a \qquad a$$

Esempio $\displaystyle\lim_{x\to 0^+} \frac{1}{x^4} \int_{\sin^2 x}^{\sin x} \frac{2 - t\sin t - 2\cos t}{e^t - 1}\, dt = (*)$

$$f(x) = \frac{2 - x\sin x - 2\cos x}{e^x - 1} = \frac{\cancel{2} - x\left(\cancel{x} - \frac{x^3}{6} + o(x^3)\right) - 2\left(\cancel{1} - \cancel{\frac{x^2}{2}} + \frac{x^4}{4!}\right)}{x + o(x)}$$

$$= \frac{\frac{x^4}{6} - \frac{x^4}{12} + o(x^4)}{x + o(x)} = \frac{x^3}{12} + o(x^3)$$

$$(*) \lim_{x\to 0^+} \frac{1}{x^4} \int_{x^2 + o(x^2)}^{x + o(x)} \left(\frac{x^3}{12} + o(x^3)\right) dx$$

Teoremino II

$$\boxed{\int_0^x o(t^\alpha)\, dt = o(x^{\alpha+1})}$$ se $o(t^\alpha)$ è continua.

$0/0$

Esercizi per casa:

① $\displaystyle\int_0^{+\infty} \frac{1}{(x^2-1)\ln^2 x}\,dx$ converge?

② $\displaystyle\int_2^{+\infty} \frac{\ln^2 x}{x^2-1}\,dx$ converge?

③ $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}-\ln x}{(x-\sin x)^p}\,dx$ per quali $p\in\mathbb{R}$ converge?