ANALISI MATEMATICA B
LEZIONE 19 - 6.11.2020

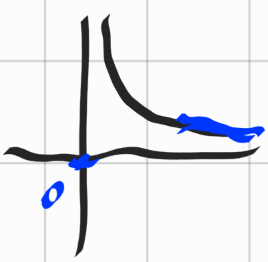
$$\lim_{x\to x_0} f(x) = \ell$$

$f(x)\to \ell$ per $x\to x_0$

Osservazione

$$\frac{1}{x}\to 0 \quad \text{per } x\to+\infty$$

$$\frac{1}{x}\to 0^+ \quad \text{per } x\to+\infty$$



Limiti e continuità

x_0 punto di acc. per dom f

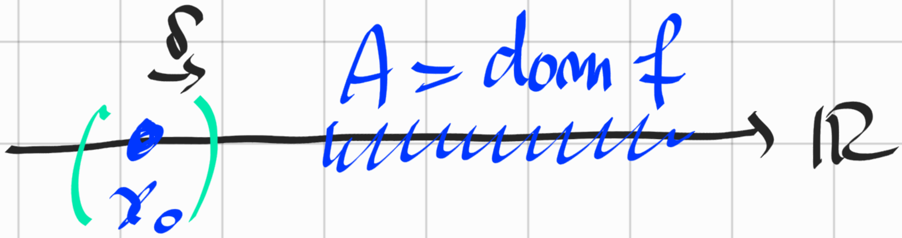
f è continua in x_0 $\Leftrightarrow$ $\boxed{\lim_{x\to x_0} f(x)=f(x_0)}$

$\Updownarrow$

$x\neq x_0$

$$\forall \varepsilon>0\ \exists\delta>0 : |x-x_0|<\delta \Rightarrow |f(x)-\underbrace{f(x_0)}_{\ell}|<\varepsilon$$

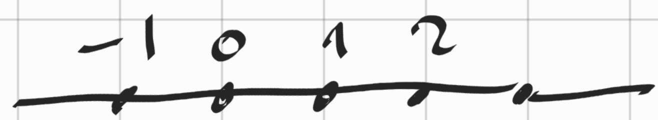
Se $x_0 \in \text{dom } f$ ma non di accumulazione



x_0 si chiama <u>punto isolato</u>

la funzione f è sempre continua nei punti isolati.

<u>Es</u> $f : \mathbb{Z} \to \mathbb{R}$



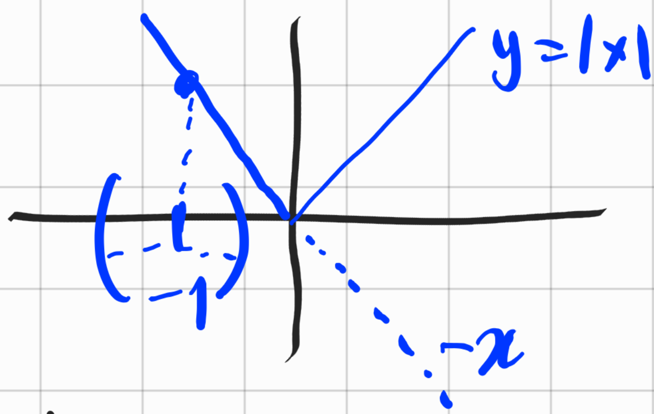
f è continua.

<u>Proprietà del limite</u>

1) località ← il limite per $x \to x_0$ dipende da f solo in un intorno di x_0.

Es $\lim_{x\to -1} |x| = \lim_{x\to -1} -x$

$$|x| = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } \boxed{x<0} \end{cases}$$



2) Restrizione $\qquad f: A \to \mathbb{R}$

Se $\lim_{x\to x_0} f(x) = \ell$ $\qquad g: B \to \mathbb{R}$

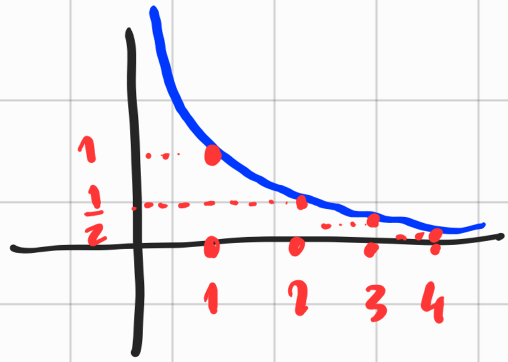
Allora $\lim_{x\to x_0} g(x) = \ell$ $\qquad B \subseteq A$

$g(x) = f(x) \quad \forall x \in B$

(x_0 pto di accumulazione per B è quindi anche per A)

ES

$$f(x) = \frac{1}{x}$$



$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$g(n) = \frac{1}{n} \qquad n \in \mathbb{N}$$

$\frac{1}{n} \to 0$ per $n \to +\infty$, $n \in \mathbb{N}$.

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$\forall x \neq 0$

$$\forall \varepsilon > 0 : \exists \alpha : x > \alpha \Rightarrow |f(x) - 0| < \varepsilon$$

$U = (-\varepsilon, \varepsilon)$ $V = (\alpha, +\infty]$

$$\lim_{n \to +\infty} \frac{1}{n} = 0$$
$(n \in \mathbb{N})$

$$\forall \varepsilon > 0 : \exists \alpha : \forall n \in \mathbb{N} : n > \alpha \Rightarrow |f(n) - 0| < \varepsilon$$

$n \in \mathbb{R}$.

$$\forall x \in A : \ldots\ldots$$
$$\Downarrow$$
$$\forall x \in B : \ldots\ldots$$

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legame tra limite e limite destro/sinistro

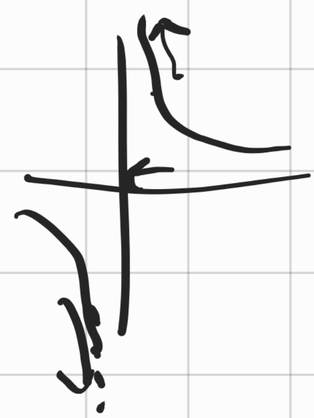
$$\mathcal{B}_{x_0} = \{ (x_0-\varepsilon, x_0+\varepsilon) : \varepsilon>0 \}$$
$$\mathcal{B}_{x_0^+} = \{ [x_0, x_0+\varepsilon) : \varepsilon>0 \}$$
$$\mathcal{B}_{x_0^-} = \{ (x_0-\varepsilon, x_0] : \varepsilon>0 \}$$
$$(x_0-\varepsilon, x_0+\varepsilon) = (x_0-\varepsilon, x_0] \cup [x_0, x_0+\varepsilon)$$

$$\lim_{x\to x_0} f(x) = l \iff \begin{cases} \lim_{x\to x_0^+} f(x) = l \\ \lim_{x\to x_0^-} f(x) = l \end{cases}$$

$$\lim_{x\to x_0} f(x) = \ell \iff \begin{cases} \lim_{x\to x_0^+} f(x)=\ell \\ \lim_{x\to x_0^-} f(x)=\ell \end{cases}$$

(Se x_0^+ e x_0^- sono punti di accumulazione).

Es $\boxed{\lim_{x\to 0} \frac{1}{x}}$ non esiste perché



$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Es $\lim_{x\to 0} \sqrt{x} = \lim_{x\to 0^+} \sqrt{x}$

$\mathrm{dom}\,\sqrt{x} = [0,+\infty)$

$$\lim_{\boxed{x \to 0^+}} -x = 0^- \qquad \lim_{\boxed{x \to 0^-}} -x = 0^+$$

$x \geq 0$

$-x \leq 0$

$-x \to 0^-$ per $x \to 0^+$

$-x \to 0^+$ per $x \to 0^-$

$$\lim_{x \to +\infty} \frac{1}{x} = 0^+ \leftarrow \left(\begin{array}{l} -0^- = 0^+ \\ \dfrac{1}{0^+} = +\infty \end{array} \right)$$

$\displaystyle\lim_{x \to +\infty} \frac{1}{x} = 0$, e $\dfrac{1}{x} \geq 0$ per $x > 0$.

$x_0^+ \in \mathbb{R}^+$ $\qquad \dfrac{1}{0}$ $\qquad +\infty$

$x_0^- \in \mathbb{R}^-$ $\qquad -\infty$

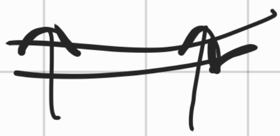
Teorema (cambio di variabile / limite della fn composta)

$f: A \to B$, x_0 pto di accumulazione di A

$g: B \to \mathbb{R}$, y_0 pto di accumulazione di B

$f(A \setminus \{x_0\}) \subseteq B \setminus \{y_0\}$, $\left(f(x) \neq y_0 \ \text{se} \ x \neq x_0 \right)$

$\lim_{x \to x_0} f(x) = y_0$ $\qquad$ $\lim_{\substack{y \to y_0 \\ (y \neq y_0)}} g(y) = \ell$



Allora:

$$\lim_{x \to x_0} g(f(x)) = \ell$$

$(\ y = f(x)$ se $x \to x_0$ $\quad y \to y_0$

dim $\forall U \in \mathcal{B}_{y_0} \ \exists V \in \mathcal{B}_{x_0} : f(A \cap V \setminus \{x_0\}) \subseteq U$ $\quad x \neq x_0$

$\forall W \in \mathcal{B}_\ell \ \exists U \in \mathcal{B}_{y_0} : g(B \cap U \setminus \{y_0\}) \subseteq W$ $\quad y \neq y_0$

$\forall W \in \mathcal{B}_\ell \ \exists V \in \mathcal{B}_{x_0} : g(f(A \cap V \setminus \{x_0\})) \subseteq W$ $\square$

se $f(A \cap V \setminus \{x_0\}) \overset{?}{\subseteq} \overset{ok}{B} \cap \overset{ok}{U} \setminus \{y_0\}$ ipotesi

Es $\lim_{x\to+\infty} \overset{y}{x+1} = +\infty$ $\lim_{y\to+\infty} \frac{1}{y} = 0$

Allora $\lim_{\boxed{x\to+\infty}} \frac{1}{x+1} = 0$

$y = x+1$ | se $x\to+\infty$

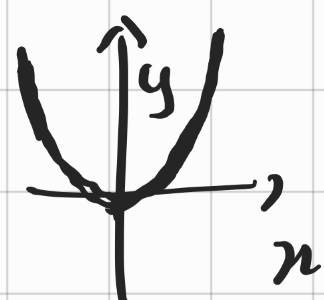
$y = x+1 \to +\infty$

$$\lim_{y\to+\infty} \frac{1}{y} = 0$$

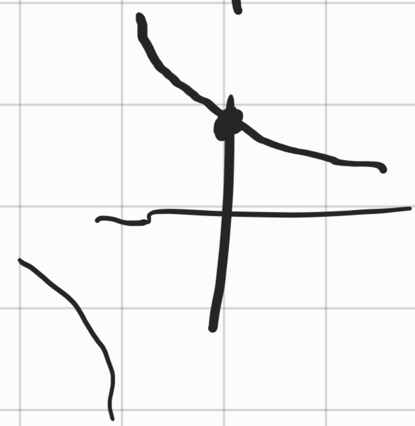
$x\to+\infty$ $x+1\to+\infty$ $y=x+1\to+\infty$

$\frac{1}{x+1} = \frac{1}{y} \to 0$ se $y\to+\infty$.

Es $\lim_{x\to 0} x^2 = 0 = y_0$



$\lim_{y\to \underset{y_0}{0}} \frac{1}{1+y} = 1$



allora $y = x^2$

$$\lim_{x \to 0} \boxed{\frac{1}{1+x^2}} = 1$$

$f(x) = x^2$ $\qquad f : \mathbb{R} \to \overset{A}{\overset{\shortparallel}{[0,+\infty)}}$

$g(y) = \dfrac{1}{1+y}$ $\qquad g : \underset{B}{\underset{\shortparallel}{\mathbb{R}\setminus\{-1\}}} \to \mathbb{R}$

$f(A \setminus \{\underset{x_0}{0}\}) \subseteq B \setminus \{\underset{y_0}{0}\} \Leftarrow x^2 > 0$ se $x \neq 0$.

$x \to 0 \quad y = x^2 \to 0^2 = 0 \qquad \underline{y \to 0}$

$z = 1 + x^2 \to 1 + 0^2 = 1 \qquad \underline{\underline{z \to 1}}$

$u = \dfrac{1}{z} = \dfrac{1}{\overset{=1+y}{1+x^2}} \to \dfrac{1}{1+0^2} = 1.$ per $z \to 1$

$\frac{1}{z} \to 1 \quad$ per $z \to 1$

Es

Hyp $\begin{cases} \lim_{x \to 1} x^2 = 1 \\ \lim_{y \to 1} \frac{1}{y} = 1 \end{cases}$

Th $\lim_{x \to 1} \frac{1}{x^2} = 1$

$y = f(x) = x^2 \qquad x \to 1$
$\qquad\qquad\qquad\quad y \to 1$ $\|$

$g(y) = \frac{1}{y}$ ⚠ $f(x) \neq 1$ se $x \neq 1$

$\frac{1}{x^2} = g(f(x))$

Posso comunque applicare il teorema

restringendosi a $x > 0$.

Esempio $f : [0,1] \to [0,1]$

$g : [0,1] \to \mathbb{R}$

$$f(x) = 0, \quad g(y) = \begin{cases} 1 & \text{se } y = 0 \\ 0 & \text{se } y \neq 0 \end{cases}$$

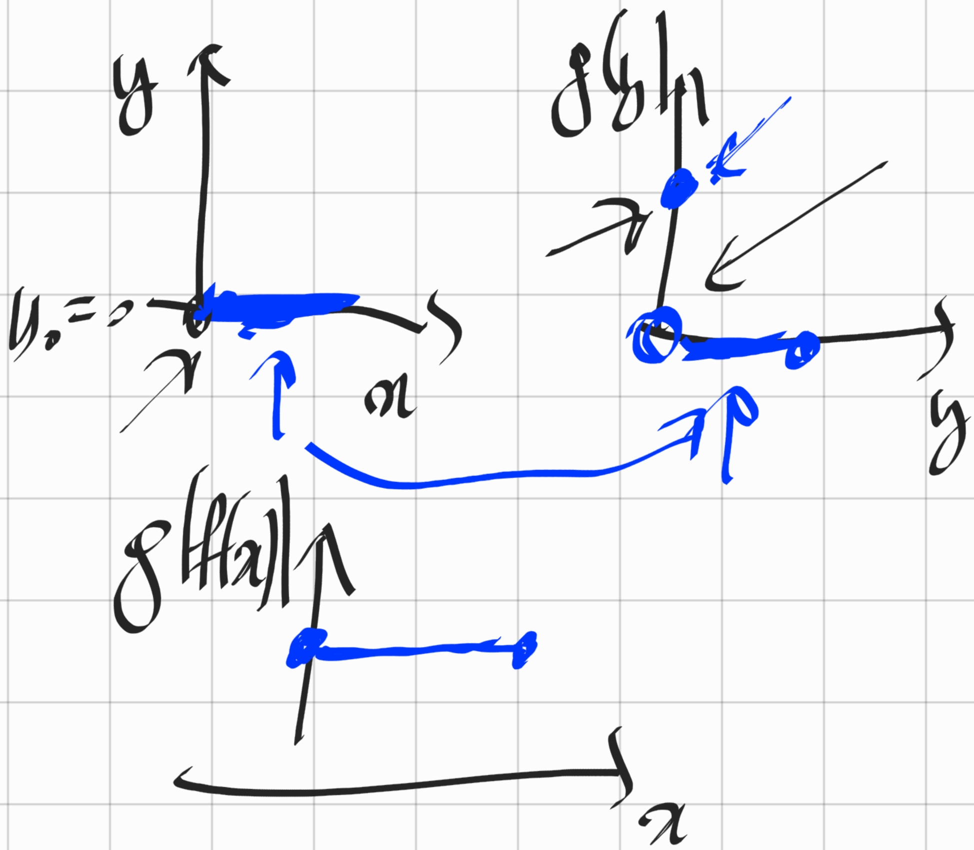
$$\lim_{x \to 0} f(x) = 0 = y_0 \qquad \lim_{y \to 0} g(y) = 0$$

$$\lim_{x \to 0} g(f(x)) \overset{??}{=} \cancel{0} \quad \text{No!!}$$

$y = f(x) \to 0$ per $x \to 0$

$$g(f(x)) = 1 \quad \forall x,$$

$$\lim g(f(x)) = 1.$$



$f(x) = 0$

$g(f(x)) \equiv g(0)$

$\lim\limits_{x \to 0} g(f(x)) = \lim\limits_{x \to 0} g(0) = g(0) = 1$

$\neq \lim\limits_{y \to 0} g(y) = 0$ $\square$

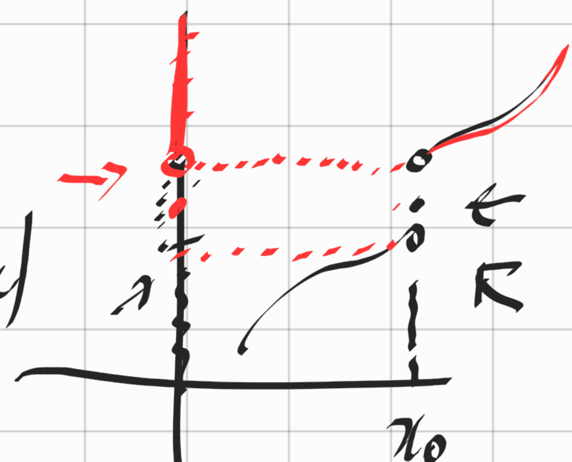
Teo (limiti di funzione monotona)

Sia $f : A \subseteq \mathbb{R} \to \mathbb{R}$ crescente

$$\lim_{x \to x_0^+} f(x) = \inf_{x > x_0} f(x) \quad (\text{se } x_0^+ \text{ pto di acc. } A)$$

$$\lim_{x \to x_0^-} f(x) = \sup_{x < x_0} f(x) \quad (\text{se } x_0^- \text{ pto di acc. } A)$$

Se f decrescente



$$\lim_{x \to x_0^+} f(x) = \sup_{x > x_0} f(x)$$

$$\lim_{x \to x_0^-} f(x) = \inf_{x < x_0} f(x)$$

$\square$

- "Le funzioni monotone hanno discontinuità a salto"

Es $\lim_{x\to+\infty} a^x$ $\qquad a>1$.

$f(x)=a^x$ è strett. crescente

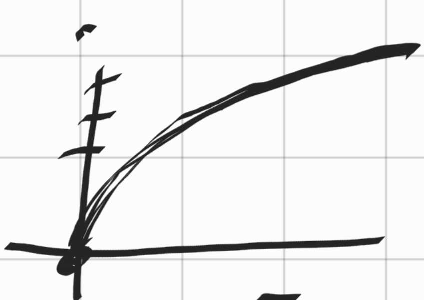
$+\infty$ è pto di acc.

$+\infty = +\infty^-$

$$\lim_{x\to+\infty} a^x = \sup_{x\in\mathbb{R}} a^x = \sup\,(0,+\infty) = +\infty.$$

Es $\lim_{x\to 0}\sqrt{x} = \lim_{x\to 0^+}\sqrt{x}$

$$= \inf_{x>0}\sqrt{x} = 0$$



$\sqrt{x}=y \qquad x=y^2$