ANALISI MATEMATICA B
LEZIONE 61 - 8.3.2023

ESERCIZI di RISCALDAMENTO o per casa

$$\int \frac{x}{\sqrt{4-x^2}}\,dx \qquad \int \frac{1}{\sqrt{4-x^2}}\,dx \qquad \int \frac{1}{x\ln x}\,dx$$

$$\int \frac{x}{\sqrt{x^2+2x}}\,dx \qquad \int \frac{1}{\sqrt{x^2+2x}}\,dx \qquad \int x\sin(x^2)\,dx$$

$$\int x^3 \cdot e^{(x^2)}\,dx$$

INTEGRALI di FUNZIONI RAZIONALI

$\deg R < \deg Q$

$$\int \frac{P(x)}{Q(x)}$$

$$P(x) = Q(x)\cdot T(x) + R(x)$$

$$\frac{P(x)}{Q(x)} = T(x) + \frac{R(x)}{Q(x)}$$

ES 1 $\displaystyle\int \frac{x^3+1}{x-1}\,dx$

Se $\deg P \geq \deg Q$ faccio la divisione con resto

$$\frac{x^3+1}{x-1} = \frac{\cancel{x^3}+1-x^2(\cancel{x}-1)}{x-1} + x^2$$

$$= \frac{x^2+1}{x-1} + x^2 = \frac{\cancel{x^2}+1-x(\cancel{x}-1)}{x-1} + x + x^2$$

$$= \frac{x+1}{x-1} + x + x^2 = \frac{x-1+2}{x-1} + x + x^2$$

$$= \frac{2}{x-1} + \underbrace{1 + x + x^2}$$

$$\int \frac{x^3+1}{x-1}\,dx = \int \frac{2}{x-1}\,dx + \int (1+x+x^2)\,dx$$

$$= 2\ln|x-1| + x + \frac{x^2}{2} + \frac{x^3}{3}$$

$$\begin{array}{l|l} x^3+1 & x-1 \\ \hline x^3-x^2 & x^2+x+1 \\ \hline x^2+1 & \\ x^2-x & \\ \hline x+1 & \\ x-1 & \\ \hline 2 & \end{array}$$

ALGORITMO IN COLONNA

Es 2

$$\int \frac{x+1}{x^2-x}\,dx$$

trucco * moltiplico per $(x-1)$ e faccio $x \to 1$ $B=2$. $A=-1$.

$$\frac{x+1}{x^2-x} = \frac{x+1}{x(x-1)} \stackrel{?}{=} \frac{A}{x} + \frac{B}{x-1} \stackrel{?}{=} \mathrm{span}\left\{\frac{1}{x}, \frac{1}{x-1}\right\}$$

$$\frac{A(x-1)+Bx}{x\cdot(x-1)}$$

$x+1 \in \mathrm{span}\{(x-1), x\}$

$\mathbb{R}_{\leq 1}[x] = \mathrm{span}\{(x-1), x\}$

ha dim 2

$x-\lambda$, $x-\mu$ se $\lambda \neq \mu$ sono indipendenti

$$A(x-1)+Bx = x+1$$

$$(A+B)x - A = x+1$$

$$\begin{cases} A+B=1 \\ -A=1 \end{cases} \qquad \begin{cases} A=-1 \\ B=2 \end{cases}$$

$$\frac{x+1}{x^2-x} = -\frac{1}{x} + \frac{2}{x-1} \qquad \left[\text{verifica } \frac{-(x-1)+2x}{x(x-1)} = \frac{x+1}{x^2-x} \quad \text{ok}\right]$$

$$\int \frac{x+1}{x^2-x}\,dx = -\int \frac{1}{x}\,dx + 2\int \frac{1}{x-1}\,dx$$

$$= -\ln|x| + 2\ln|x-1|$$

$$= \ln\frac{(x-1)^2}{|x|}$$

Es 3

$$\int \frac{x-1}{x^2+2x+1}\,dx$$

*Trucco moltiplico per $(x+1)^2$, $x \to -1$ $B=-2$

$$\frac{x-1}{x^2+2x+1} = \frac{x-1}{(x+1)^2} = \frac{x-1}{(x+1)(x+1)} \stackrel{?}{=} \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

ovvero $\dfrac{Cx+D}{(x+1)^2}$ #

$$= \frac{A(x+1)+B}{(x+1)^2} \overset{!}{=} \frac{x-1}{(x+1)^2}$$

$$A(x+1)+B \overset{!}{=} x-1$$

$$\parallel$$

$$Ax + A + B = x - 1$$

$$\begin{cases} A = 1 \\ A+B = -1 \end{cases} \qquad \begin{cases} A = 1 \\ B = -2 \end{cases}$$

$$\frac{x-1}{(x+1)^2} = \frac{1}{(x+1)} - \frac{2}{(x+1)^2}$$

$$\int \frac{x-1}{(x+1)^2}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{2}{(x+1)^2}\,dx$$

$$= \ln|x+1| + \frac{2}{x+1}$$

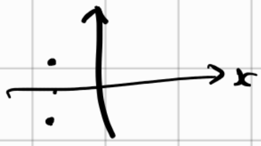
ES 4

$$\int \frac{x+2}{x^2+2x+3}\,dx$$

$$\frac{x+2}{x^2+2x+3} \overset{?}{=} \frac{A(x^2+2x+3)'}{x^2+2x+3} + \frac{B}{x^2+2x+3}$$

$$\frac{\Delta}{4} = 1-3 < 0$$

$$z_{12} = -1 \pm \sqrt{-2} = -1 \pm i\sqrt{2}$$



$$z_2 = \overline{z_1}$$

Fatto generale.

Se $P \in \mathbb{R}[x]$ se $P(z)=0$ anche $P(\bar{z})=0$.

$$P(z) = \sum a_k z^k$$

$$P(\bar{z}) = \sum a_k \bar{z}^k = \sum a_k \overline{z^k}$$

$$= \sum \overline{a_k z^k}$$

$$= \overline{\sum a_k z^k} = \overline{P(z)} = 0$$

$$\frac{x+2}{x^2+2x+3} = \frac{1}{2} \cdot \frac{2x+2}{x^2+2x+3} + \frac{1}{x^2+2x+3}$$

$$\int \frac{1}{2} \frac{2x+2}{x^2+2x+3} dx = \frac{1}{2} \cdot \ln(x^2+2x+3) = \ln\sqrt{x^2+2x+3}$$

$$\int \frac{1}{x^2+2x+3} dx \rightsquigarrow \int \frac{1}{y^2+1}$$

completamento del quadrato: $x^2+2x+3 = (x+1)^2 + 2$

raccolgo 2 $\quad (x+1)^2+2 = 2\left[\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1\right]$

$$y = \frac{x+1}{\sqrt{2}}$$

$$\int \frac{1}{x^2+2x+3} dx = \int \frac{1}{2\left[\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right]} dx = \frac{\sqrt{2}}{2} \operatorname{arctg} \frac{x+1}{\sqrt{2}}$$

$$\int \frac{x+2}{x^2+2x+3} dx = \ln\sqrt{x^2+2x+3} + \frac{\sqrt{2}}{2} \operatorname{arctg} \frac{x+1}{\sqrt{2}} \qquad \square$$

Teo (scomposizione di Hermite) P, Q polinomi, $\deg P < \deg Q$

Q monico

$$Q(x) = (x-\lambda_1)^{p_1} \cdot (x-\lambda_2)^{p_2} \cdots (x-\lambda_n)^{p_n} \cdot (x^2+\alpha_1 x+\beta_1)^{q_1} \cdots (x^2+\alpha_m x+\beta_m)^{q_m}$$

$\deg Q = p_1 + \dots + p_n + 2(q_1 + \dots + q_m)$

$$(z-\lambda)(z-\bar\lambda) = \underbrace{z^2 - \underbrace{(\lambda+\bar\lambda)}_{2\operatorname{Re}\lambda} z + \underbrace{\lambda \cdot \bar\lambda}_{|\lambda|^2}}_{\text{ha coefficienti reali}}$$

Allora

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{m} \frac{A_k}{x-\lambda_k} + \sum_{k=1}^{m} \frac{B_k x + C_k}{(x^2+\alpha_k x+\beta_k)} + \left[\frac{\tilde{P}(x)}{\tilde{Q}(x)}\right]'$$

$$\tilde{Q}(x) = \prod_{k=1}^{m}(x-\lambda_k)^{p_k-1} \prod_{k=1}^{m}(x^2+\alpha_k x+\beta_k)^{q_k-1}$$

con $\deg \tilde{P} < \deg \tilde{Q}$. D.

ES 3 avevamo: $\frac{x-1}{(x+1)^2} = \frac{1}{x+1} - \frac{2}{(x+1)^2}$

Hermite dice: $\frac{x-1}{(x+1)^2} = \frac{A}{x+1} + \left(\frac{B}{x+1}\right)'$

ES 5 $\int \frac{1}{(1+x^2)^2}\,dx$

$$\frac{1}{(1+x^2)^2} \overset{?}{=} \boxed{\frac{Ax+B}{1+x^2} + \left(\frac{Cx+D}{1+x^2}\right)'}$$

$$= \frac{Ax+B}{1+x^2} + \frac{C(1+x^2)-(Cx+D)\cdot 2x}{(1+x^2)^2}$$

$$= \frac{(Ax+B)(1+x^2) + C + Cx^2 - 2Cx^2 - 2Dx}{(1+x^2)^2}$$

$$Ax^3 + Bx^2 + Ax + B + C + Cx^2 - 2Cx^2 - 2Dx = 1$$

$$\begin{cases} A = 0 \\ B - C = 0 \\ A - 2D = 0 \\ B + C = 1 \end{cases} \qquad \begin{cases} A = 0 \\ D = 0 \\ B = C \\ B = \frac{1}{2} \end{cases} \qquad \begin{cases} A = 0 \\ D = 0 \\ B = \frac{1}{2} \\ C = \frac{1}{2} \end{cases}$$

$$\frac{1}{(1+x^2)^2} = \boxed{\frac{\frac{1}{2}}{1+x^2} + \left(\frac{\frac{1}{2}x}{1+x^2}\right)'}$$

$$\int \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\operatorname{arctg} x + \frac{1}{2}\frac{x}{1+x^2}$$

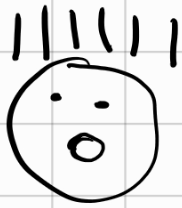
VERIFICA: $$D(\ ) = \frac{1}{2}\frac{1}{1+x^2} + \frac{1}{2}\frac{(1+x^2)-x\cdot 2x}{(1+x^2)^2}$$

$$= \frac{1}{2}\frac{1}{1+x^2} + \frac{1}{2}\frac{1-x^2}{(1+x^2)^2}$$

$$= \frac{1}{2}\frac{1+x^2+1-x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2} \quad \square$$

sugli appunti:

$$\int \frac{x^{10}+x^8-x^6+1}{x^8-x^7+2x^6-2x^5+x^4-x^3}\,dx$$



idea della dimostrazione

$$u_\lambda = \frac{1}{2-\lambda}$$ ← sono indipendenti

$$u_\lambda^p = \left(\frac{1}{2-\lambda}\right)^p$$

$$D : \operatorname{span}\{u_\lambda\} \to \operatorname{span}\{u_\lambda^2\}$$