SOMMA PER PARTI e CRITERIO DI DIRICHLET

Esempio: $\sum \frac{\cos(\alpha n)}{n}$ $\alpha \in \mathbb{R}$.

c'è convergenza assoluta? $\left|\frac{\cos(\alpha n)}{n}\right| \leq \frac{1}{n}$

ma $\sum \frac{1}{n} = +\infty$

(Esercizio: non c'è convergenza assoluta)

Se $\alpha = 0$ $\cos(\alpha n) = 1$ $\sum \frac{1}{n} = +\infty$

Se $\alpha = \pi$ $\cos(\alpha n) = (-1)^n$ $\sum \frac{(-1)^n}{n}$ converge

per il criterio di Leibniz.

$\frac{1}{n}$ è decrescente e infinitesima.

Negli altri casi?

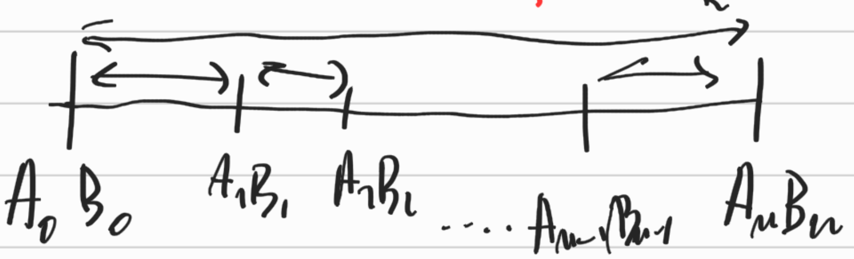
SOMMA PER PARTI

a_k successione qualunque definiamo: $A_n = \sum_{k=0}^{n-1} a_k$

allora $a_n = A_{n+1} - A_n$ $\qquad (A_0 = 0)$

b_k : $B_k = \sum_{k=0}^{n-1} b_k$, $\qquad b_n = B_{n+1} - B_n$. $\quad (B_0 = 0)$

$$A_n \cdot B_n - \cancel{A_0 \cdot B_0} = \sum_{k=0}^{n-1} \left(A_{k+1} B_{k+1} - A_k B_k\right)$$



$$A_n \cdot B_n = \sum_{k=0}^{n-1} \left(A_{k+1} \cdot B_{k+1} - A_{k+1} B_k + A_{k+1} \cdot B_k - A_k \cdot B_k\right)$$

$$= \sum_{k=0}^{n-1} A_{k+1} \cdot \left(B_{k+1} - B_k\right) + \sum_{k=0}^{n-1} \left(A_{k+1} - A_k\right) \cdot B_k$$

$$= \sum_{k=0}^{n-1} A_{k+1} \cdot b_k + \sum_{k=0}^{n-1} a_k \cdot B_k$$

Teorema (somma per parti)

$$\boxed{\sum_{k=0}^{n-1} \underbrace{a_k \cdot B_k} = \underbrace{A_n B_n}_{\to 0} - \sum_{k=0}^{n-1} A_{k+1} \cdot b_k}$$

assolutamente convergente

se: a_k qualunque, $A_n = \sum_{k=0}^{n-1} a_k$

B_k qualunque, $B_0 = 0$, $b_n = B_{n+1} - B_n$. $\square$

Teorema (criterio di Dirichlet)

(in Leibniz $a_k = (-1)^k$, B_k decrescente, infinitesima!)

Hyp:

1. B_n infinitesima, $B_0 = 0$
2. A_n limitata
3. $\sum |b_k|$ convergente

Tesi: $\sum_{k=0}^{+\infty} a_k B_k$ converge

dim

$$|A_{k+1} \cdot b_k| \leq |A_{k+1}| \cdot |b_k| \leq C \cdot |b_k|$$

$$\sum |A_{k+1} \cdot b_k| \leq C \sum |b_k| \quad \square$$

Ana'

$$\sum_{k=0}^{+\infty} a_k B_k = -\sum_{k=0}^{+\infty} A_{k+1} b_k$$

(con l'ipotesi $B_0 = 0$)

Esempio $\sum \frac{\cos(\alpha n)}{n}$ $a_k = \cos(\alpha k)$

$B_n = \begin{cases} \frac{1}{n} & \text{se } n \neq 0 \\ 0 & \text{se } n = 0 \end{cases}$

① $B_n \to 0$ ✓

③ $\sum |b_k| = \sum |B_{k+1} - B_n|$

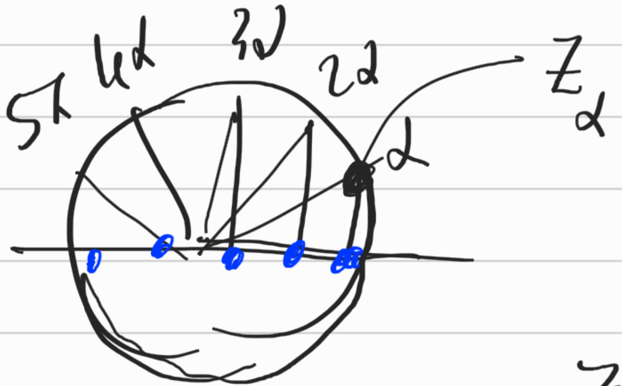
$= \sum \left| \frac{1}{k+1} - \frac{1}{k} \right|$ è convergente.

$= \sum \left| \frac{\cancel{k} - (\cancel{k}+1)}{k^2+k} \right| = \sum \frac{1}{k^2+k}$

$\frac{1}{k^2+k} \sim \frac{1}{k^2}$ $\sum \frac{1}{k^2}$ convergente

② A_n è limitata?

$A_n = \sum_{k=0}^{n-1} \cos(k \cdot \alpha)$



$$Z_\alpha = \cos\alpha + i\sin\alpha$$

$$|z| = 1$$

$$Z_\alpha^n = \cos(n\alpha) + i\sin(n\alpha)$$

$$a_n = \cos(n\alpha) = \mathrm{Re}(Z_\alpha^n) = \frac{Z_\alpha^n + \overline{Z_\alpha}^n}{2}$$

$Z_\alpha \neq 1$

$\alpha \neq 2m\pi$

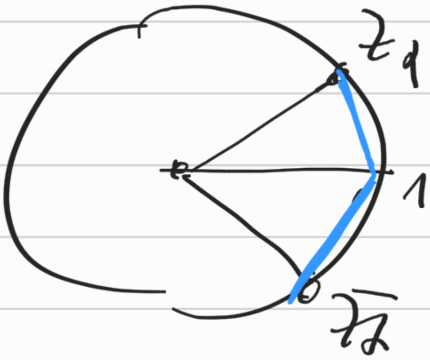
$$A_n = \sum_{k=0}^{n-1} a_k = \sum_{k=0}^{n-1} \frac{Z_\alpha^k + \overline{Z_\alpha}^k}{2}$$

$$= \frac{1}{2}\sum_{k=0}^{n-1} Z_\alpha^k + \frac{1}{2}\sum_{k=0}^{n-1} \overline{Z_\alpha}^k$$

$$= \frac{1}{2}\,\frac{1 - Z_\alpha^n}{1 - Z_\alpha} + \frac{1}{2}\,\frac{1 - \overline{Z_\alpha}^n}{1 - \overline{Z_\alpha}}$$

$$|A_n| \leq \frac{1}{2}\,\frac{|1 - Z_\alpha^n|}{|1 - Z_\alpha|} + \frac{1}{2}\,\frac{|1 - \overline{Z_\alpha}^n|}{|1 - \overline{Z_\alpha}|}$$

$$|1-\overline{z_\alpha}| = |1-z_\alpha|$$



$$|z_\alpha^n| = |z_\alpha|^n = 1^n = 1$$

$$|1-z_\alpha^n| \leq |1| + |-z_\alpha^n| = 2$$

$$= \frac{2+2}{2|1-z_\alpha|} = \frac{2}{|1-z_\alpha|} \quad \square$$

Se $\alpha \neq 2m\pi$ la serie $\sum \frac{\cos(n\alpha)}{n}$ è convergente.