Hypercomputability.
An application of non standard methods to recursion theory and computable analysis

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Abstract

The aim of this talk is to apply non standard methods to recursion theory [5] and computable analysis [7].
Some direct consequences of this approach are a (new) notion of *hypercomputable function* and the extension of the theory of representations (see [7]) to non standard universes.
The hypercomputable functions we will introduce can be considered as functions computed by abstract Turing machines which transform internal digital hypersequences. Such hypersequences can be either hyperfinite strings of symbols from a given alphabet, or elements of the enlarged Baire space.
In the first case we work essentially in the field of recursion theory, more precisely we study computable functions on $^*\mathbb{N}$.
In the second case hypersequences can be used to denote individuals in enlargements of mathematical structures (like $\mathbb{R}$) whose elements are usually coded by infinite sequences in classical computable analysis. In this way we obtain a concept of computability for such enlarged structures.
Consequently, we show that some basic classical results of non standard analysis ([6]) may be expressed in terms of *hypercomputation*, in the same way (standard) computable analysis provides effective versions of standard analysis theorems.

1 Hypercomputable functions

We apply conceptual instruments of non standard analysis to recursion theory and computable analysis. In this way we obtain a non standard extension of the classical notion of computability.

Following the terminology used by R. Goldblatt in his introduction to non standard analysis ([2]), we sometimes use the prefix "*hyper*" to refer to non standard extensions of standard notions. In this sense we will speak of *hypercomputability*. The word "hypercomputability" has been used already in other contexts, usually to denote abstract notions of computability beyond the limit of Church-Turing's thesis. These concepts share with our notion a high level of abstraction not reducible to the classical concrete notion of computability. Therefore, we think we can use the same word also in our context.

We work with topological spaces used in recursion theory and computable analysis:

- the discrete topology on $\Sigma^{<\mathbb{N}}$, i.e. the set of all finite words from a countable alphabet Σ containing 0,1. We use alphabets $2 = \{0, 1\}$, $4 = \{0, 1, -, /\}$, $\mathbb{N}$;
- the Baire space $\mathbb{B}$, i.e. the product topology on $\mathbb{N}^{\mathbb{N}}$.

These are elements of any *universe* (or *"superstrucure"*) $\mathbb{U}(X)$ such that $\mathbb{R} \subseteq X$ (without loss of generalization this holds also for $4^{<\mathbb{N}}$). We enlarge this universe by the usual ultrafilter method, obtaining $\mathbb{U}({}^*X)$, which is the *enlargement* of $\mathbb{U}(X)$. See [2] or [4] for more details.

As in [4], we assume that the elements of X are *atomic*, thus if $b \in X$ there is no $a \in \mathbb{U}(X)$ such that $a \in b$.
Any element ${}^*x \in \mathbb{U}({}^*X)$, for $x \in \mathbb{U}(X)$, is called *standard*. When $x \in X$, we often identify x with its image ${}^*x \in {}^*X$.
Given an $a \in \mathbb{U}(X) \setminus X$, we say that ${}^*a \in \mathbb{U}({}^*X)$ is the *enlargement* of a.
We recall that an element $a \in \mathbb{U}({}^*X)$ is *internal* if there is $b \in \mathbb{U}(X)$ such that $a \in {}^*b$.
An element that is not internal is *external*.

Observe that we have previously called $\mathbb{U}({}^*X)$ the "enlargement" of $\mathbb{U}(X)$. We are thus using the word "enlargement" with two slightly different meanings, for elements $a \in \mathbb{U}(X)$ and for $\mathbb{U}(X)$ respectively. This produces no ambiguity, since $\mathbb{U}(X)$ is not an element of the universe, and so ${}^*\mathbb{U}(X)$ is not the enlargement of $\mathbb{U}(X)$ (in the second meaning of the word). We use the same word, since in both cases we refer to extensions (indeed "enlargement") of mathematical objects.

We use a language $\mathcal{L}_X$ defined as in [2] or [4] to speak of $\mathbb{U}(X)$. All elements $a \in \mathbb{U}(X)$ are constants of the language, and all quantifiers occurring in $\mathcal{L}_X$-formulas are of the type $\exists x \in t$ and $\forall x \in s$, where t, s are $\mathcal{L}_X$-terms. The only predicates of the language are $\in$ and $=$.
Mathematical structures and entities (metric spaces, topological spaces, continuous functions...) have corresponding extensions ("enlargements") in $\mathbb{U}(^*X)$. In particular, this holds for notions of computable analysis, like "computable metric space", "realization", "representation", and so on. We refer the reader to [7] and [1] for a satisfactory introduction to these concepts, but we will provide here some basic definition. These concepts can be then enlarged in $\mathbb{U}(^*X)$, satisfying the transfer principle. We won't provide here the necessary technical details, but we assume that the transfer principle assures that all enlarged notions satisfy the same $\mathcal{L}_X$-properties of the corresponding original standard notions.
We will refer to enlarged notions through the expression "*internally*". In this sense we will speak for example of "internally computable function", "internally continuous function", or "internally computable metric space".
For the sake of linguistic soundness, we will sometimes use the simpler expression "*internal*", instead of "internally". For example, we will speak of "internal realization", rather than "internally realization", even if, properly speaking, an internal realization of a set X would be an internal surjective function $\delta :\subseteq \mathbb{B} \to X$, whereas we mean a(n internal) surjective functions $\delta :\subseteq {}^*\mathbb{B} \to X$. Similar arrangements will be taken for analogous cases.

Recall that ${}^*\mathbb{N} = \mathbb{N} \cup \mathbb{N}^\infty$, where $\mathbb{N}^\infty$ is the external set of unlimited hypernatural numbers.
We use lower-case letters n, m, k, i, j for standard natural numbers, and corresponding capital letters for unlimited or generic hypernatural numbers.

The Baire topology is a metric space: for $p \neq q \in \mathbb{B}$, $d(p,q) = 2^{-y}$, where $y = \mu z \in \mathbb{N} : p(z) \neq q(z)$.
${}^*\mathbb{B}$ is the internal(ly) metric space of all *internal(ly)* sequences $p : {}^*\mathbb{N} \to {}^*\mathbb{N}$.

Given any two hypersequences $p, q \in {}^*\mathbb{B}$, $d(p,q)$ is infinitesimal ($d(p,q) \simeq 0$) if and only if either $p = q$ or the least z such that $p(z) \neq q(z)$ is unlimited (thus $z \in \mathbb{N}^\infty$).

Therefore for any $p \in {}^*\mathbb{B}$, its monad $\mu(p)$ is the set:

$$\{q \in {}^*\mathbb{B} : \forall y \in {}^*\mathbb{N}[q(y) \neq p(y) \Rightarrow y \in \mathbb{N}^\infty]\}.$$

Since $\mathbb{B}$ is a limited metric spaces, the same is ${}^*\mathbb{B}$ ($d(p,q) \leq 1$ for all $p, q \in {}^*\mathbb{B}$). Nevertheless, the Baire space is not compact and its enlargement ${}^*\mathbb{B}$ contains elements which are not near-standard (all sequences p for which there is a standard n such that $p(n) \in \mathbb{N}^\infty$).

The enlargement ${}^*(\Sigma^{<\mathbb{N}}) = {}^*\Sigma^{<{}^*\mathbb{N}}$ is the set of all internal words of hyperfinite length. If Σ is finite, then ${}^*\Sigma = \Sigma$.

Notation 1.1 *Given any $p \in \mathbb{B}$ and $n \in \mathbb{N}$, we denote by $p[n]$ the initial segment of p of length n.*
For $p \in \mathbb{B}$ and $n \in \mathbb{N}$, we write "$n \lhd p$" if $n \in \mathrm{range}(p)$ (thus p lists n).
The same conventions are used for $p \in {}^*\mathbb{B}$ and $N \in {}^*\mathbb{N}$.

1.1 Turing hypermachines

We use Type-2 Turing machines described in [7]. In this context we define the notion of computable function:

Definition 1.2 ***Computable function.*** *A function $G :\subseteq \mathbb{B}^n \to \mathbb{B}$, for $n \in \mathbb{N}$, is* computable *if there is a Type-2 Turing machine $\mathcal{M}$ that transforms any input $(p_1, ..., p_n) \in dom(G) \subseteq \mathbb{B}^n$ into the sequence $G(p_1, ..., p_n) \in \mathbb{B}$.*
By this definition, computations run by $\mathcal{M}$ may converge also on a larger set than $\mathrm{dom}(G)$*.*

Definition 1.3 *Comp is the set of all computable functions $F :\subseteq \mathbb{B}^n \to \mathbb{B}$, for $n \in \mathbb{N}$.*
Rec is the set of all recursive functions $f :\subseteq \mathbb{N}^m \to \mathbb{N}$, for $m \in \mathbb{N}$.

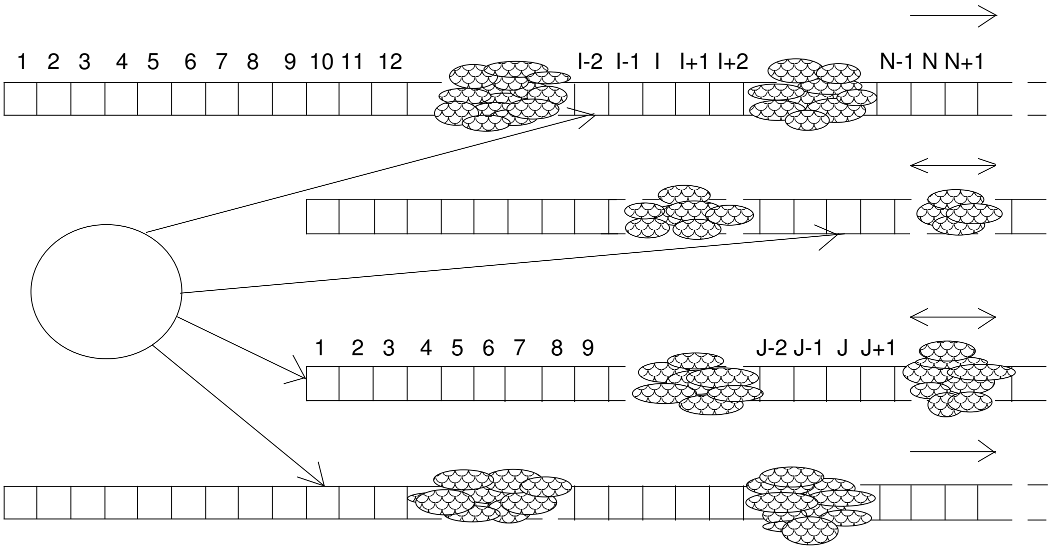
By using Cantor pairing functions, one can not be rigorous in specifying the ariety of functions in *Comp* and *Rec*.

Comp and *Rec* are elements of $\mathbb{U}({}^*X)$, and therefore are enlarged to ${}^*Comp$ and ${}^*Rec$ in $\mathbb{U}({}^*X)$.
According to our linguistic conventions, functions in ${}^*Comp$ and ${}^*Rec$ are called *internally computable* (or *hypercomputable*) and *internally recursive* (or *hyperrecursive*), respectively.

By transfer principle, all computability concepts can be then immediately extended to ${}^*Comp$ and ${}^*Rec$. For example there is a hyper-enumeration $\{\varphi_N\}_{N \in {}^*\mathbb{N}}$ of ${}^*Rec$ satisfying both the utm- and the smn-properties (see [7]) (admissible enumeration).

In analogy with standard computability theory, it may be of some help to consider hypercomputable functions of the set ${}^*Comp$ as if they were computed by suitable Turing machines. Actually, these machines should be conceived as "hyperphysical" extensions of the standard TTE-Turing-machines.
An hyper-Turing machine will be characterized by the following features:

1. hyperfinitely many input tapes, an output tape and hyperfinitely many working tapes, which are hypersequences of cells;
2. a program made by a hyperfinite list of instructions.



An example of Turing hypermachine. I and J are unlimited hypernatural numbers. Each cloud hides uncountably many cells galaxies.

A Turing hypermachine can be conceived as a device working in a "hyperphysical time and space", where temporal intervals are made of hyperfinite sets of instants.

2 Theory of naming systems

For an introduction to the theory of naming systems see [7].

Definition 2.1 ***Naming systems.***
A notation *of a set* $A \in \mathbb{U}(X)$ *is a surjective function* $\nu :\subseteq \Sigma^{<\mathbb{N}} \to A$*.*
A representation *of a set* $A \in \mathbb{U}(X)$ *is a surjective function* $\delta :\subseteq \mathbb{B} \to A$*.*
A naming system *of a set* $A \in \mathbb{U}(X)$ *is either a notation or a representation of* A*.*

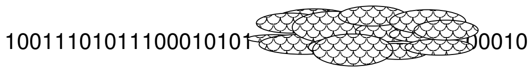
In the case $\Sigma = \mathbb{N}$, since $|\mathbb{N}^{<\mathbb{N}}| = |\mathbb{N}|$, it is enough to define notations with domain in $\mathbb{N}$.

Example 2.2 ***Some useful notations.***

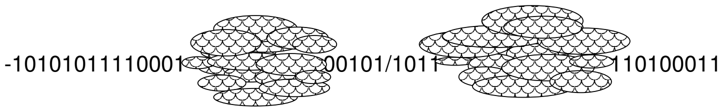
- **Natural numbers:**
  - *For* $\Sigma = 2$ *define* $\nu_{\mathbb{N}}(0) = 0$ *and* $\nu_{\mathbb{N}}(a_k, ..., a_0) = \sum_{i=0}^{k} a_i 2^i$*, for* $a_k \neq 0$*;*
  - *For* $\Sigma = \mathbb{N}$ *define* $\nu_{\mathbb{N}}(n) = n$*;*
- **Integer numbers:**
  - *For* $\Sigma = 4$ *define* $\nu_{\mathbb{Z}}(w) = \nu_{\mathbb{N}}(w)$ *and* $\nu_{\mathbb{Z}}(-w) = -\nu_{\mathbb{N}}(w)$*, for* $w \in \mathrm{dom}(\nu_{\mathbb{N}})$*;*
  - *For* $\Sigma = \mathbb{N}$ *define* $\nu_{\mathbb{Z}}(2n) = n$ *and* $\nu_{\mathbb{Z}}(2n+1) = -n$*;*
- **Rational numbers:**
  - *For* $\Sigma = 4$ *define* $\nu_{\mathbb{Q}}(u/v) = \nu_{\mathbb{Z}}(u)/\nu_{\mathbb{N}}(v)$*, for* $\nu_{\mathbb{N}}(v) \neq 0$*;*
  - *For* $\Sigma = \mathbb{N}$ *define* $\nu_{\mathbb{Q}}\langle n_1, n_2\rangle = \nu_{\mathbb{Z}}(n_1)/(\nu_{\mathbb{N}}(n_2) + 1)$*.*

All notations $\nu_{\mathbb{N}}$, $\nu_{\mathbb{Z}}$, $\nu_{\mathbb{Q}}$ are in $\mathbb{U}(X)$. Therefore they have immediate enlargements in $\mathbb{U}({}^*X)$. By transfer principle they are notations of ${}^*\mathbb{N}, {}^*\mathbb{Z}, {}^*\mathbb{Q}$, respectively, and we denote them by $\nu_{^*\mathbb{N}}$, $\nu_{^*\mathbb{Z}}$, $\nu_{^*\mathbb{Q}}$.

Observe that for $\Sigma = 2$ a $\nu_{^*\mathbb{N}}$-name of an $N \in {}^*\mathbb{N}$ is a string like:



whose length is $\lceil {}^*\log_2(N) \rceil$. The cloud hides uncountably many galaxies (see [2]) of digits in case $N \in \mathbb{N}^\infty$. For $\Sigma = 4$, a $\nu_{{}^*\mathbb{Q}}$-name of a (negative) hyperrational number is a string like:



where each cloud hides possibly uncountably many digits galaxies.
For $\Sigma = \mathbb{N}$, a $\nu_{{}^*\mathbb{N}}$-name of a non standard hyperrational number is a number $\langle N_1, N_2 \rangle \in {}^*\mathbb{N}$, where either N_1 or N_2 are unlimited.

2.1 Enlarging representations

By transfer principle, the notion of representation can extended to non standard universes, and the same is for the notion of admissible representation, which we now consider only for the case of computable metric spaces.

Definition 2.3 ***Computable metric space.*** *A* computable metric space (Y, d, ν) *is a* 3*-tuple where* (X, d) *is a nonempty complete metric space and:*

- $\nu : \mathbb{N} \to Y$ *is a dense sequence in* Y*;*
- *the set* $e_d = \{ (n, m, k, i) \in \mathbb{N}^4 : \nu_{\mathbb{Q}}(n) < d(\nu(m), \nu(k)) < \nu_{\mathbb{Q}}(i) \}$ *is r.e.*

Definition 2.4 ***Admissible representations of computable metric spaces.*** *Given a computable metric space* $\mathbf{Y} = (Y, d, \nu) \in \mathbb{U}(X)$*, let* $(I_n)_{n \in \mathbb{N}}$ *be a computable enumeration of all balls with center in* range(ν) *and radius in* $\mathbb{Q}^+$*. We define then the following representations of* Y*, for* $p \in \mathbb{B}$*:*

Standard representation: $\delta_{\mathbf{Y}}(p) = y \in Y \iff \forall n \in \mathbb{N}(y \in I_n \leftrightarrow n + 1 \lhd p)$*;*

Cauchy representation: $\delta^C_{\mathbf{Y}}(p) = y \in Y \iff \forall n \in \mathbb{N} : d(y, \nu(p(n))) < 2^{-n}$*.*

Definition 2.5 ***Realizations.*** *Let $\gamma_1 :\subseteq \mathbb{B} \to X_1$ and $\gamma_0 :\subseteq \mathbb{B} \to X_0$ be representations. Let $f :\subseteq X_1 \to X_0$ be a function.*
The function $F :\subseteq \mathbb{B} \to \mathbb{B}$ is a (γ_1, γ_0)-realization of f *if and only if $\gamma_0 \circ F(p) = f \circ \gamma_1(p)$ for all $p \in \mathbb{B}$ such that $\gamma_1(p) \in \mathrm{dom}(f)$.*
The function f is said to be (γ_1, γ_0)-continuous *if and only if it has a continuous (γ_1, γ_0)-realization.*
The function f is said to be (γ_1, γ_0)-computable *if and only if it has a computable (γ_1, γ_0)-realization.*

$$
\begin{array}{ccc}
\mathbb{B} & \xrightarrow{F} & \mathbb{B} \\
{\scriptstyle \gamma_1}\downarrow & & \downarrow{\scriptstyle \gamma_0} \\
X_1 & \xrightarrow[f]{} & X_0
\end{array}
$$

By enlargement construction, the notions of "internally (γ_1, γ_0)-continuous" and internally "(γ_1, γ_0)-computable" function are obtained.
$\mathcal{L}_X$-properties of standard computability are transferred to internal computability. Therefore, we can directly deduce the existence of a certain hypercomputable function, if we know that there is a standard function doing something similar in the standard world. On the other hand, we can prove the existence of a standard computable function, provided that we know that there is an internally computable function that behaves similarly. The following theorem shows non standard extensions of basic results of computable analysis:

Lemma 2.6 *The followings hold in $\mathbb{U}({}^*X)$:*

1. (a) *the composition $G \circ F$ of any two internally computable functions $F :\subseteq {}^*\mathbb{B} \to {}^*\mathbb{B}$, $G :\subseteq {}^*\mathbb{B} \to {}^*\mathbb{B}$ is internally computable;*

   (b) *given any two internally $(\delta_{\mathbf{Y}_1}, \delta_{\mathbf{Y}_2})$- and $(\delta_{\mathbf{Y}_2}, \delta_{\mathbf{Y}_3})$-computable functions $f :\subseteq Y_1 \to Y_2$ and $g :\subseteq Y_2 \to Y_3$, for $\mathbf{Y}_i$ $(1 \leq i \leq 3)$ an internally computable metric space, the composition $g \circ f$ is internally $(\delta_{\mathbf{Y}_1}, \delta_{\mathbf{Y}_3})$-computable;*

2. (a) *any internally computable function $F :\subseteq {}^*\mathbb{B} \to {}^*\mathbb{B}$ is internally continuous;*

   (b) *given any two internally computable metric spaces $\mathbf{Y}_1, \mathbf{Y}_0$ and any internal function $f :\subseteq Y_1 \to Y_0$, if f is internally $(\delta_{\mathbf{Y}_1}, \delta_{\mathbf{Y}_0})$-computable then it is internally $(\delta_{\mathbf{Y}_1}, \delta_{\mathbf{Y}_0})$-continuous;*

3. *given any two internally computable metric spaces* $\mathbf{Y}_1, \mathbf{Y}_0$ *and any internal function* $f :\subseteq Y_1 \to Y_0$*, the function* f *is internally continuous if and only if it has an internally continuous* $(\delta_{\mathbf{Y}_1}, \delta_{\mathbf{Y}_0})$*-realization.*

Point 3 is the non standard extension of the Main Theorem in [7] (originally formulated in [3]).

3 Non standard effective versions of non standard classical theorems

Non standard analysis deals mainly with external concepts: all basic notions of the theory (infinitesimal numbers, unbounded numbers, monads...) are external entities. Therefore, to prove hypercomputable versions of classical non standard analysis theorems we will need to focus on the behavior of hypercomputable functions on external subsets of their domains.
In this way we will be able to *effectivize* (in a non standard way) some classical results. Among these, some theorems by the founder of nonstandard analysis A. Robinson (see [6]). We begin with the following:

Theorem 3.1 *[A. Robinson] Let a topological space* $(Y, \Upsilon) \in \mathbb{U}(X)$ *be given, for* Υ *a topology on a set* Y*, and let* $({}^*Y, {}^*\Upsilon) \in \mathbb{U}({}^*X)$ *be its enlargement. If a point* $y \in Y$ *belongs to the closure* $\overline{A}$ *of a set* $A \subseteq Y$ *then* $\mu({}^*y) \cap {}^*A \neq \emptyset$ *(for* $\mu({}^*y)$ *the monad of* ${}^*y$*).*

We formulate an hypercomputable version of this result with respect to metric spaces and open sets, through the following representation:

Definition 3.2 *Let a computable metric space* $\mathbf{Y} = (Y, d, \nu)$ *be given. Let* $\mathcal{O}(Y)$ *be the topology generated by its metric* d*. We define then* θ_+ *as the following representation of* $\mathcal{O}(Y)$*:*

$$\theta_+(p) = O \in \mathcal{O}(Y) \Longleftrightarrow O = \bigcup_{n+1 \lhd p} I_n$$

for $p \in \mathbb{B}$ *(see Definition 2.4).*

Theorem 3.3 *Let* $\mathbf{Y} = (Y, d, \nu) \in \mathbb{U}(X)$ *be a computable metric space and let* ${}^*\mathbf{Y} \in \mathbb{U}({}^*X)$ *be its enlargement. There is an internally computable function* $F :\subseteq {}^*\mathbb{B} \times {}^*\mathbb{B} \to {}^*\mathbb{B}$ *such that for any open* $O \subseteq Y$*, any* $y \in \overline{O}$*, and any standard* $p, q \in \mathbb{B}$ *with* $\delta_{\mathbf{Y}}(p) = y$*,* $\theta_+(q) = O$*:*

$$\delta_{{}^*\mathbf{Y}}(F({}^*p, {}^*q)) \in \mu({}^*y) \cap {}^*O.$$

3.0.1 Computability properties of overflow and spillover principles

We now show some applications of hypercomputability to two basic non standard principles: overflow and spillover (see [2]).
The following theorem is a well known example of overflow:

Theorem 3.4 ***Robinson's Sequential Lemma.*** *Let $(s_N)_{N\in{}^*\mathbb{N}}$ be an internal sequence in ${}^*\mathbb{R}$ with $s_n \simeq 0$ for all $n \in \mathbb{N}$. There is then an $M \in \mathbb{N}^\infty$ such that $s_N \simeq 0$ for all $N \leq M$.*

This statement has been proved in [4] in an "almost hypercomputable" way. An immediate effective version is the following:

Theorem 3.5 *There is an internally $(\rho^{{}^*\mathbb{N}}, \rho)$-computable function f mapping each internal sequence $(s_N)_{N\in{}^*\mathbb{N}}$ in ${}^*\mathbb{R}$ with $s_n \simeq 0$ for all $n \in \mathbb{N}$ to an $M \in \mathbb{N}^\infty$ such that $s_N \simeq 0$ for all $N \leq M$.*

We now consider an example for the spillover principle taken from [2]:

Theorem 3.6 *Let an internal set $A \subseteq {}^*\mathbb{R}$ be given. If A has arbitrarily large limited members, then it has a positive unlimited member.*

Classically, this is a direct consequence of the internal Dedekind completeness. Differently from Theorem 3.4, but similarly to Theorem 3.1, we consider only a special application of the statement. We need the following representation:

Definition 3.7 *Let a computable metric space $\mathbf{Y} = (Y, d, \nu)$ be given. Let $\mathcal{A}(Y)$ be the class of all closed subsets of Y. We define the following representations ψ_+, ψ of $\mathcal{A}(Y)$:*

$$\begin{aligned} \psi_+(p) = A \in \mathcal{A}(Y) &\iff (A \cap I_n \neq \emptyset \leftrightarrow n+1 \lhd p) \\ \psi(p) = A \in \mathcal{A}(Y) &\iff (p = \langle q, r\rangle \wedge \psi_+(q) = A \wedge \theta_+(r) = Y \setminus A) \end{aligned}$$

for $p, q, r \in \mathbb{B}$, $n \in \mathbb{N}$, and $\lambda(p,q).\langle p, q\rangle$ a Cantor pairing function.

We have then:

Theorem 3.8 *There is an internally (ψ, ρ)-computable function f mapping any hyperclosed set $A \subseteq {}^*\mathbb{R}$ with arbitrarily large limited members to a positive unlimited member in it.*

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