14 Calcolare

$$\lim_{x \to 0} \frac{\sin\left(1-\sqrt[3]{\cos(x^2)}\right)}{\operatorname{arctg}^4 x}$$

$$\cancel{\cos(x^2)} \quad \cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$\cos x^2 = 1 - \frac{x^4}{2} + o(x^4)$$

$$(1+x)^{\frac{1}{3}} = 1 + \frac{x}{3} + o(x)$$

$$\sqrt[3]{\cos(x^2)} = \left(1 - \frac{x^4}{2} + o(x^4)\right)^{\frac{1}{3}} = 1 + \frac{\left(-\frac{x^4}{2} + o(x^4)\right)}{3} + o(x^4)$$

$$= 1 - \frac{x^4}{6} + o(x^4).$$

$$\sin\left(1-\sqrt[3]{\cos(x^2)}\right) = \sin\left(\frac{x^4}{6} + o(x^4)\right) = \frac{x^4}{6} + o(x^4).$$

$$\operatorname{arctg} x = x + o(x)$$

$$(\operatorname{arctg} x)^4 = (x + o(x))^4 = x^4 + o(x^4)$$

$$\frac{\sin\left(1-\sqrt[3]{\cos(x^2)}\right)}{\operatorname{arctg}^4 x} = \frac{\frac{x^4}{6} + o(x^4)}{x^4 + o(x^4)} = \frac{\frac{1}{6} + \underbrace{\frac{o(x^4)}{x^4}}_{0}}{1 + \underbrace{\frac{o(x^4)}{x^4}}_{0}} \to \frac{1}{6}$$

(1A)

$$\lim_{x\to 0} \frac{1-\sqrt{\cos(\sin^2 x)}}{\operatorname{arctg}^4 x}$$

$$\sin^2 x = (x+o(x))^2 = x^2 + o(x^2)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$\cos(\sin^2 x) = 1 - \frac{(x^2+o(x^2))^2}{2} + o(x^4)$$

$$= 1 - \frac{x^4}{2} + o(x^4).$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$$

$$\sqrt{\cos(\sin^2 x)} = \sqrt{1 - \frac{x^4}{2} + o(x^4)} = 1 - \frac{x^4}{4} + o(x^4)$$

$$\operatorname{arctg}^4 x = x^4 + o(x^4) \quad (\text{vedi } \boxed{1C}).$$

$$\frac{1-\sqrt{\cos(\sin^2 x)}}{\operatorname{arctg}^4 x} = \frac{\frac{x^4}{4} + o(x^4)}{x^4 + o(x^4)} = \frac{\frac{1}{4} + \cancel{\frac{o(x^4)}{x^4}}}{1 + \cancel{\frac{o(x^4)}{x^4}}} \to \frac{1}{4}$$

$$0$$

$\boxed{1D}$

$$\lim_{x\to 0} \frac{\sin^2\left(1-\sqrt[3]{\cos x}\right)}{\operatorname{arctg}^4 x}$$

$$\sqrt[3]{1+x} = 1+\frac{x}{3}+o(x)$$

$$\cos x = 1-\frac{x^2}{2}+o(x^2),$$

$$\sqrt[3]{\cos x} = \left(1-\frac{x^2}{2}+o(x^2)\right)^{\frac{1}{3}} = 1-\frac{x^2}{6}+o(x^2)$$

$$1-\sqrt[3]{\cos x} = \frac{x^2}{6}+o(x^2)$$

$$\sin x = x+o(x)$$

$$\sin\left(1-\sqrt[3]{\cos x}\right) = \frac{x^2}{6}+o(x^2)+o\left(\frac{x^2}{6}+o(x^2)\right)$$

$$= \frac{x^2}{6}+o(x^2)$$

$$\sin^2\left(1-\sqrt[3]{\cos x}\right) = \left(\frac{x^2}{6}+o(x^2)\right)^2 = \frac{x^4}{36}+o(x^4)$$

$$\operatorname{arctg}^4 x = x^4+o(x^4) \quad \left(\text{Vedi } \boxed{1C}\right).$$

$$\frac{\sin^2\left(1-\sqrt[3]{\cos x}\right)}{\operatorname{arctg}^4 x} = \frac{\frac{x^4}{36}+o(x^4)}{x^4+o(x^4)} = \frac{\frac{1}{36}+\cancel{\frac{o(x^4)}{x^4}}^{\,0}}{1+\cancel{\frac{o(x^4)}{x^4}}_{\,0}} \to \frac{1}{36}$$

$\boxed{1B}$

$$\lim_{x\to 0} \frac{1-\sqrt[3]{\cos(\sin(x^2))}}{\arctan^4 x}$$

$$\sin x = x + o(x)$$

$$\sin(x^2) = x^2 + o(x^2)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$\cos(\sin(x^2)) = 1 - \frac{(x^2+o(x^2))^2}{2} + o\left((x^2+o(x^2))^2\right)$$

$$= 1 - \frac{x^4}{2} + o(x^4) + \cancel{o(x^4)}.$$

$$\sqrt[3]{1+x} = 1 + \frac{1}{3}x + o(x)$$

$$\sqrt[3]{\cos(\sin(x^2))} = \sqrt[3]{1-\frac{x^4}{2}+o(x^4)} = -\frac{x^4}{6} + o(x^4).$$

$$\arctan^4 x = x^4 + o(x^4) \quad (\text{VEDI } \boxed{1C}).$$

$$\frac{1-\sqrt[3]{\cos(\sin(x^2))}}{\arctan^4 x} = \frac{\frac{x^4}{6}+o(x^4)}{x^4+o(x^4)} = \frac{\frac{1}{6}+\cancel{\frac{o(x^4)}{x^4}}^{\,0}}{1+\cancel{\frac{o(x^4)}{x^4}}_{\,0}} \to \frac{1}{6}$$

$\boxed{2A}$

$$f(x) = \frac{\cos x}{1+x^2} = (\cos x)\cdot(1+x^2)^{-1}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + o(x^2) \cancel{+ \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3}$$

$$\cancel{+ \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24}x^4 + o(x^4)}$$

$$(1+x^2)^{-1} = 1 - x^2 + x^4 + o(x^4)$$

$$f(x) = (\cos x)(1+x^2)^{-1} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)\left(1 - x^2 + x^4 + o(x^4)\right)$$

$$= 1 - \left(\frac{1}{2}+1\right)x^2 + \left(\frac{1}{24}+\frac{1}{2}+1\right)x^4 + o(x^4)$$

$$= 1 - \frac{3}{2}x^2 + \frac{1+12+24}{24}x^4 + o(x^4)$$

$$= 1 - \frac{3}{2}x^2 + \frac{37}{24}x^4 + o(x^4)$$

$$P_4(x) = 1 - \frac{3}{2}x^2 + \frac{37}{24}x^4$$ ✓

$\boxed{2B}$ è molto simile a $\boxed{2A}$

$$\frac{\cos x}{1-x^2} = 1 + \frac{x^2}{2} + \frac{13}{24}x^4$$

$\boxed{2C}$

($\boxed{2D}$ è MOLTO SIMILE)

$$\frac{1-x^2}{\cos x} = 1 - \frac{x^2}{2} - \frac{7x^4}{24} + o(x^4)$$

$$f(x) = \frac{1+x^2}{\cos x} = (1+x^2)(\cos x)^{-1}$$

$$= (1+x^2)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^{-1}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + o(x^2)$$

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^{-1} =$$

$$= 1 - \left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) +$$

$$+ \left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^2 + o(x^4)$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{24} + o(x^4)$$

$$+ \frac{x^4}{4} + o(x^4) = 1 + \frac{x^2}{2} + \frac{6-1}{24} x^4 + o(x^4)$$

$$= 1 + \frac{x^2}{2} + \frac{5}{24} x^4 + o(x^4)$$

$$f(x) = (1+x^2)\left(1 + \frac{x^2}{2} + \frac{5}{24} x^4 + o(x^4)\right)$$

$$= 1 + \frac{3}{2} x^2 + \left(\frac{1}{2} + \frac{5}{24}\right) x^4 + o(x^4)$$

$$= 1 + \frac{3}{2} x^2 + \frac{17}{24} x^4 + o(x^4) \qquad P_4(x) = 1 + \frac{3}{2} x^2 + \frac{17}{24} x^4$$

$\boxed{3A}$

$$f = \frac{x^2y - xy}{1+y^2}$$

$$\frac{\partial f}{\partial x} = \frac{2xy - y}{1+y^2} = \frac{y(2x-1)}{1+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2-x)(1+y^2) - (x^2y - xy)\,2y}{(1+y^2)^2}$$

$$= \frac{x\left[(x-1)(1+y^2) - (xy-y)2y\right]}{(1+y^2)^2}$$

$$= \frac{x\left[x - 1 + xy^2 - y^2 - 2xy^2 + 2y^2\right]}{(1+y^2)^2}$$

$$= \frac{x\left[x - 1 - xy^2 + y^2\right]}{(1+y^2)^2}$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \quad \begin{cases} y(2x-1) = 0 \\ x(x-1-xy^2+y^2) = 0 \end{cases} \quad \begin{cases} y = 0 \\ x(x-1) = 0 \end{cases} \qquad \begin{cases} x = \frac{1}{2} \\ \frac{1}{2}\left(\frac{1}{2} - 1 - \frac{y^2}{2} + y^2\right) = 0 \end{cases}$$

$(1+y^2) \neq 0 \quad \forall y.$

$$\begin{cases} y = 0 \\ x(x-1) = 0 \end{cases} \qquad \begin{cases} y = 0 \\ x = 0 \end{cases} \qquad \begin{cases} y = 0 \\ x = 1 \end{cases}$$

$$\begin{cases} x = \frac{1}{2} \\ -1 + y^2 = 0 \end{cases} \qquad \begin{cases} x = \frac{1}{2} \\ y = 1 \end{cases} \qquad \begin{cases} x = \frac{1}{2} \\ y = -1 \end{cases}$$

4 punti critici: $(0,0)$, $(1,0)$, $(\frac{1}{2},1)$, $(\frac{1}{2},-1)$.

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{2xy - y}{1+y^2} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{x[x-1-xy^2+y^2]}{(1+y^2)^2} \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2y}{1+y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{2x(1+y^2) - (2xy-y)2y}{(1+y^2)^2} = \frac{2x + 2xy^2 - 4xy^2 + 2y^2}{(1+y^2)^2} = \frac{2x - 2xy^2 + 2y^2}{(1+y^2)^2}$$

(3A)

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\,\frac{2xy - y}{1+y^2} = \frac{2y}{1+y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(2x-1)(1+y^2) - (2xy-y)\,2y}{(1+y^2)^2}$$

$$= \begin{cases} (\text{se } y=0) = \dfrac{2x-1}{1} = 2x-1 \\[2mm] (\text{se } x=\frac{1}{2}) = 0 \end{cases}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\,\frac{x^2 - x - x^2y^2 + xy^2}{(1+y^2)^2}$$

$$= \frac{(-2x^2y + 2xy)(1+y^2)^2 - (x^2 - x - x^2y^2 + xy^2)\,2(1+y^2)\cdot 2y}{(1+y^2)^4}$$

$$= \begin{cases} (\text{se } y=0) = 0 \\[2mm] (\text{se } y^2=1) = \end{cases}$$

$$= \frac{2xy(1-x)\cdot 2^2 - 0}{2^4}$$

$$= \frac{xy(1-x)}{2} = \left(x=\frac{1}{2}\right) = \frac{y}{8} = \pm\frac{1}{8}$$

$y = \pm 1$

$$D^2f(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \begin{matrix} \det = -1 \\ tr = 0 \end{matrix} \quad \text{sella}$$

$$D^2f(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{matrix} \det = -1 \\ tr = 0 \end{matrix} \quad \text{sella}$$

$$D^2f\left(\frac{1}{2},1\right) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{8} \end{pmatrix} \quad \begin{matrix} \det = \frac{1}{8} \\ tr > 0 \end{matrix} \quad \text{min}$$

$$D^2f\left(\frac{1}{2},-1\right) = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{8} \end{pmatrix} \quad \begin{matrix} \det = \frac{1}{8} \\ tr < 0 \end{matrix} \quad \text{max}$$

$\boxed{3B}$ $\boxed{3C}$ e $\boxed{3D}$ sono molto SIMILI

$\boxed{4A}$ $f = \dfrac{x^3 y + x^2}{1+x^2}$

$$\frac{\partial}{\partial x}\left(x \frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(x \frac{\partial f}{\partial x}\right) =$$

$$= \frac{\partial f}{\partial y} + \cancel{x \frac{\partial^2 f}{\partial x \partial y}} - \cancel{x \frac{\partial^2 f}{\partial x \partial y}} = \frac{\partial f}{\partial y}$$

$$= \frac{x^3}{1+x^2}$$

$\boxed{4B}$ ... $= \dfrac{\partial f}{\partial y} = \dfrac{x^2}{1+x^4}$

$\boxed{4C}$ ... $= \dfrac{\partial f}{\partial y} = \dfrac{x^2}{1+x^2}$

$\boxed{4D}$ ... $= \dfrac{\partial f}{\partial y} = \dfrac{x^3}{1+x^4}$