Singular Perturbation Methods in Control Theory

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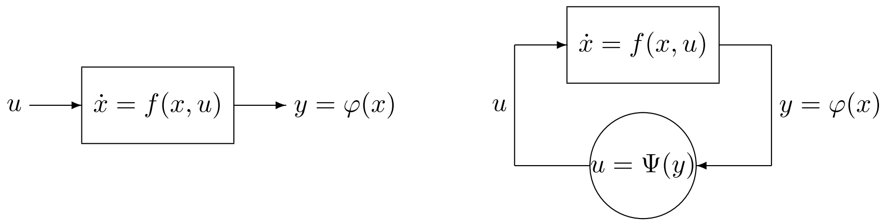
Open-loop and closed-loop systems

Open-loop system :

$$\dot{x} = f(x,u), \qquad y = \varphi(x).$$

$x \in \mathbf{R}^n$: is the *state vector*,
$u \in \mathbf{R}^p$: the *input vector*,
$y \in \mathbf{R}^q$: the *output vector*.
$\Psi : \mathbf{R}^q \longrightarrow \mathbf{R}^p$, $y \mapsto u = \Psi(y)$: a *static* feedback
Closed-loop system :

$$\dot{x} = f(x, \Psi(\varphi(x)))$$



Feedback Stabilization

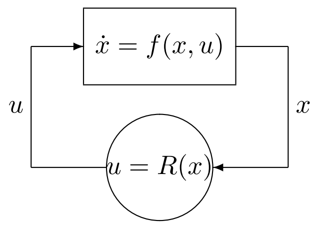
$$\dot{x} = f(x, u)$$

Assume that $f(0,0) = 0$.

Find a feedback $u = R(x)$ such that $R(0) = 0$ and the origin of the closed loop system

$$\dot{x} = f(x, R(x))$$

is GLOBALLY ASYMPTOTICALLY STABLE (GAS).



Global asymptotic stability

$$\dot{x} = F(x), \qquad F(0) = 0.$$

$x = 0$ is GAS $\Leftrightarrow x = 0$ is stable and globally attractive

Definition 1

stable $\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \forall x(t) (\|x(0)\| < \delta \Rightarrow \forall t > 0 \, \|x(t)\| < \varepsilon)$
globally attractive $\Leftrightarrow \forall x(t) \lim_{t \to +\infty} x(t) = 0$

Nonstandard characterization :
ASSUME THAT F IS STANDARD THEN

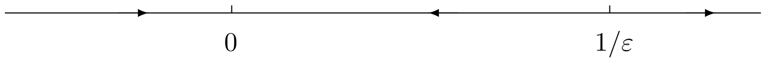
$$x = 0 \text{ is GAS } \Leftrightarrow x = 0 \text{ is S-GAS}$$

Definition 2 $x = 0$ *is* S-*GAS if and only if*

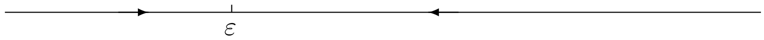
$$\forall x(t)\ \forall t\ (x(0) \textit{ limited and } t \simeq +\infty \Rightarrow x(t) \simeq 0)$$

Examples : the origin of the following systems, where $\varepsilon \simeq 0$ is S-GAS.

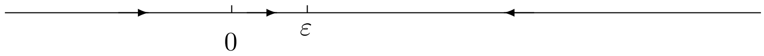
- $\dot{x} = x(\varepsilon x - 1)$. The origin is stable but not GAS.



- $\dot{x} = \varepsilon - x$. The origin is not an equilibrium.



- $\dot{x} = x^2(\varepsilon - x)$. The origin is unstable.



Practical semi-global stability

$$\dot{x} = F(x, \varepsilon)$$

Definition 3 $x = 0$ *is practically semi-globally asymptotically stable (PSGAS) when* $\varepsilon \to 0$ *if for all* $A > 0$ *and* $r > 0$ *there exist* $\varepsilon_0 > 0$ *and* $T > 0$ *such that for all* ε*, for all solution* $x(t, \varepsilon)$ *and for all time* t

$$\varepsilon < \varepsilon_0 \ \ \|x(0, \varepsilon)\| < A \textit{ and } t > T \Rightarrow \|x(t, \varepsilon)\| < r$$

Remark In the case of uniqueness of the solution $x(t, x_0, \varepsilon)$ with initial condition $x(0, x_0, \varepsilon) = x_0$, the origin $x = 0$ is PS-GAS if and only if

$$\lim_{t \to +\infty, \varepsilon \to 0} x(t, x_0, \varepsilon) = 0,$$

the limit being uniform for x_0 in any prescribed bounded domain.

Proposition 1 *If F is standard then the origin of*

$$\dot{x} = F(x, \varepsilon)$$

is PSGAS when $\varepsilon \to 0$ if and only if it is S*-GAS for all $\varepsilon \simeq 0$.*

Stabilization of slow and fast systems

The state vector (x, z) has slow components x and fast components z.

$$\dot{x} = f(x, z, u), \qquad \varepsilon\dot{z} = g(x, z, u).$$

with $f(0, 0, 0) = 0$ and $g(0, 0, 0) = 0$.

Problem : design a control

$$u = R(x, z), \text{ such that } R(0, 0) = 0$$

and the equilibrium $(0, 0)$ of the closed loop system

$$\dot{x} = f(x, z, R(x, z)), \qquad \varepsilon\dot{z} = g(x, z, R(x, z)).$$

is asymptotically stable for small ε.

The problem of singular perturbations

$$\begin{aligned} \dot{x} &= F(x, z, \varepsilon), \quad && x \in \mathbb{R}^n \qquad && \dot{x} = \tfrac{dx}{dt} \\ \varepsilon \dot{z} &= G(x, z, \varepsilon), \quad && z \in \mathbb{R}^m \qquad && \dot{z} = \tfrac{dz}{dt}. \end{aligned}$$

What is the asymptotic behavior of solutions as $\varepsilon \to 0$ and $t \in [0, T]$?

$$\lim_{\varepsilon \to 0} x(t, \varepsilon) \quad \text{and} \quad \lim_{\varepsilon \to 0} z(t, \varepsilon) \quad \text{for} \quad t \in [0, T].$$

Tykhonov's theory

$$\begin{aligned} \dot{x} &= F(x,z,\varepsilon), \;\; x(0)=\xi, \\ \varepsilon\dot{z} &= G(x,z,\varepsilon), \;\; z(0)=\zeta. \end{aligned} \tag{1}$$

We write the system at time scale $\tau = \dfrac{t}{\varepsilon}$. We obtain

$$\begin{aligned} x' &= \varepsilon F(x,z,\varepsilon), \quad \text{where } x' = \tfrac{dx}{d\tau} = \varepsilon \tfrac{dx}{dt} \\ z' &= G(x,z,\varepsilon), \quad \text{where } z' = \tfrac{dz}{d\tau} = \varepsilon \tfrac{dz}{dt} \end{aligned}$$

Now the continuous dependance of solutions with respect to the parameter ε applies :
THE SOLUTIONS ARE APPROXIMATED FOR $\tau \in [0, L]$ BY THE SOLUTIONS OF SYSTEM

$$\begin{aligned} x' &= 0 \\ z' &= G(x,z,0) \end{aligned}$$

The fast equation

$$z' = G(x, z, 0).$$

The slow manifold

$$G(x, z, 0) = 0 \Leftrightarrow z = h(x).$$

THE EQUILIBRIUM $z = h(x)$ OF THE FAST EQUTION IS ASYMPTOTICALLY STABLE UNIFORMLY IN $x \in X$

The Reduced Problem

$$\dot{x} = F(x, h(x), 0) \qquad x(0) = \xi.$$

HAS A UNIQUE SOLUTION $x_0(t) \in X$, for $0 \leq t \leq T$.

Tykhonov's theorem

Under some regularity conditions in the domain

$$0 \leq t \leq T, \quad x \in X, \quad \|z - h(x)\| \leq r, \quad 0 < \varepsilon \leq \varepsilon_0$$

every solution of (1) is defined at least on $[0, T]$ and satisfies :

$$\lim_{\varepsilon \to 0} x(t, \varepsilon) = x_0(t) \qquad \text{uniformly on } [0, T]$$

$$\lim_{\varepsilon \to 0} z(t, \varepsilon) = h(x_0(t)) \qquad \text{uniformly on } 0 < [t_0, T]$$

THERE IS A BOUNDARY LAYER IN $z(t, \varepsilon)$.
Let $\tilde{z}(\tau)$, be the solution of

$$z' = G(\xi, z, 0), \qquad z(0) = \zeta$$

We have

$$\lim_{\varepsilon \to 0} (z(t, \varepsilon) - \tilde{z}(t/\varepsilon)) = h(x_0(t)) - h(\xi) \text{ on } [0, T]$$

An example : Predators and Preys

$$\dot{x} = xz - x, \qquad \varepsilon \dot{z} = z(2 - z) - xz.$$

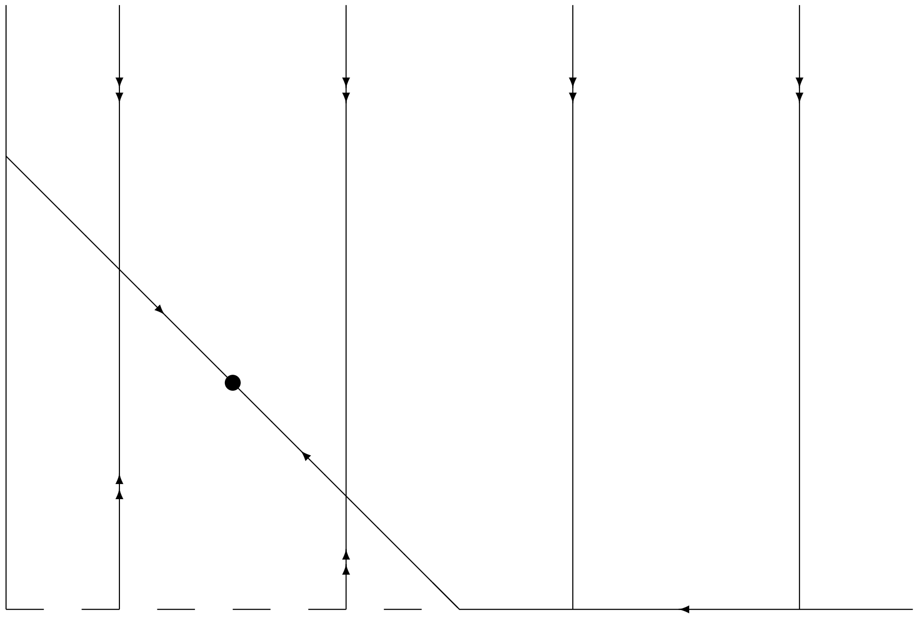
- **The fast equation is :** $z' = z(2 - z) - xz$.
- **The slow manifold is :** $z = 0$ or $z = 2 - x$.
  The component $z = 0$ is asymptotically stable if $x > 2$ and unstable if $0 < x < 2$
  The component $z = 2 - x$ is asymptotically stable if $0 < x < 2$.
- **The Reduced equation on $z = 0$ is :**

$$\dot{x} = -x.$$

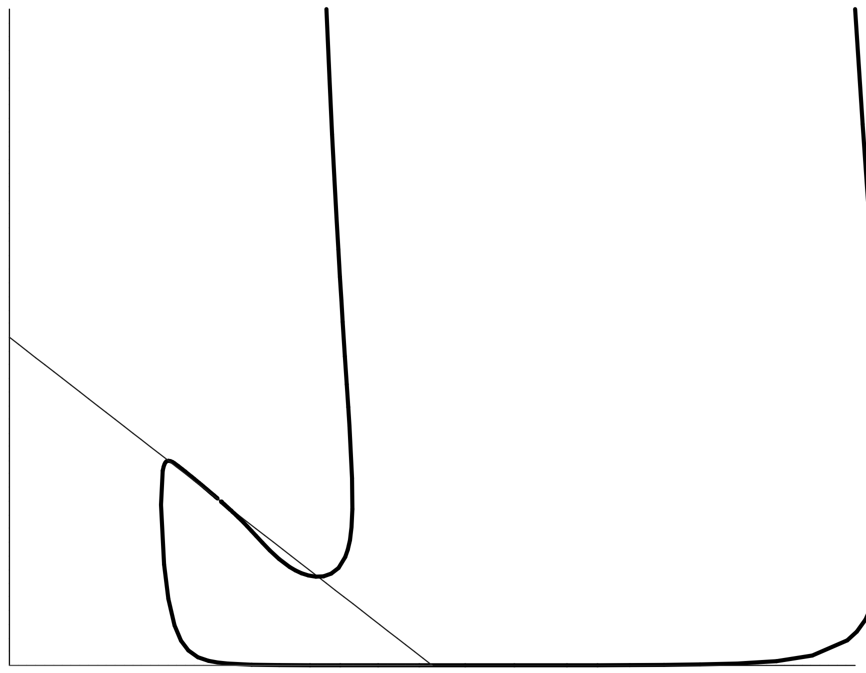
- **The Reduced equation on $z = 2 - x$ is :**

$$\dot{x} = x(1 - x).$$

Symbolical representation of the orbits of prey-predator system, according to Zeeman conventions.



Numerical orbits for $\varepsilon = 0.1$.



SINGULAR PERTURBATION THEORY
CONSIDERS ONLY
one parameter deformations

$$\dot{x} = F(x, z, \varepsilon), \qquad \varepsilon\dot{z} = G(x, z, \varepsilon).$$

and there no notion of "perturbation" in *Singular Perturbation Theory*

Actually, as noticed by Arnold :

The behaviour of the perturbed problem solutions "takes place in all systems that are close to the original unperturbed system. Consequently, one should simply study neighbourhoods of the unperturbed problem in a suitable function space. However, here and in other problems of perturbation theory, for the sake of mathematical convenience, in the statements of the results of an investigation such as an asymptotic result, we introduce (more or less artificially) a small parameter ε and, instead of neighborhoods, we consider one-parameter deformations of the perturbed systems. The situation here is as with variational concepts: the directional derivative (Gateaux differential) historically preceded the derivative of a mapping (the Fréchet differential)".

- V.I. Arnold (Ed.), *Dynamical Systems* V, Encyclopedia of Mathematical Sciences, Vol. 5, Springer-Verlag, 1994, footnote page 157.

The nonstandard notion of perturbation

Let U_0 is a standard open subset of $\mathbb{R}^d$. Let

$$f_0 : U_0 \to \mathbb{R}^m$$

be a standard function.

A point x is said to be nearstandard in U_0 if there exists a standard $x_0 \in U_0$ such that $x \simeq x_0$.

Definition 4 *A continuous function $f : U \to \mathbb{R}^m$ is said to be a perturbation of f_0, which is denoted by $f \simeq f_0$, if*

- *U contains all the nearstandard points in U_0,*
- *$f(x) \simeq f_0(x)$ for all nearstandard x in U_0.*

In other words $f \simeq f_0$ if and only if for all standard compact subset $K \subset U_0$ and for all standard $\varepsilon > 0$,

$$K \subset U \text{ and } \sup_{x \in K} \| f(x) - f_0(x) \| < \varepsilon$$

Nonstandard singular perturbation theory

Instead of considering one parameter deformations

$$\dot{x} = F(x, z, \varepsilon), \qquad \varepsilon\dot{z} = G(x, z, \varepsilon),$$

we consider perturbations

$$\dot{x} = F(x, z), \qquad \varepsilon\dot{z} = G(x, z).$$

where the vector field

$$(F, G) : D \subset \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$$

is a perturbation of a standard vector field

$$(F_0, G_0) : D_0 \subset \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$$

that is to say

$$\forall^{\mathrm{st}}(x_0, y_0) \in D_0\ \forall(x, y) \in D[x \simeq x_0 \text{ and } y \simeq y_0$$
$$\Rightarrow F(x, y) \simeq F_0(x, y) \text{ and } G(x, y) \simeq G_0(x, y)]$$

Behavior of solutions when $\varepsilon \simeq 0$ and $t \in [0, T]$

$$\begin{aligned} \dot{x} &= F(x, z), \quad x(0) = \xi, \\ \varepsilon \dot{z} &= G(x, z), \quad z(0) = \zeta. \end{aligned}$$

The fast equation

$$z' = G_0(x, z).$$

The slow manifold

$$G_0(x, z) = 0 \Leftrightarrow z = h(x).$$

THE EQUILIBRIUM $z = h(x)$ OF THE FAST EQUTION IS ASYMPTOTICALLY STABLE UNIFORMLY IN $x \in X$

The Reduced Problem

$$\dot{x} = F_0(x, h(x)) \qquad x(0) = \xi_0 := \mathrm{st}(\xi).$$

HAS A UNIQUE SOLUTION $x_0(t) \in X$, for $0 \le t \le T$.

Nonstandard Tykhonov's theorem

$$\dot{x} = F(x,z), \quad \varepsilon\dot{z} = G(x,z), \quad x(0) = \xi, \quad z(0) = \zeta. \quad (2)$$

$$\dot{x} = F_0(x, h(x)) \qquad x(0) = \xi_0 := \mathrm{st}(\xi).$$

$$z' = G_0(\xi_0, z), \qquad z(0) = \zeta_0 := \mathrm{st}(\zeta)$$

Theorem 1 *Every solution of (2) is defined at least on* $[0,T]$ *and there exists* $L \simeq +\infty$ *such that* $\varepsilon L \simeq 0$ *and we have :*

$$\begin{array}{lll} x(t,\varepsilon) \simeq x_0(t) & \textit{for all} & t \in [0,T] \\ z(t,\varepsilon) \simeq h(x_0(t)) & \textit{for all} & t \in [\varepsilon L, T] \\ z(t,\varepsilon) \simeq \tilde{z}(t/\varepsilon) & \textit{for all} & t \in [0, \varepsilon L] \end{array}$$

UNIFORM ASYMPTOTIC STABILITY

Definition 5 *The equilibrium* $z = h(x)$ *of*

$$z' = G(x, z, 0)$$

is said to be asymptotically stable uniformly for $x \in X$ *if*

$$\begin{aligned}&\forall \mu > 0\ \exists \eta > 0\ \forall x \in X\ \forall z(\tau, x)\\ &\|z(0,x) - h(x)\| < \eta \ \Rightarrow\ \forall \tau > 0\ \|z(\tau, x) - h(x)\| < \mu\\ &\text{and } \lim_{\tau \to +\infty} z(\tau, x) = h(x)\end{aligned}$$

Proposition 2 *Assume that* G, h *and* X *are standard. Then* $z = h(x)$ *asymptotically stable uniformly for* $x \in X$ *if and only if there exists* $\eta > 0$ *standard such that for all* $x \in X$, *any solution* $z(\tau, x)$ *with* $\|z(0,x) - h(x)\| < \eta$ *satisfies*

$$z(\tau, x) \simeq h(x) \text{ for all } \tau \simeq +\infty.$$

Approximations for all $0 \leq t < \infty$

$$\dot{x} = F(x, z), \quad \varepsilon \dot{z} = G(x, z), \quad x(0) = \xi, \quad z(0) = \zeta.$$

$$\dot{x} = F_0(x, h(x)) \qquad x(0) = \xi_0 := \mathrm{st}(\xi).$$

Supplementary assumption

- $F_0(0, 0) = 0$, $h(0) = 0$
- the equilibrium $x = 0$ of the reduced equation is asymptotically stable and the initial condition ξ_0 is in its basin of attraction.

Theorem 2 *The solution $x(t, \varepsilon)$, $(z(t, \varepsilon)$ is defined for all $t \geq 0$ and satisfies*

$$\begin{array}{ll} x(t, \varepsilon) = x_0(t), & \textit{for all } t \geq 0 \\ z(t, \varepsilon) = h(x_0(t)), & \textit{for all noninfinitesimal } t > 0. \end{array}$$

HENCE

$$t \simeq +\infty \;\Rightarrow\; x(t,\varepsilon) \simeq 0, \quad z(t,\varepsilon) \simeq 0.$$

BUT this result does not imply that the origin of the system is asymptotically stable.

$$\begin{aligned} \dot{x} &= x^2(\varepsilon - x) \\ \varepsilon \dot{z} &= -z \end{aligned} \tag{3}$$

- $z = 0$ is GAS for the fast equation $z' = -z$
- the slow manifold is $z = 0$
- $x = 0$ is GAS for the reduced equation $\dot{x} = -x^3$
- the origine is unstable for (3)

Khalil's Theorem : Asymptotic Stability

$$\dot{x} = F(x, z, \varepsilon), \qquad \varepsilon\dot{z} = G(x, z, \varepsilon).$$

$$F(0, 0, \varepsilon) = 0, \qquad G(0, 0, \varepsilon) = 0, \qquad h(0) = 0$$

hence the origin is an equilibrium for all $\varepsilon > 0$.

- regularitry conditions on

$$0 \leq t < \infty, \quad x \in X, \quad \|z - h(t, x)\| \leq r, \quad 0 < \varepsilon \leq \varepsilon_0$$

- $x = 0$ is exponentially stable for

$$\dot{x} = F(x, h(x), 0)$$

- $z = h(x)$ is uniformly exponentially stable on $x \in X$ for

$$z' = G(x, z, 0)$$

Definition 6 *$z = h(x)$ is exponentially stable uniformly in $x \in X$ for*

$$z' = G(x, z, 0)$$

if there exist $k > 0$, $\gamma > 0$, and $r > 0$ such that for all $x \in X$ any solution $z(\tau, x)$ with $\|z(0, x) - h(x)\| \leq r$ satisfies

$$\|z(\tau, x)\| \leq k\|z(0)\|\mathrm{e}^{-\gamma\tau} \text{ for all } \tau \geq 0.$$

Theorem 3 *There exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the origin is exponentially stable for*

$$\dot{x} = F(x, z, \varepsilon), \qquad \varepsilon\dot{z} = G(x, z, \varepsilon).$$

EXPONENTIAL STABILTY CANNOT BE REPLACED BY ASYMPTOTIC STABILITY

$$\dot{x} = x^2(\varepsilon - x), \qquad \varepsilon\dot{z} = -z$$

$x = 0$ is GAS but not exponentially stable for the reduced equation $\dot{x} = -x^3$ and the origin is unstable for the complete system.

The result holds only for $\varepsilon < \varepsilon^*$

$$\begin{cases} \dot{x} &= -z_2 \\ \varepsilon \dot{z}_1 &= -z_1 + x \\ \varepsilon \dot{z}_2 &= -z_2 + z_1 \end{cases} \tag{4}$$

- $(z_1, z_2) = (x, x)$ is an exponentially stable equilibrium of

$$z_1' = -z_1 + x$$
$$z_2' = -z_2 + z_1$$

- $x = 0$ is exponentially stable for $\dot{x} = -x$.

By Khalil's theorem, the origin of (4) is exponentially stable for small ε. When $\varepsilon > 2$, the origin is unstable.

The attractivity is not global

$$\dot{x} = -x + x^2 z, \qquad \varepsilon \dot{z} = -z. \tag{5}$$

- $z = 0$ is globally exponentially stable for the fast equation

$$z' = -z$$

- $x = 0$ globally exponentially stable for the reduced problem

$$\dot{x} = -x$$

- By Khalil's theorem, the origin of (5) is exponentially stable for small ε.

- Actually the origin is exponentially stable for all $\varepsilon > 0$, but the attractivity is not global since

$$\frac{d}{dt}(xz) = xz(xz - 1 - 1/\varepsilon),$$

shows that the hyperbola$xz = 1 + 1/\varepsilon$ is invariant.

- The basin of attraction of the origin is the set

$$B := \{(x, z) \in \mathbb{R}^2 : xz < 1 + 1/\varepsilon\}.$$

Thus the origin is not GAS for (5).

- The the origin of (5) is PSGAS when $\varepsilon \to 0$.

Practical semi global stability

We do not assume that

$$F(0,0,\varepsilon) = 0, \quad \text{et} \quad G(0,0,\varepsilon) = 0$$

Hence the origin is not an equilibrium of

$$\dot{x} = F(x,z,\varepsilon), \qquad \varepsilon\dot{z} = G(x,z,\varepsilon). \tag{6}$$

Theorem 4 *Assume that*

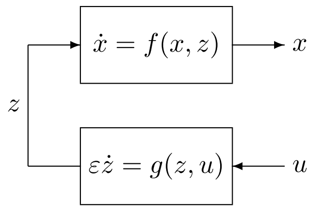
- $F(0,0,0) = 0$, $G(0,0,0) = 0$, $h(0) = 0$
- $x = 0$ *is GAS for* $\dot{x} = F(x,h(x),0)$
- $z = h(x)$ *is GAS for* $z' = G(x,z,0)$.

The origin of (6) is PSGAS as $\varepsilon \to 0$.

Stabilization of cascade systems

$$\dot{x} = f(x, z), \qquad \varepsilon\dot{z} = g(z, u).$$

with $f(0,0) = 0$ and $g(0,0) = 0$.



THE OUTPUT OF THE FAST EQUATION IS THE INPUT OF THE SLOW EQUATION

Problem : design a control $u = R(x, z)$ such that the origin of the closed loop system

$$\dot{x} = f(x, z), \qquad \varepsilon\dot{z} = g(z, R(x, z)).$$

is asymptotically stable for small ε.

A simple case (that never occur in practice !) : we assume that the slow manifold $z = h(u)$ is an exponentially stable equilibrium of the fast equation

$$z' = g(z, u).$$

- Design a control $u = u_s(x)$ such that the origin of the reduced system

  $$\dot{x} = f(x, h(u_s(x)))$$

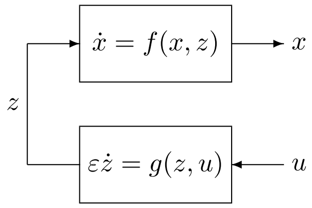
  is exponentially stable uniformly in x.

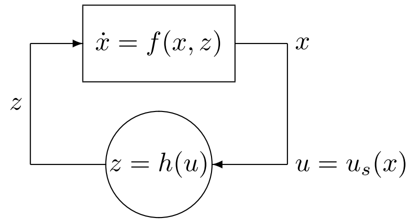
Theorem 5 *The feedback control*

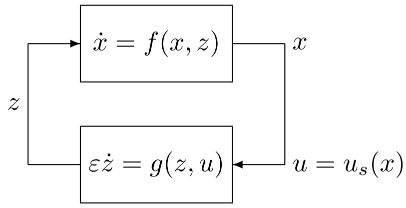
$$u = u_s(x)$$

will stabilizise the system, that is, for small ε, the origin is an exponentially stable equilibrium of the closed-loop system

$$\dot{x} = f(x, z), \qquad \varepsilon\dot{z} = g(z, u_s(x)).$$







The more realistic case : we do not assume that the slow manifold $z = h(u)$ is an exponentially stable equilibrium of the fast equation

$$z' = g(z, u).$$

Since we have the control u to our disposal, we can choose it such that the slow manifold becomes exponentially stable.

Composite control : $u = u_s(x) + u_f(x, z)$

Step 1 Design a control $u = u_s(x)$ such that the origin of the reduced system

$$\dot{x} = f(x, h(u_s(x)))$$

is exponentially stable uniformly in x.

Step 2 With the knowledge of u_s design a control law $u = u_f(x, z)$, such that $u_f(x, h(u_s(x))) = 0$, which stabilizises the fast equation

$$z' = g(z, u_s(x) + u)$$

at $z = h(u_s(x))$, that is to say the equilibrium point $z = h(u_s(x))$ of the closed-loop system

$$z' = g(z, u_s(x) + u_f(x, z))$$

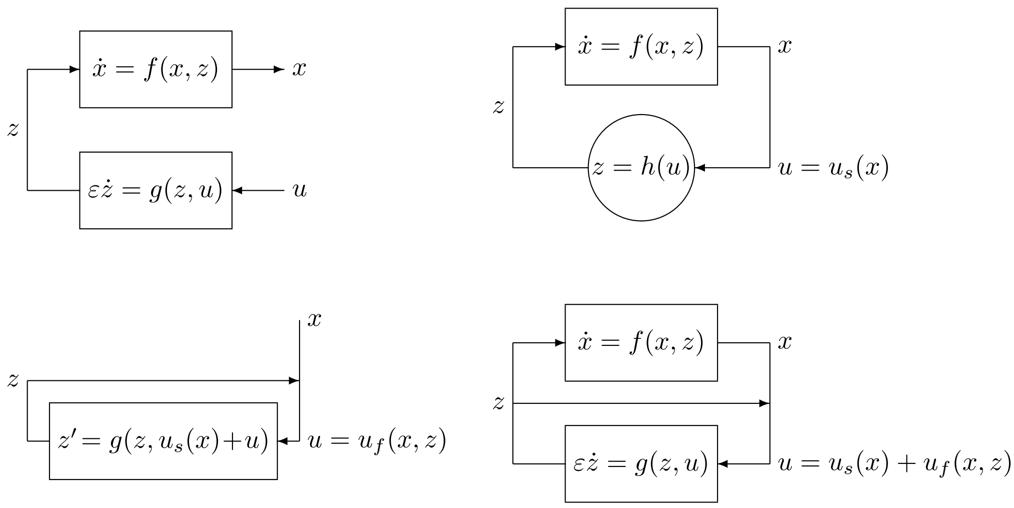
is exponentially stable uniformly in x.

Theorem 6 *The composite feedback control*

$$u = u_s(x) + u_f(x, z)$$

will stabilizise the system, that is, for small ε*, the origin is an exponentially stable equilibrium of the closed-loop system*

$$\dot{x} = f(x, z), \qquad \varepsilon\dot{z} = g(z, u_s(x) + u_f(x, z)).$$



The Peaking phenomenon

$$\begin{aligned} \dot{x} &= f(x, z), \\ \dot{z}_1 &= z_2 \\ \dot{z}_3 &= -z_1/\varepsilon^2 - 2z_2/\varepsilon. \end{aligned} \tag{7}$$

We assume that $x = 0$ is a GAS equilibrium for the zero input system

$$\dot{x} = f(x, 0).$$

The solutions of the linear equation

$$\dot{z} = G(\varepsilon)z, \qquad \text{where } G(\varepsilon) = \begin{pmatrix} 0 & 1 \\ -1/\varepsilon^2 & -2/\varepsilon \end{pmatrix}.$$

are given by $z(t, \varepsilon) = \mathrm{e}^{tG(\varepsilon)} z_0$ where

$$\mathrm{e}^{tG(\varepsilon)} = \begin{pmatrix} 1 + t/\varepsilon & t \\ -t/\varepsilon^2 & 1 - t/\varepsilon \end{pmatrix} \mathrm{e}^{-t/\varepsilon}$$

Assume that $\varepsilon \simeq 0$.

- If t is noninfinitesimal then $z(t, \varepsilon) \simeq 0$

FALSE REASONING : Since the solutions of $\dot{z} = G(z, \varepsilon)$ tend to 0 arbitrarily fast in t when $\varepsilon \to 0$, then the zero-input system

$$\dot{x} = f(x, 0)$$

takes over and drives x to zero.

- $\max_{t \geq 0} \left(\frac{t}{\varepsilon^2} \mathrm{e}^{-t/\varepsilon} \right) = \frac{1}{\varepsilon \mathrm{e}}$ is reached for $t = \varepsilon$

The interaction of this peaking with the nonlinear growth in the first equation in system (7) could destabilize system (7)

Let $f(x,z) = -(1+z_2)\frac{x^2}{2}$

$$\dot{x} = -(1+z_2)\frac{x^2}{2},$$

with initial conditions $x(0) = x_0$, $z(0) = (1,0)$. The solution is

- $x(t,\varepsilon) = \frac{x_0}{\sqrt{1+x_0^2\left[t-1+(1+t/\varepsilon)\mathrm{e}^{-t/\varepsilon}\right]}}$
- If $x_0^2 > 1$ the solution explose in a finite time $t_e(\varepsilon) > 0$ et $t_e(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Instantaneous Stability

Definition 7 *The origin of system*

$$\dot{z} = g(z)$$

is said to be instantaneously stable (IS) if for all solution $z(t)$ *and all* $t > 0$*,*

$$z(0) \text{ limited and } t \text{ noninfinitesimal} \;\Rightarrow\; z(t) \simeq 0.$$

Example If $\varepsilon \simeq 0$ then the origin of the following system is IS

$$\dot{z} = G(\varepsilon)z, \qquad \text{where} \qquad G(\varepsilon) = \begin{pmatrix} 0 & 1 \\ -1/\varepsilon^2 & -2/\varepsilon \end{pmatrix}.$$

Uniform infinitesimal boundedness (UIB)

$$\dot{x} = f(x,z), \qquad \dot{z} = g(z) \tag{8}$$

Definition 8 *System (8) is UIB if*

$$x(0), z(0) \text{ limited, and } 0 < t \simeq 0 \Rightarrow x(t) \text{ is limited}$$

Theorem 7 *Assume that f is standard and*

- *H1 : $x = 0$ is GAS for $\dot{x} = f(x,0)$,*
- *H2 : The system (8) is UIB*
- *H3 : The origin of $\dot{z} = g(z)$ is IS.*

Then the origin of (8) is S*-GAS, that is*

$$x(0), z(0) \text{ limited, and } t \simeq +\infty \Rightarrow x(t) \simeq 0 \text{ and } z(t) \simeq 0$$

Internal formulations

$$\dot{x} = f(x,z), \qquad \dot{z} = g(z,\varepsilon) \tag{9}$$

Definition 9 *The origin of system $\dot{z} = g(z,\varepsilon)$ is IS as $\varepsilon \to 0$, if for all $\delta > 0$, $A > 0$ and $T > 0$, there exists $\varepsilon_0 > 0$ such that for any solution $z(t,\varepsilon)$, if $\|z(0,\varepsilon)\| \leq A$, then $\|z(t,\varepsilon)\| < \delta$ for all $t \geq T$ and all $0 < \varepsilon < \varepsilon_0$.*

Definition 10 *The system (9) is UIB as $\varepsilon \to 0$, if for all $A > 0$, there exist $B > 0$, $t_0 > 0$ and $\varepsilon_0 > 0$ such that for any solution $x(t,\varepsilon)$, $x(t,\varepsilon)$ of system (9), if $\|x(0,\varepsilon)\| \leq A$ and $\|z(0,\varepsilon)\| \leq A$, then $\|x(t,\varepsilon)\| \leq B$ for all $t \in [0,t_0]$ and all $0 < \varepsilon < \varepsilon_0$.*

Theorem 8 *Assume that*

- *H1 :* $x = 0$ *is GAS for* $\dot{x} = f(x, 0)$,
- *H2 : system (10) is UIB as* $\varepsilon \to 0$
- *H3 : the origin of* $\dot{z} = g(z, \varepsilon)$ *is IS as* $\varepsilon \to 0$.

Then the origin of

$$\dot{x} = f(x, z), \qquad \dot{z} = g(z, \varepsilon) \tag{10}$$

is PSGAS as $\varepsilon \to 0$, *that is to say, for all* $A > 0$ *and* $r > 0$ *there exist* $\varepsilon_0 > 0$ *and* $T > 0$ *such that, for all solution* $x(t, \varepsilon)$, $z(t, \varepsilon)$, *and for all time* t *if* $\|x(0, \varepsilon)\| < A$, $\|z(0, \varepsilon)\| < A$ *then* $\|x(t, \varepsilon)\| < r$, $\|z(t, \varepsilon)\| < r$ *for all* $t > T$ *and all* $0 < \varepsilon < \varepsilon_0$.

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