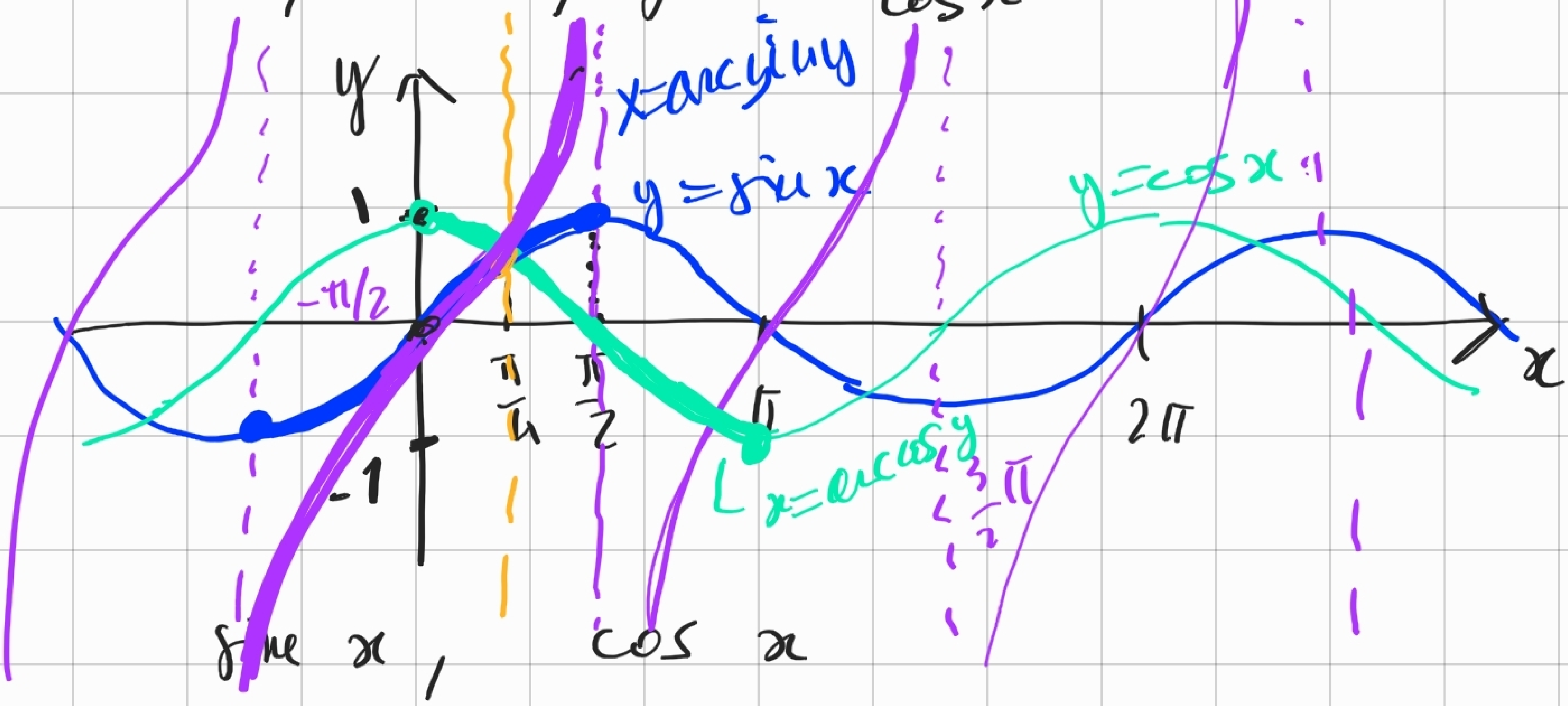
ANALISI MATEMATICA B
LEZIONE 42 – 20.1.2021

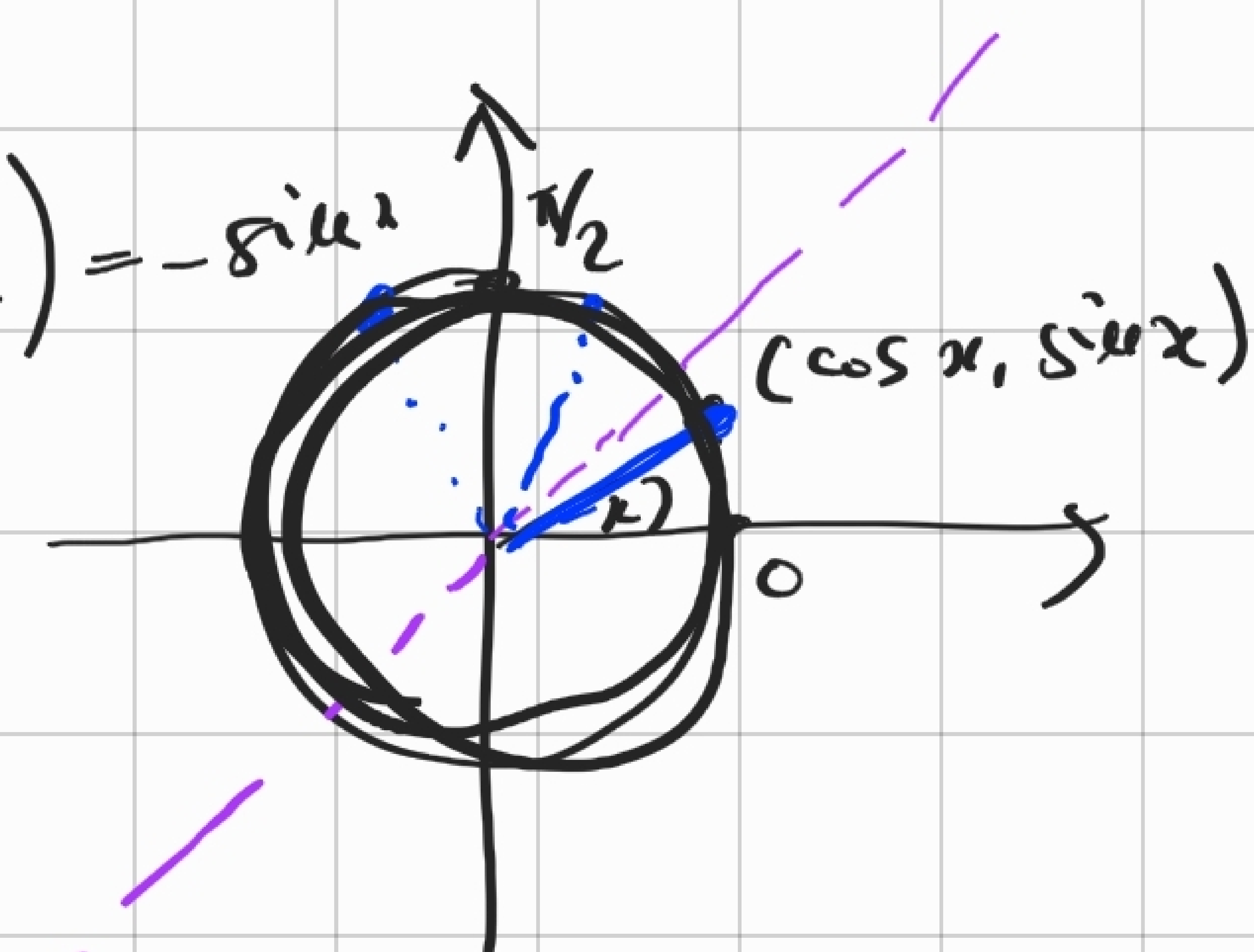
$\sin x$, $\cos x$, $\operatorname{tg} x = \dfrac{\sin x}{\cos x}$



$\sin x$, $\cos x$

$$\sin\left(\frac{\pi}{2}-x\right) = \cos x \qquad \cos\left(\frac{\pi}{2}-x\right) = \sin x$$

$$\sin\left(\frac{\pi}{2}+x\right) = \cos x, \quad \cos\left(\frac{\pi}{2}+x\right) = -\sin x$$



$\sin(\pi - x) \qquad \cos(\pi - x)$

$\pi + x \qquad$ – –

$\frac{3}{2}\pi \pm x \qquad$ – – –

$2\pi - x \qquad$ – –

$$\sin(2\pi n + x) = \sin x$$
$$\cos(2\pi n + x) = \cos x$$

Funzioni trigonometriche inverse

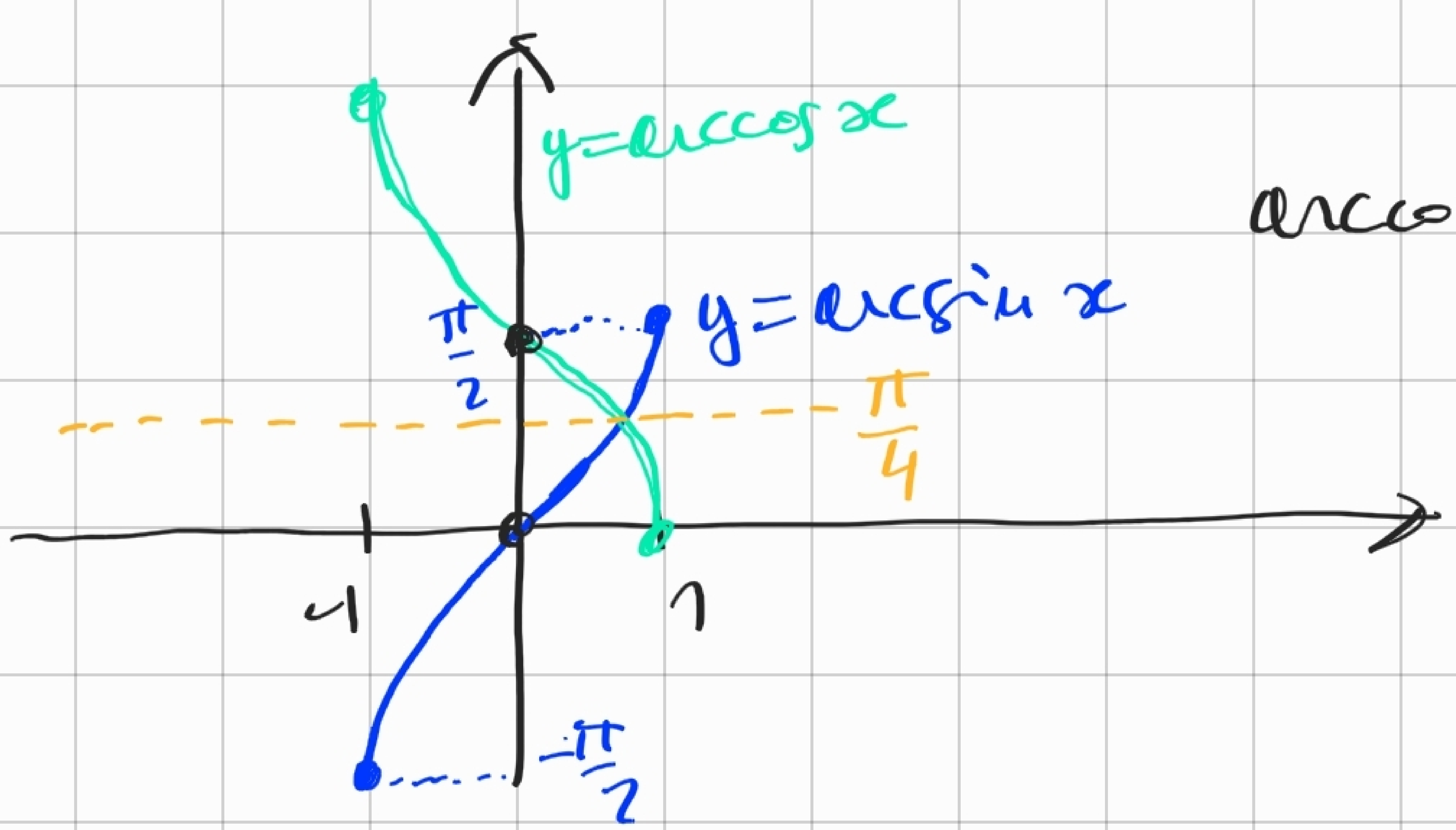
$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ è bigettiva

$\cos : [0, \pi] \longrightarrow [-1, 1]$ è bigettiva

Le inverse si chiamano:

$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arccos : [-1, 1] \longrightarrow [0, \pi]$



$\arccos x = \frac{\pi}{2} - \arcsin x$

$\cos \updownarrow$

$x = x$

$$\cos(\arccos x) = x \qquad \forall x \in [-1,1]$$

$$\sin(\arcsin x) = x \qquad \forall x \in [-1,1]$$

<u>Attenzione</u>

$$f(x) = \arcsin(\sin x)$$

$$f(x) = x \quad \text{se} \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

<u>Esercizio</u> $\boxed{\triangle}$ Disegnare il grafico della

funzione $f(x) = \arcsin(\sin x)$
$g(x) = \arccos(\cos x)$.

$$\operatorname{tg} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

è strettamente crescente (verificare!)
(dispari) $\Rightarrow$ iniettiva.

$$\lim_{x \to \frac{\pi}{2}^-} \operatorname{tg} x = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = +\infty$$

$$\lim_{x \to \frac{\pi}{2}^+} \operatorname{tg} x = -\infty$$

$\operatorname{tg} x$ è continua.

Per il teorema dei valori intermedi

$$\operatorname{tg} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$$

è suriettiva. Infatti:

$$\forall y \in \mathbb{R} \quad \exists ?\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) : \operatorname{tg} x = y$$

Se $y > 0$

$\exists x_0 : \operatorname{tg} x_0 > y$

perchè

$$\lim_{x \to \frac{\pi}{2}^-} \operatorname{tg} x = +\infty$$

Per il teorema dei valori intermedi

$\exists\, x \in [0, x_0]$ t.c. $\operatorname{tg} x = y$

$\operatorname{tg} 0 = 0 < y$

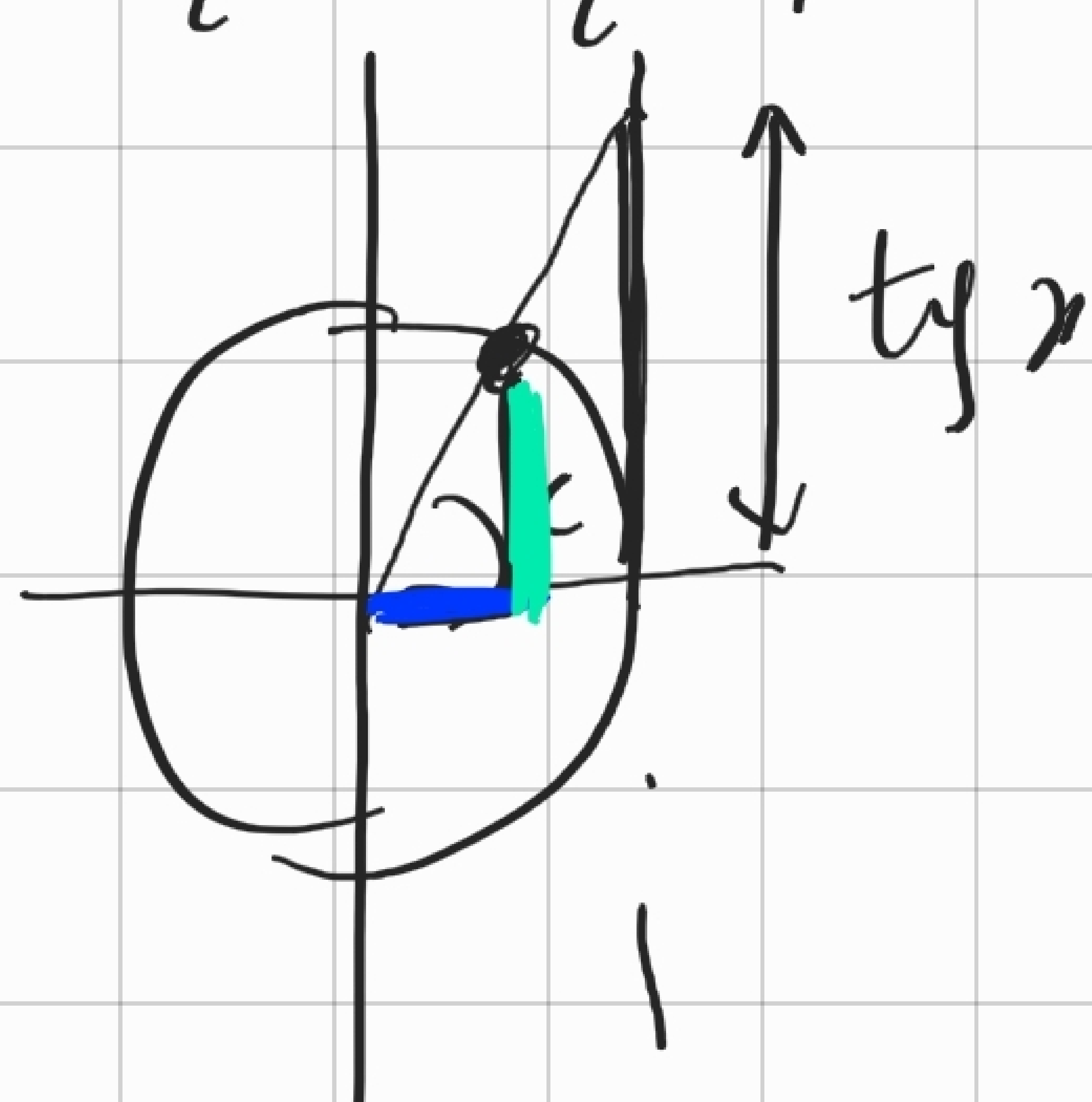
$\operatorname{tg} x_0 > y$.

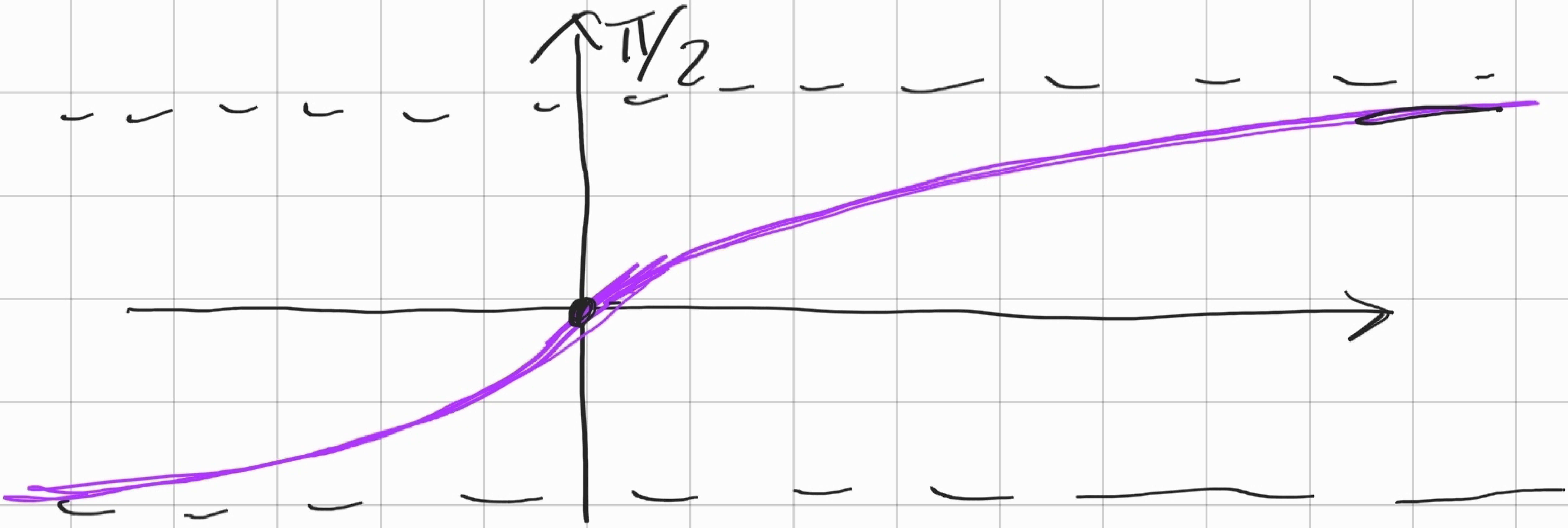
$\operatorname{tg} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

è biiettiva.

la funzione inversa si chiama

$\operatorname{arctg} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.





$$\lim_{x\to+\infty} \operatorname{arctg} x = \frac{\pi}{2}$$

$$\lim_{x\to+\infty} \left(\frac{\pi}{2} - \operatorname{arctg} x\right)\cdot x = ?$$

se $x>0$

$$\boxed{\frac{\pi}{2} - \operatorname{arctg} x = \operatorname{arctg} \frac{1}{x}}$$

$y = mx$

$\uparrow$

$m = \operatorname{tg} \alpha$

$\alpha = \operatorname{arctg} m$

$$\left(\frac{\pi}{2} - \operatorname{arctg} x\right) \cdot x = \frac{\operatorname{arctg} \frac{1}{x}}{\frac{1}{x}} \longrightarrow 1$$

$$\boxed{\lim_{t \to 0} \frac{\operatorname{arctg} t}{t} = 1}$$

$$\boxed{\frac{\sin x}{x} \to 1} \quad \text{pour } x \to 0$$

$$\frac{\operatorname{tg} x}{x} \to 1 \quad \text{pour } x \to 0$$

$$\frac{y}{\operatorname{arctg} y} \to 1 \quad \text{pour } x \to 0$$

FUNZIONI IPERBOLICHE

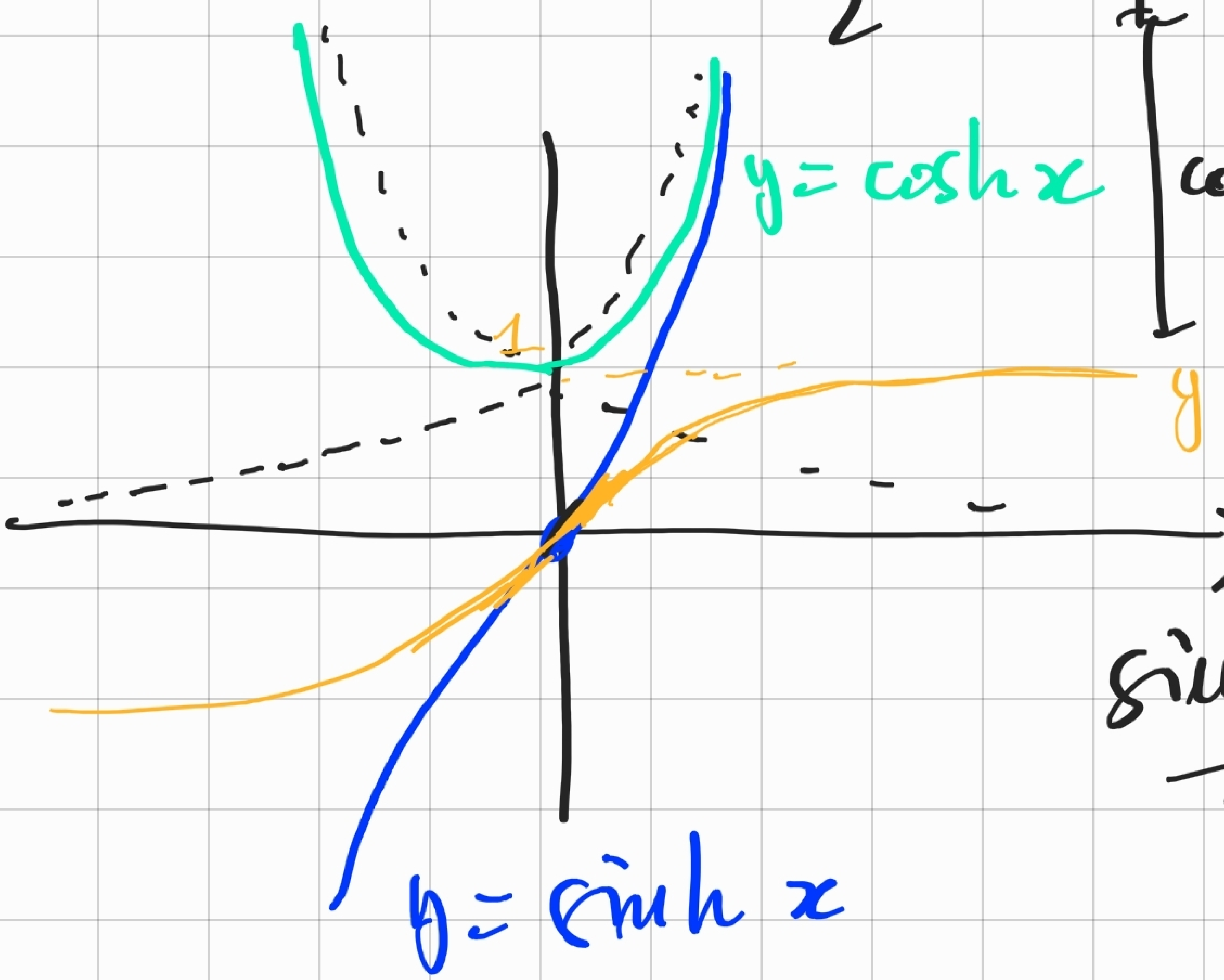
$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\cos^2 x + \sin^2 x = 1$$



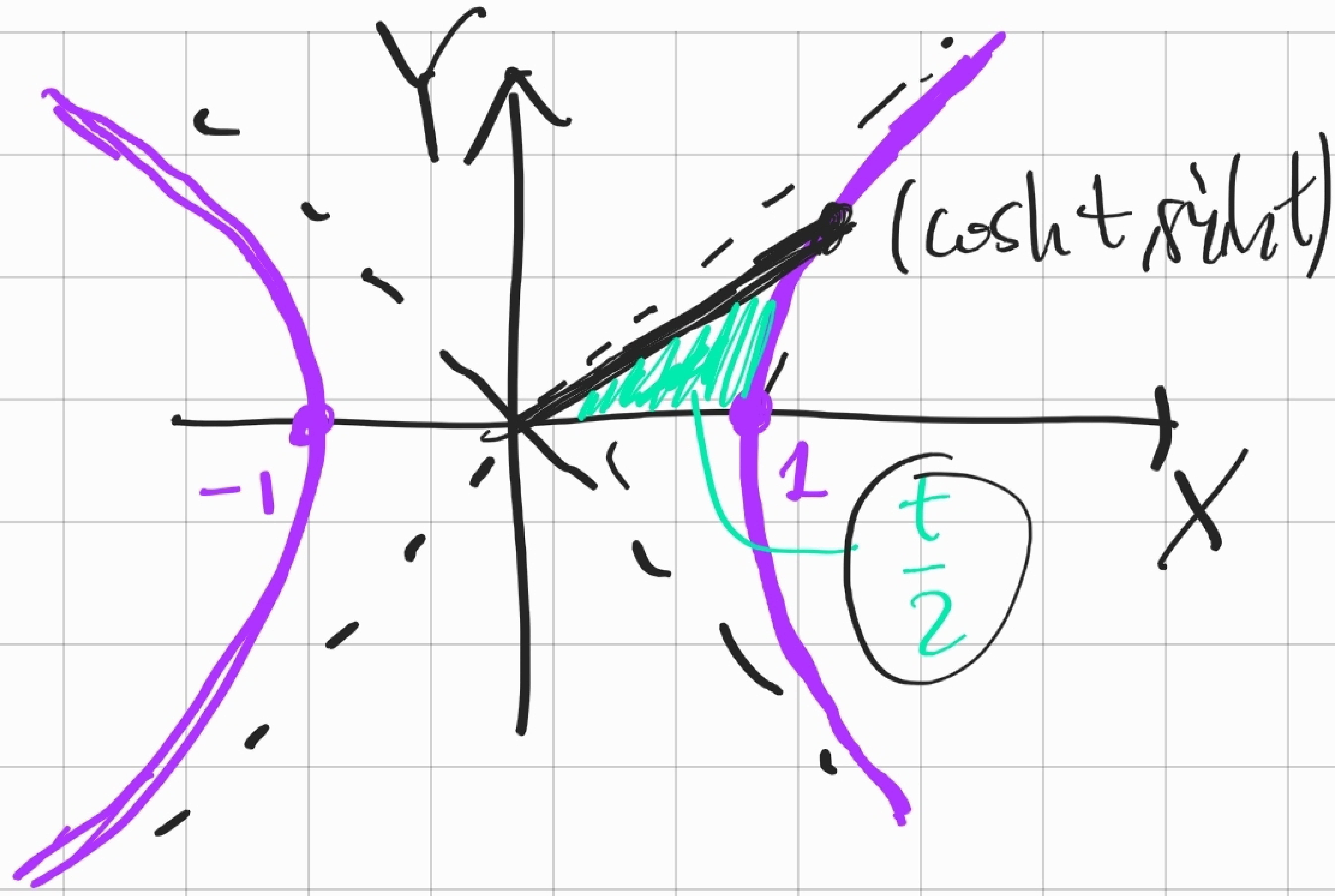
$$\frac{\sinh x}{x} \to 1$$

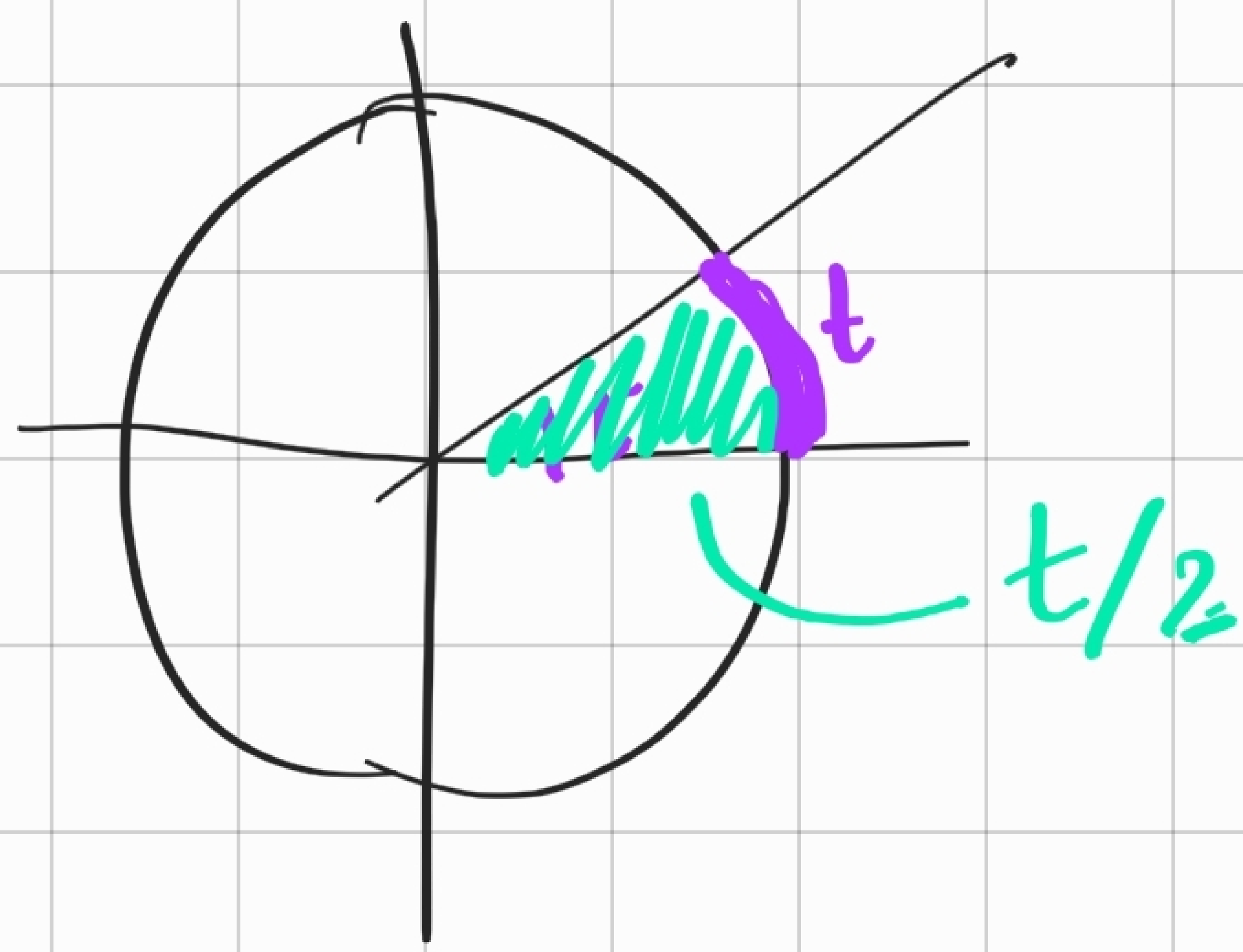
per $x \to 0$

(4) $$\boxed{\cosh^2 t - \sinh^2 t = 1}$$

$$\boxed{x^2 - y^2 = 1}$$

$$\begin{cases} x = \cosh t \\ y = \sinh t \end{cases}$$





Verifichiamo (*)

$$\cosh^2 t - \operatorname{senh}^2 t = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2$$

$$= \frac{\cancel{e^{2t}} + \cancel{e^{-2t}} + 2}{4} - \frac{\cancel{e^{2t}} + \cancel{e^{-2t}} - 2}{4}$$

$$= \frac{2}{4} - \frac{-2}{4} = 1.$$

Formule di addizione

$$\cosh(\alpha+\beta) = \cosh\alpha \cosh\beta + \sinh\alpha \sinh\beta$$

$$\sinh(\alpha+\beta) = \sinh\alpha \cosh\beta + \sinh\alpha \cosh\beta$$

$$\longrightarrow \quad e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$$

$$\cosh x = \sum_{k=0}^{+\infty} \frac{x^{2k}}{2k!}$$

$$\sinh x = \sum_{k=0}^{+\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum (-1)^k \frac{x^{2k}}{2k!}$$

$$\sin x = \sum (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\text{tgh} = \frac{\sinh}{\cosh}$$

settsenh è la funzione inversa di $\text{senh} : \mathbb{R} \to \mathbb{R}$

$\text{settcosh} : [1, +\infty) \to [0, +\infty)$

è l'inversa di

$\cosh : [0, +\infty) \to [1, +\infty)$

sett tgh : $(-1, 1) \to \mathbb{R}$.

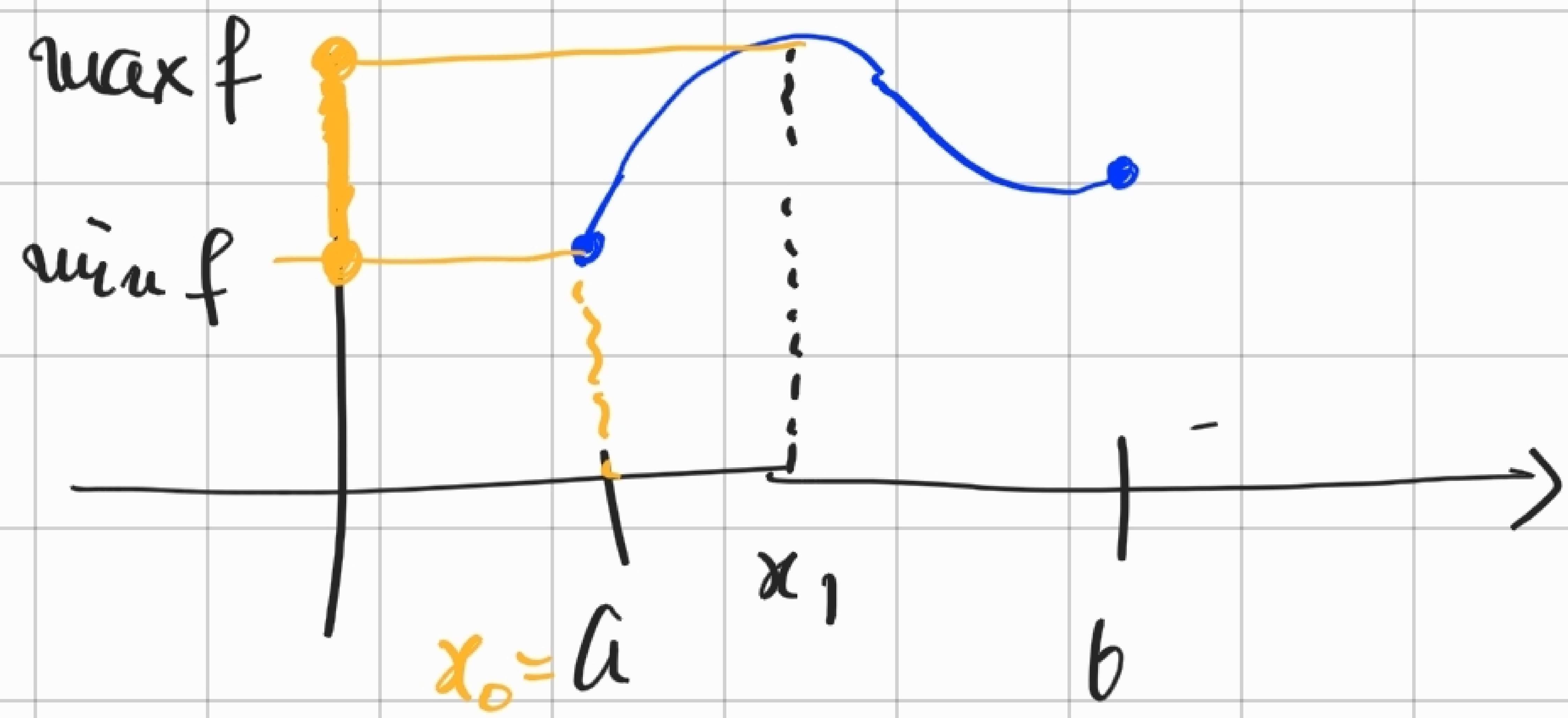
<u>Esercizio</u> trovare una formula per scrivere settsenh x tramite logaritmi.

Teorema di Weierstraß

Teorema (Weierstraß) $f:[a,b]\to\mathbb{R}$, f continua.

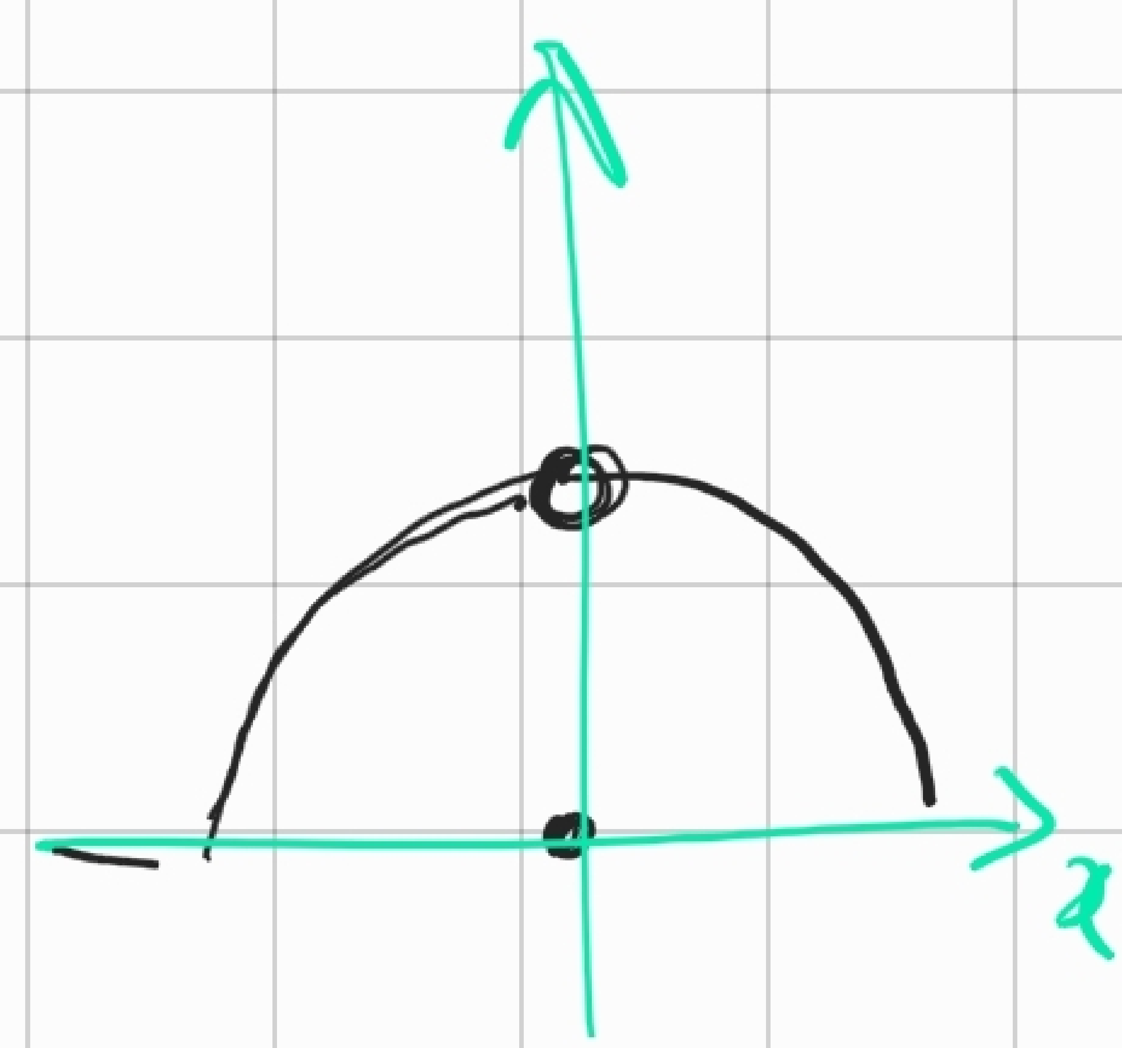
Allora $\exists\ x_0\in[a,b]$: $f(x_0)=\min\limits_{x\in[a,b]} f$

$\exists\ x_1\in[a,b]$: $f(x_1)=\max\limits_{x\in[a,b]} f$



Esempio se f non è continua:

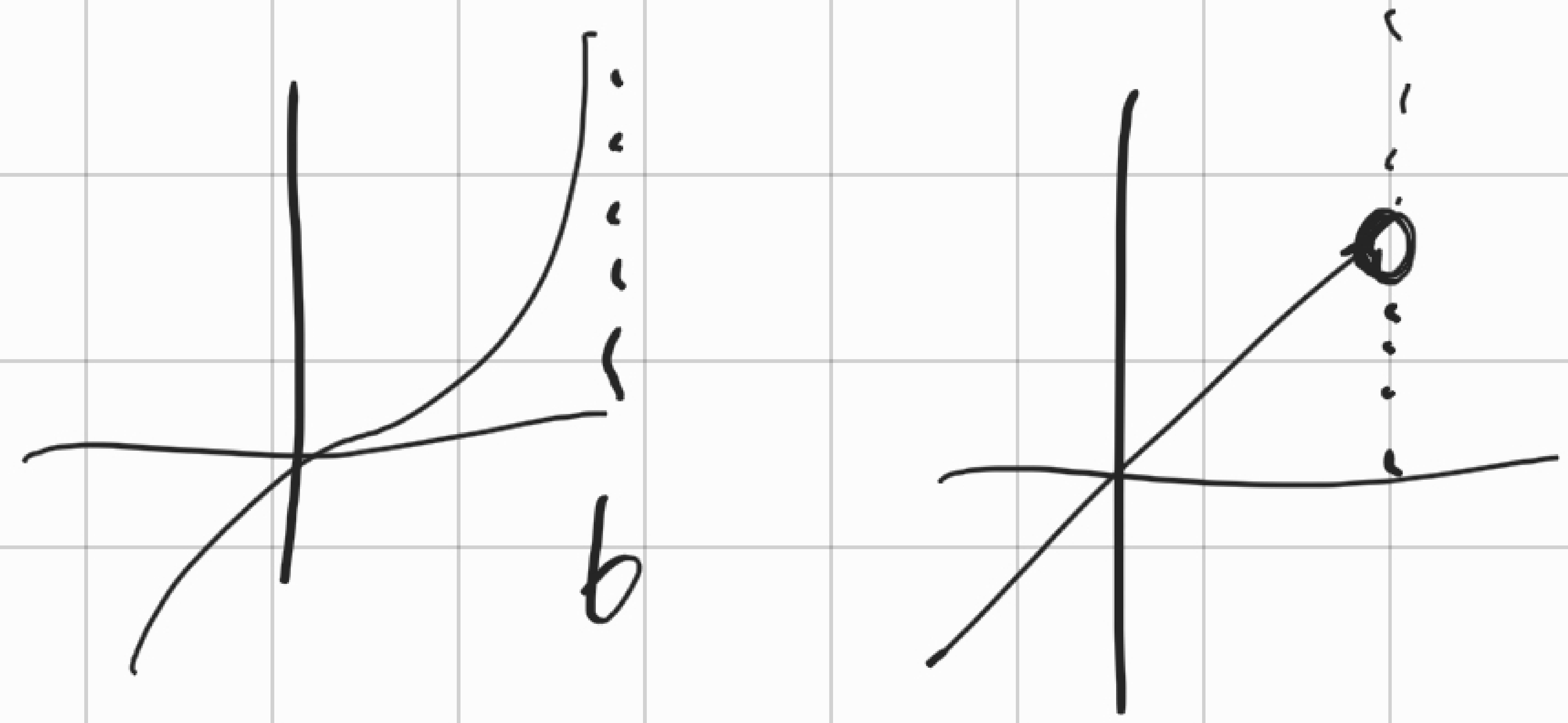
$$f(x)=\begin{cases} 1-x^2 & \text{se } x\neq 0 \\ 0 & \text{se } x=0 \end{cases}$$



non ha massimo. $\sup f = 1$

ma $f<1$.

Esempio se $f:[a,b)\to\mathbb{R}$ continua

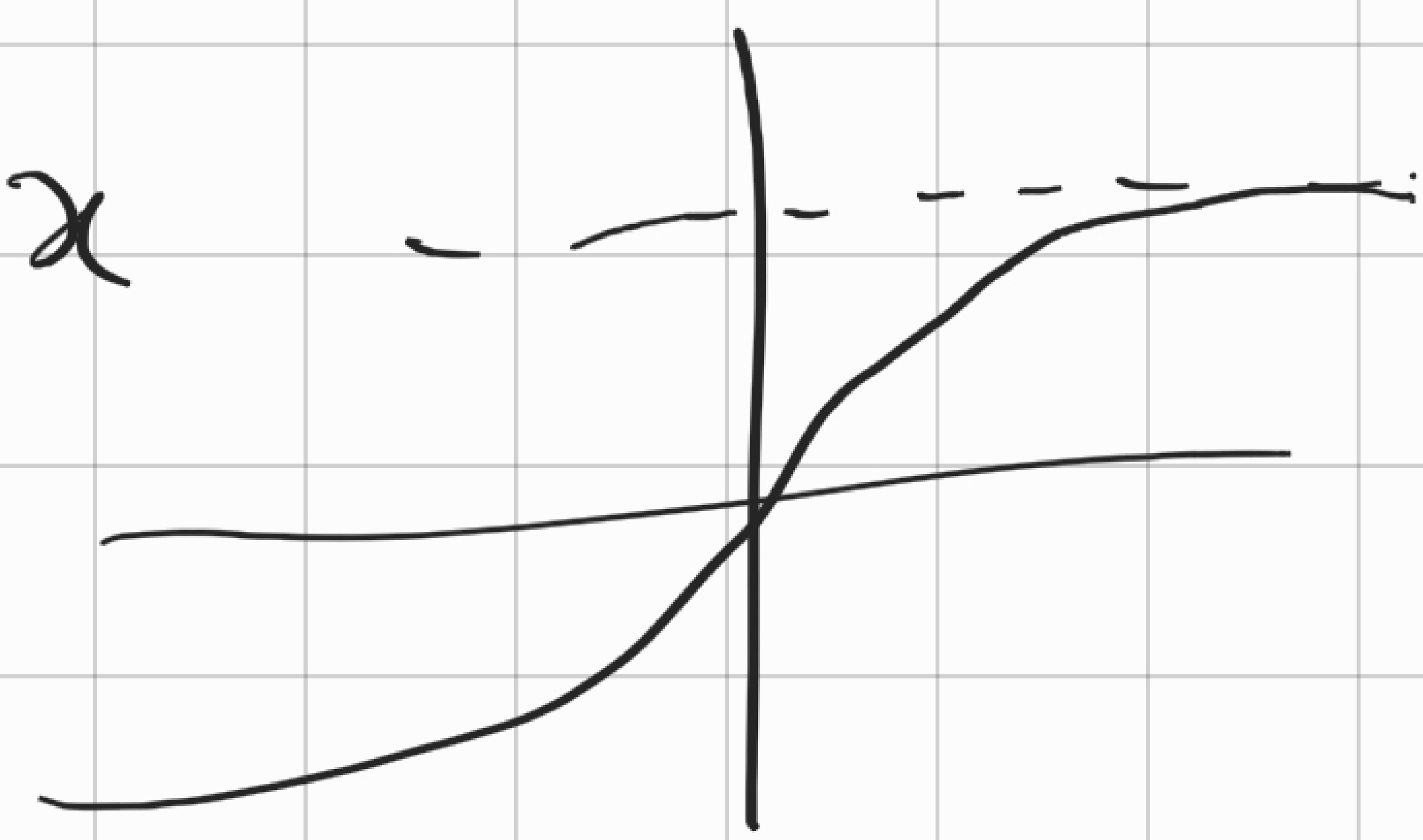


$f(x) = x \qquad f : [0,1) \to \mathbb{R}$ continua

non ha massimo.

Esempio $\quad f : \mathbb{R} \to \mathbb{R}$ continua

$f(x) = \operatorname{arctg} x$



ha $\sup f = \frac{\pi}{2}$

ma $f < \frac{\pi}{2}$.

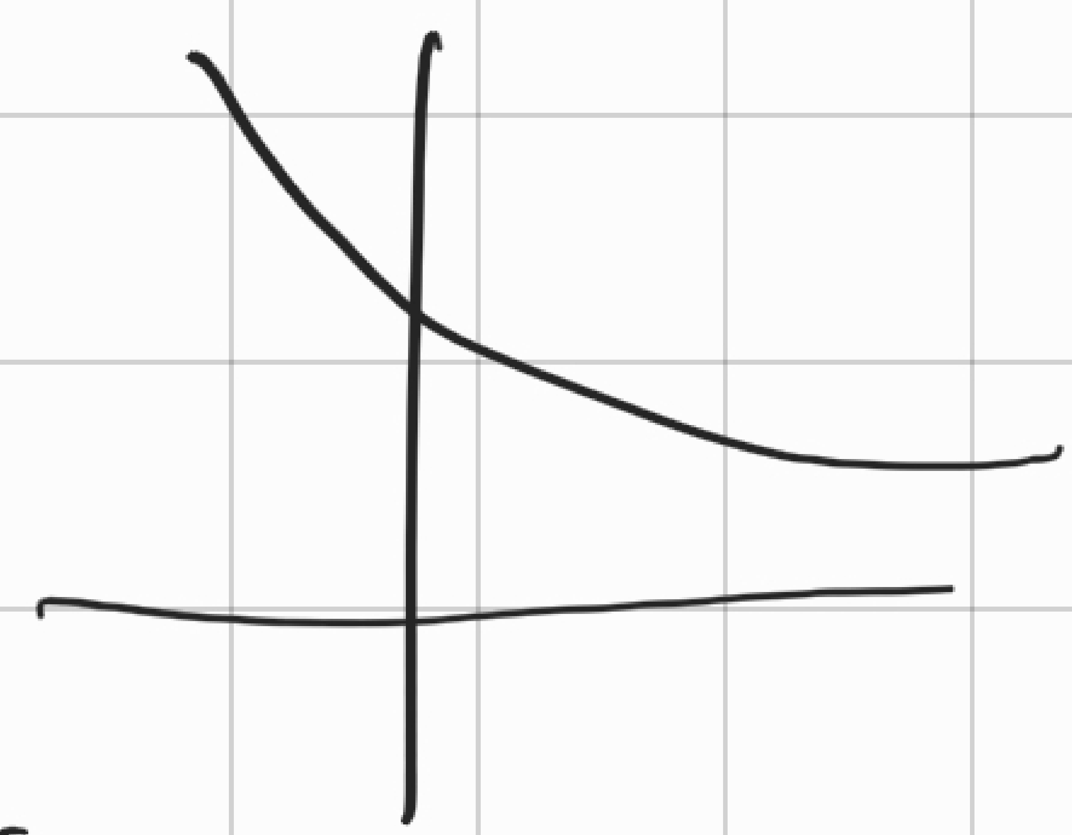
dim (Weierstrass)

In generale se $f : A \to \mathbb{R}$

$$s = \inf f = \inf f(A)$$

$$= \inf_{x \in A} f(x)$$

$\exists \quad s \in [-\infty, +\infty)$



$\exists \; a_n \in A$ t.c. $f(a_n) \to s$

perché s è il massimo dei minoranti

Se $y > s$ y non è un minorante

$\forall y > s$ $\underline{\exists a \in A}$ t.c. $f(a) < y$

$s \le f(a) \quad \forall a \in A$

$s \in \mathbb{R}$ $\quad y = s + \frac{1}{n}$ $\quad s \leq f(a_n) < s + \frac{1}{n}$

$$\downarrow$$

$$s$$

$$f(a_n) \to s$$

$s = -\infty$ $\quad y = -n$ $\quad f(a_n) < -n$

$$f(a_n) \to -\infty$$

a_n è una successione minimizzante

$$f(a_n) \to \inf f.$$

$a_n \in [a, b]$ $\quad A = [a, b]$

B.W. $\exists\, a_{n_k} \to a$

$$a \in [a, b]$$

f continue in a

$$f(a_{n_k}) \to f(a)$$

$$f(a_n) \downarrow \inf f$$

$$f(a) = \inf f$$

a è un punto di minimo

$$f(a) = \min f$$

$\square$

(1) $f : [a,b] \to \mathbb{R}$ $\quad f(a) \le 0$

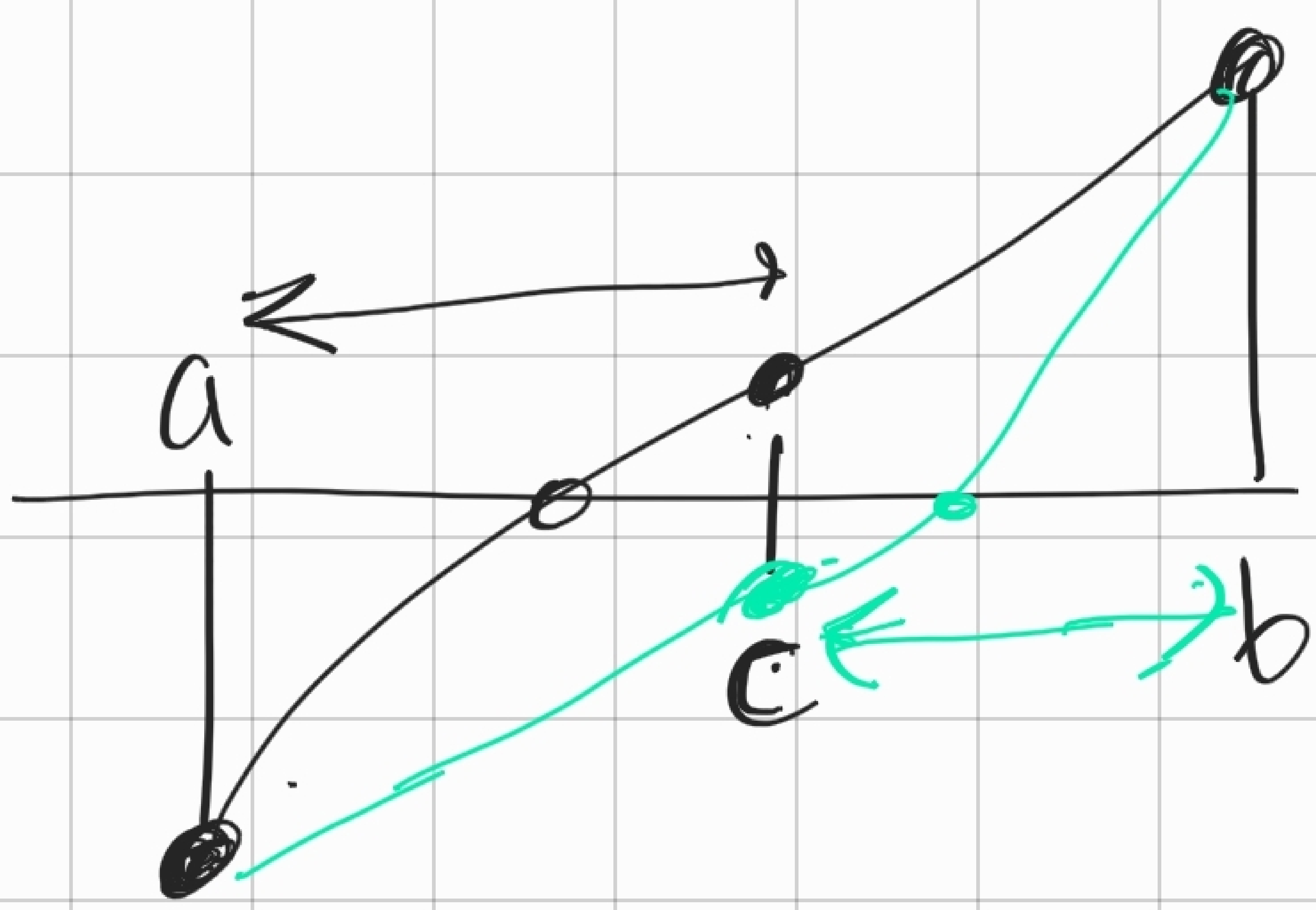
$f(b) \geq 0$

I intervallo

(2) $f : I \to \mathbb{R}$, $a,b \in I$, $f(a) \le 0$

$f(b) \geq 0$

$\underbrace{[a,b] \subseteq I}$



$c = \frac{a+b}{2}$