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Department of Mathematics
University of Catania

Provincia Regionale
di Catania

Azienda Provinciale Turismo
di Catania

Stability of bifurcation delay in delay differential equations

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(joint work with
Rinko Miyazaki)

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Kiyoyuki Tchizawa

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Bifurcation Delay

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-y+\mu x - x(x^2+y^2)\\ x+\mu y - y(x^2+y^2)\end{pmatrix} \equiv f(x,y,\mu)$$

μ : parameter

$$\dot{x} = \frac{dx}{dt}$$

$$\begin{cases}\dot{x} = -y+\mu x - x(x^2+y^2)\\ \dot{y} = x+\mu y - y(x^2+y^2)\\ \boxed{\dot{\mu} = \varepsilon} \qquad (\underset{\fallingdotseq 0}{\varepsilon > 0})\end{cases}$$

Consider the system in polar coordinates.

$$\begin{cases}\dot{\theta} = 1\\ \dot{\rho} = \rho(\mu-\rho^2)\\ \dot{\mu} = \varepsilon\end{cases}$$

$$\mu = \underline{\mu_0} + \varepsilon t \qquad (\underline{\mu_0 < 0})$$

We see that $\boxed{\rho=0}$ is a <u>trivial sol</u>.

Every sol. starting from (ρ_0, μ_0) first tends in finite time to a point which is very close to $(\underline{0}, \mu_0)$. Note that $\underline{0}$ is a <u>stable equilibrium</u> for $\boxed{\rho' = \rho(\mu_0 - \rho^2)}$. ↑

<u>Thm</u> (C. Lobry)

> It is clear that ρ will remain close to $\underline{0}$ until $\mu(t)$ will be positive.

☺

Changing coordinates:

$$y = \ln(\rho)$$

$$y' = \frac{1}{\rho}\frac{d\rho}{dt}$$

$$= \frac{1}{\cancel{\rho}}\cancel{\rho}(\mu - \rho^2)$$

$$= \mu - \rho^2, \qquad \boxed{\mu = \mu_0 + \varepsilon t}$$

$$\begin{cases} \boxed{y(t)' = \mu(t)} - \rho^2(t) \\ y(0) = 0 \end{cases}$$

(as soon as ρ takes the value 1

As long as

$\mu = \mu_0 + \varepsilon t$ is negative, then

$\rho^2(t)$ is negligible and

$$y(t) \simeq \mu_0 t + \varepsilon \frac{t^2}{2} \qquad (\mu = \overset{<0}{\mu_0} + \varepsilon t)$$

When $\underline{\underline{\mu = 0}}$: $\underline{t = -\frac{\mu_0}{\varepsilon}} > 0$,

$$y\left(\frac{-\mu_0}{\varepsilon}\right) \simeq \frac{-\mu_0^{2}}{2\varepsilon} .$$

when $\underline{\underline{\mu = -\mu_0}}$: $t = \frac{-2\mu_0}{\varepsilon} > 0$,

$$\underbrace{y\left(\frac{-2\mu_0}{\varepsilon}\right)} \simeq \mu_0 \frac{-2\mu_0}{\varepsilon} + \frac{\varepsilon}{2} \frac{4\mu_0^2}{\varepsilon^2} = \boxed{0}$$

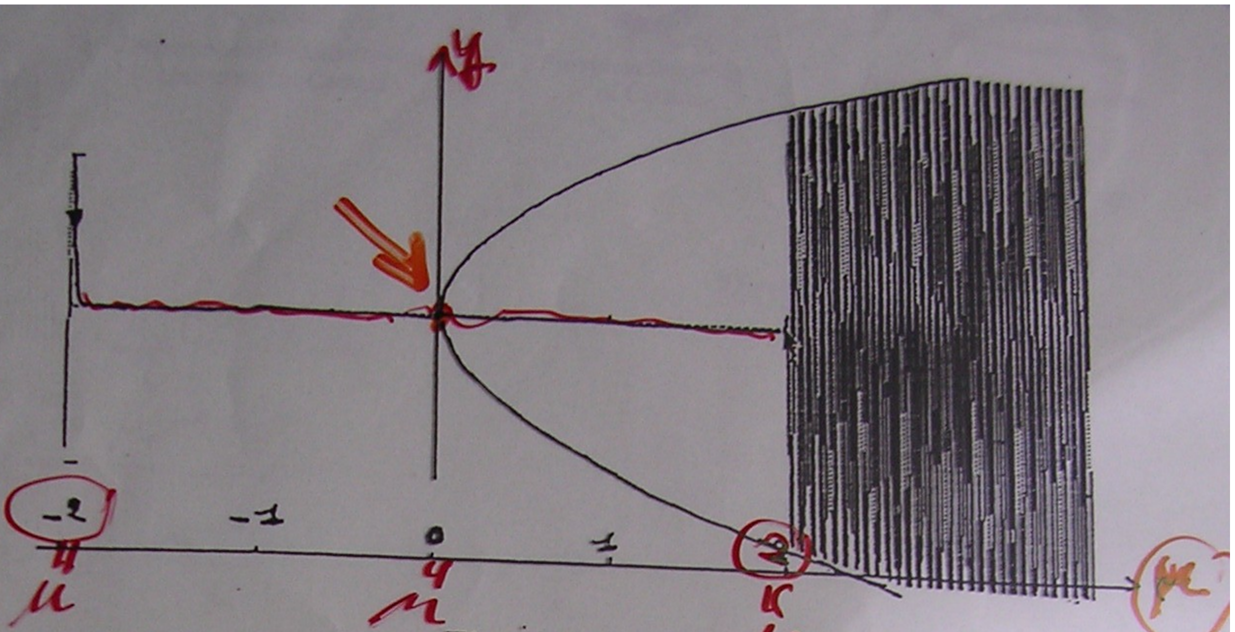


Figure 3: $\varepsilon = 0.05$ μ

This has nothing to do with the canonical description ! The mathematical explana~~tion is~~ rather simple. Consider the equations in polar coordinates. We have :

$\dot{y} =$

$\varepsilon \dot{z} = \boxed{h(x,y,z)}$ $\varepsilon \to 0$

$$\begin{cases} \theta' &= 1 \\ \rho' &= \rho(\mu - \rho^2) \\ \mu' &= \varepsilon \end{cases}$$

Forget the angular coordinate. We see that $\rho = 0$ is a trivial solution. Every solutio which starts from (ρ_0, μ_0) with $\mu_0 < 0$ first tends in finite time to a point which is ver close to $(0, \mu_0)$ because 0 is a stable equilibrium point for the equation :

$h(x,y,z) = 0$

$$\rho' = \rho(\mu_0 - \rho^2)$$

and the motion is infinitesimally slow with respect to μ. It is clear that ρ will remain close to 0 until $\mu(t)$ will be positive. Now we perform the following change o coordinates :

$$y = \ln(\rho)$$

as soon as ρ takes the value 1. We have :

$$\begin{aligned} y(t)' &= \mu(t) - \rho^2(t) \\ y(0) &= 0 \end{aligned}$$

As long as $\mu_0 + \varepsilon t$ is negative the term $\rho^2(t)$ is negligible and one has :

$$y(t) \simeq \mu_0 t + \varepsilon \frac{t^2}{2}$$

and thus, for $t = -\frac{\mu_0}{\varepsilon}$ we get :

$$y\left(\frac{-\mu_0}{\varepsilon}\right) \simeq \frac{-\mu_0^2}{2\varepsilon}$$

Let us consider the following system for $x \in \mathbb{R}^{\boxed{2}}$ with a control $\boxed{u} \in \mathbb{R}^{\boxed{1}}$:

$$(*)\begin{cases} \dot{x} = F(x, \boxed{u}, \mu) \\ F(0, 0, \mu) = 0 \end{cases}$$

, where μ is a parameter.

Then, consider a feedback control system:

$$u = -k^T(x(t) - x(t - \underset{\substack{\uparrow \\ \text{time delay}}}{\boxed{\tau}})),$$

2π

$2n\pi$

$$\frac{\partial F}{\partial u} = b,$$

where k, b are constant vectors.

Simplifying we put

$$F(x, u, \mu) \equiv f(x, \mu) - bk^T(x(t) - x(t-\tau))$$

$$f(x, \mu) = \begin{pmatrix} -x_2 + \mu x_1 - x_1(x_1^2 + x_2^2) \\ x_1 + \mu x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

$$= \underset{\substack{\shortparallel \\ A(\mu)}}{\boxed{\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (x_1^2 + x_2^2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$(*_1)$

$$\boxed{A(\mu) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}.}$$

The characteristic eq of $(*_1)$:

Time delay

$$\underbrace{p(z)} = \det\left[zI - A + (1 - e^{-\tau z})\,\mathbb{b}\,\mathbb{k}^T \right] = 0.$$

Def.

$$\Lambda = \{ z \mid p(z) = 0 \text{ and } \underbrace{\mathrm{Re}\, z} > 0 \}$$

$$\Lambda_0 = \{ z \mid p(z) = 0 \text{ and } \underbrace{\mathrm{Re}\, z} = 0 \}$$

center mfd

Assumptions:

(1) $\mathbb{k}^T \mathbb{b} = 0$ (simplicity)

(2) $\det \underbrace{[\mathbb{k}\,\mathbb{b}]} \simeq 0$

Lemma (from Rouché's Thm) $\tau = 2n\pi$

If $\boxed{\tau = 2\pi}$, then for $\boxed{\forall \mu \simeq 0}$

(i) $\mu < 0 \rightarrow \Lambda \cup \Lambda_0 = \emptyset$

(ii) $\mu = 0 \rightarrow \Lambda = \emptyset$, $\Lambda_0 = \{\pm i\}$ and $\mathrm{Re} \left.\dfrac{dz}{d\mu}\right|_{\mu=0,\, z=\pm i} > 0$

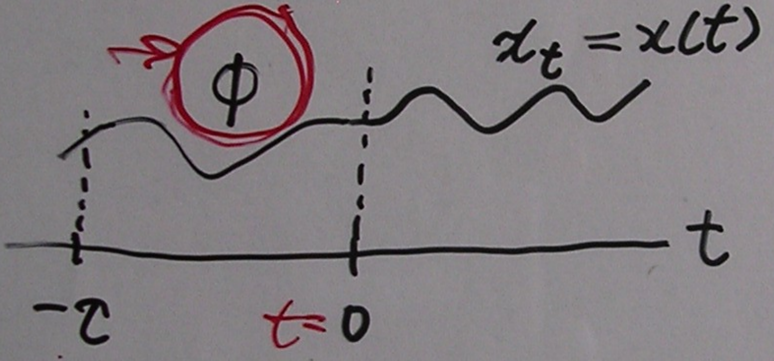
(iii) $\mu > 0 \Rightarrow \Lambda_0 = \emptyset$, Λ has 2 elements

The asymptotic behavior of sols near the equilibrium for $\underbrace{\text{small } |\mu|}_{\mu \simeq 0}$ is governed by the dynamics on the center mfd.

Def. $-\tau \leq t \leq 0$

$C(\underbrace{[-\tau, 0]}, R^2)$: functional space for $(*_1)$

$C^*(\underbrace{[0, \tau]}, R^{2^{\boxed{T}}})$: dual space of C



$$\underbrace{x_t = T(t)\phi}, \quad x_t(\theta) = x(t+\theta)$$

Def "adjoint" defined by bilinear form

$$\boxed{\phi} \in C([-\tau, 0], R^2), \quad \boxed{\psi} \in C^*([0, \tau], R^{2^T})$$

$$(\psi, \phi) = \boxed{\psi(0)}\,\boxed{\phi}(0) + \int_{-\tau}^{0} \psi(\xi + \tau)\,\boxed{\mathbb{b}\mathbb{k}^T}\,\boxed{\phi}(\xi)\,d\xi$$

$y_s = T(s)\psi$, ⟨adjoint system for $(*_1)$⟩

Let $\boxed{\mu=0}$, then the eigenspace of $(*_1)$ associated with Λ_0 is denoted by P,

$$\boxed{C = P \oplus Q} \quad , \quad \underline{Q = \emptyset}$$

A basis of P:

$$\Phi(s) = \left(\phi_1\boxed{(s)} \ \phi_2\boxed{(s)} \right)$$

$$\phi_1(s) = \begin{pmatrix} \cos s \\ -\sin s \end{pmatrix} , \quad \phi_2(s) = \begin{pmatrix} \sin s \\ \cos s \end{pmatrix}$$

function

If $\phi \in P$, then

$$\underline{\exists_1 a} \in R^2 ; \quad \phi = \Phi a$$

and the sol of $(*_1)$ through $\underline{\phi}$ at $t=0$ satisfies $x(t) = \underline{e^{A(0)t}}\,\underline{\Phi a}$

$$A(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In other words, the sol. of $(*_1)$ on the eigenspace of P behaves essentially an ord. diff. eq

$$\boxed{\frac{dy_t}{dt} = A(0)\, y_t}$$

The basis of the eigenspace of $(*_2)$ associated with Λ_0

$$\Psi(s) = \begin{pmatrix} \boxed{\psi_1(s)} \\ \boxed{\psi_2(s)} \end{pmatrix},$$

$$\psi_1(s) = (\cos s, -\sin s)$$

$$\psi_2(s) = (\sin s, \cos s)$$

$\boxed{\mu \neq 0}$

$$\left(\begin{array}{l} \overset{C}{\downarrow} \qquad\quad \overset{P}{\downarrow} \qquad\quad \overset{Q}{\downarrow} \\ \phi = \boxed{\Psi c} + \boxed{\phi^Q} \leftarrow \underline{\tau} \\ (\Psi, \phi) = \underbrace{\boxed{(\Psi, \Psi)}}_{\text{matrix}} c + \overset{\text{vector}}{\underbrace{(\Psi, \phi^Q)}_{\substack{\| \\ 0}}} \\ \Psi_0 \underbrace{\boxed{(\Psi, \phi)}}_{\text{vector}} = c \ ; \ \boxed{\Psi_0} \underset{\text{def}}{=} \boxed{(\Psi, \Psi)^{-1}} \end{array} \right)$$

$$bk = O(\varepsilon)$$

$$\underset{\begin{pmatrix}(\psi_1,\varphi_1) & (\psi_1,\varphi_2)\\ (\psi_2,\varphi_1) & (\psi_2,\varphi_2)\end{pmatrix}}{\overset{\shortparallel}{(\Psi,\Phi)}} = \underset{I}{\overset{\shortparallel}{\Psi(0)}}\,\underset{I}{\overset{\shortparallel}{\Phi(0)}} + \int_{-\tau}^{0} \underbrace{\Psi(\xi+\tau)}_{\overset{\shortparallel}{R(\xi+\tau)}}\,\boxed{\mathbf{b}}\,\boxed{\mathbf{k}^T}\,\underbrace{\Phi(\xi)}_{\overset{\shortparallel}{R(-\xi)}}\,d\xi$$

put $\tau = \boxed{2\pi}$

$$(\Psi,\Phi) = I + \int_{-2\pi}^{0} \overbrace{R(\xi)}\mathbf{b}\,\left(\overbrace{R(\xi)}\mathbf{k}\right)^T d\xi$$

$$= I + \int_{-2\pi}^{0} b\,\overbrace{\begin{pmatrix}\cos(\xi-\delta)\\ \sin(\xi-\delta)\end{pmatrix}}\,k\,\overbrace{\left(-\sin(\xi-\delta), \cos(\xi-\delta)\right.}$$

$$= I + \frac{bk}{2}\underbrace{\int_{-2\pi}^{0}}_{\shortparallel}\begin{pmatrix}-\sin 2(\xi-\delta) & \cos 2(\xi-\delta)+1\\ \cos 2(\xi-\delta)-1 & \sin 2(\xi-\delta)\end{pmatrix} d$$

$$= I + bk\pi\underbrace{\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}}$$

$$= I + bk\pi\, R\left(-\frac{\pi}{2}\right)$$

$$\left(I + bk\pi\, R\left(-\frac{\pi}{2}\right)\right)^{-1} = \frac{1}{q}\left(I + bk\pi\, R\left(\frac{\pi}{2}\right)\right) = \Psi_0$$

$$\boxed{q = 1 + (bk\pi)^2} > 0$$

$$\phi = \Phi \mathbb{C} + \phi^Q$$

$$\mathbb{C} = \overset{=(\Psi,\Phi)^{-1}}{\boxed{\Psi_0}}(\Psi, \phi)$$

$$\underset{T(t)\phi}{\underset{\shortparallel}{\boxed{x_t}}} = \Phi y_t + \underset{\downarrow\ 0}{\boxed{x_t}^Q}$$

$$y_t \overset{def}{=} \underset{(\Psi,\Phi)^{-1}}{\boxed{\Psi_0}}(\Psi, x_t)$$

Using the same arguments in [2] by J.K. Hale, $x_t^Q = O(\mu)$ $(\mu \to 0)$ whenever $|x_t|^2 < \mu$, so that the basic problem lies in $(*_3)$.

$$(*_3)\quad \frac{dy_t}{dt} = \underbrace{\underbrace{A(0)}\, y_t}_{\substack{\Lambda_0 \\ P}} + \underbrace{\boxed{\Psi_0}[\underline{\mu} y_t - |y_t|^2 y_t]}_{\substack{\Lambda \\ Q}}$$

$$= A(0) y_t + \Psi_0 [(\mu - |y_t|^2) y_t]$$

We introduce polar coordinates again

$$x_t = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ -r\sin\theta \end{pmatrix}$$

$$\begin{pmatrix} \frac{dr}{dt} \\ -r\frac{d\theta}{dt} \end{pmatrix} = R(\theta)\begin{pmatrix} \dot{x}_t \end{pmatrix} = R(\theta)\,\Phi\begin{pmatrix} \dot{y}_t \end{pmatrix}$$

Ψ_0

u

$$= R(\theta)\,\Phi\left[A(0)y_t + \boxed{\frac{\mu - r^2}{q}\left(I + bk\pi R\left(\frac{\pi}{2}\right)\right)} y_t\right]$$

$$= \begin{pmatrix} 0 \\ -r \end{pmatrix} + \frac{(\mu - r^2)r}{q}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + bk\pi\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$\begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix} = \begin{pmatrix} \frac{1}{\circled{q}}(\mu - r^2)r \\ 1 - \boxed{\frac{bk\pi}{q}(\mu - r^2)} \end{pmatrix} \quad \begin{matrix} \downarrow \\ \downarrow \end{matrix}$$

$\to 1$

$$\frac{d\mu}{dt} = \varepsilon$$

$u - r^2 < 0$

$u - r^2 > 0$

Def.

Let $N(\varepsilon)$ be an ε-nbh of the equilibrium pt.
If $\exists D \subset N(\varepsilon)$; the value of $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ decrease $(q>1)$ in D, then the system $(*)$ has strong permanence in D. feedback system

If not, it has weak permanence.

Thm

$\exists D \subset N(\varepsilon)$ ensuring strong permanence in the feedback system.

∵

$$\mathbb{b} = b\begin{pmatrix}\cos\delta \\ -\sin\delta\end{pmatrix}, \quad \mathbb{k} = k\begin{pmatrix}\sin\delta \\ \cos\delta\end{pmatrix}$$

Choose $0<\delta<\frac{\pi}{2}$ and $b=k=\sqrt{\varepsilon}$ and put

$$D = \{(r,\theta) \mid \mu - r^2 > 0\}.$$

Cor

In D^c, it has weak permanence.