Relative Infinitesimals:

Their Use in High-School mathematics

Richard O'Donovan
Collège André-Chavanne
Geneva

This is a joint project with Karel Hrbacek and this presentation should be considered as a sequel to his.

So far, we had used a simplified version of nonstandard analysis for teaching at pre-university level. We worked in an unformalised setting of $^*\mathbb{R}$

After several years, some questions arose:

The questions:

1. If $\mathbf{st}(x)$ can be used to define the derivative, why is the "updown" function

   $$f : x \mapsto 2 \cdot \mathbf{st}(x) - x$$

   not acceptable?

2. How is $f'(2 + \delta)$ calculated?

with answers that can be used in high-school...

The curse:

“We warn the reader that getting familiar with the distinction between internal and external objects is probably the hardest step in learning nonstandard analysis.”

(Benci, Forti, Di Nasso)

We tried to teach without ever explicitly using transfer or star-map, since these need some understanding of what models are: an impossible task at pre-university level.

An answer:

Relative infinitesimals
with a switch from ${}^*\mathbb{R}$ to $\mathbb{R}+$ FRIST.

If we want students to understand the concept of relative infinitesimals, we need a “story” which should help guide their intuition.

It must be noted that the understanding of these new concepts is harder for the trained mathematician who has conflicting knowledge.

Levels

We have been working so far at a very coarse level in the reals, one where there are no infinitesimals nor unlimited numbers.

Infinitesimals and unlimited numbers are at a finer level.
These numbers are infinitesimal/unlimited *with respect* to the coarser level.

And there are finer levels still.

The level of x,

noted $\mathbf{v}(x)$

is the coarsest level containing x.

$\mathbf{v}(x, y, z)$ is the coarsest level containing x, y and z

- At the coarsest level there are no infinitesimals.

- For any given level, there are numbers not at that level, there are numbers at a finer level.

- "a is at the level of b" is written
  $$a \in \mathbf{V}(b)$$

- If a number is at a given level it is also at all finer levels.

We have already shown that infinitesimlas are very intuitive. For almost all students, the existence of infinitesimals is “obvious”.

The next concept is fundamental:

A statement about x and parameters $p_1, \ldots, p_n$ is

acceptable

if it does not refer to levels or if it refers only to $\mathbf{v}(x, p_1, p_2, \ldots, p_n)$.

Rule

Only acceptable statements can be used to define sets and functions.

Answer to question 1:

$$f : x \mapsto 2 \cdot \mathbf{sh}_0(x) - x$$

is not an acceptable function (*reference to an absolute level*)

$$f : x \mapsto 2 \cdot \mathbf{sh}_x(x) - x \quad \text{is} \quad f : x \mapsto x$$

is not a problem...

Derivative

Let f be a function and I an interval with $a \in I$ and $\mathbf{v}(a, f) = \mathbf{v}(\alpha)$

f is differentiable at a iff there is an $L \in \mathbf{v}(\alpha)$ such that $\forall h \simeq_\alpha 0$

$$\frac{f(a+h) - f(a)}{h} \simeq_\alpha L$$

then the derivative is

$$f'(a) = L$$

Answer to question 2:

This definitions is for all points.

$f'(2+\delta)$ is calculated taking infinitesimals with respect to the level of $2+\delta$, i.e. the level of δ.

Checking whether an object is acceptable is purely syntactical — and quite simple:

Curse lifted.

Because our syllabus includes limits, it is important to be able to define them in this context.

(The mainstream teaching does not define the limit, or only in a handwaving fashion.)

Limits

Because $\mathbb{R}$ is complete, and because we do not extend this set, any increasing sequence bounded above has a limit.

For $\mathbf{v}(f, a) = \mathbf{v}(\alpha)$ and $L \in \mathbf{v}(\alpha)$

$$\lim_{x \to a} f(x) = L$$

$$\Longleftrightarrow$$

$$\forall x \simeq_\alpha a \quad f(x) \simeq_\alpha L$$

Derivative

Let f be a function and I an interval with $a \in I$

f is differentiable at a iff there is a value $f'(a)$ such that

$$\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) = f'(a)$$

Statements of most definitions and theorems can be the same as in mainstream mathematics.

... even though the limit is interpreted following a different definition: that of being infinitesimally close.

Compared to mainstream high-school teaching, the question now becomes:

Do we want a theory in which we can give a rigorous definition of limits and prove (and the students prove) the rules about computation with limits, or do we stick to fuzzy handwaving methods?

Our syllabus requires that students study real functions, limits, derivatives and integrals. With relative infinitesimals we do exactly that.

And we have infinitesimals.

And we avoid ε-δ formalism, with all its technical difficulties.

Irony

Acceptable statements and acceptable functions are those that transfer, but because acceptable properties apply to all values, for most proofs transfer is not needed!

The Closure Principle is used:

$$f(x) \in \mathbf{V}(x, f)$$

Transfer is needed (explicitly) only for the proofs of theorems about continuity: for higher level mathematics students.

This approach has been “beta-tested” during a maths weekend on two students who had already studied the usual ϵ-δ method.

Iaroslav Gaponenko and Fabian Santi

(with colleague Olivier Lessmann)