Čech cohomology of definable sets in o-minimal structures

A. Fornasiero
Università di Pisa
Dipartimento di Matematica “L. Tonelli”
fornasiero@mail.dm.unipi.it

International Congress
”NonStandard Methods and Applications in Mathematics”
Pisa, May 25-31, 2006

Joint work with A. Berarducci

O-minimal spectrum

M is an o-minimal structure expanding an ordered field (in some language $\mathcal{L}$).

Definable := “definable with parameters”,
$\emptyset$-definable := “definable without parameters”.

Let $A \subseteq M^n$ be definable. The o-minimal spectrum of A is $\widetilde{A}$, the set of complete types of A. Basis of the spectral topology: the sets of the form

$$\widetilde{U} := \{\, q \in \widetilde{A} : U \in q \,\},$$

where U is a definable subset of M^n which is open in the topology of M^n.

The o-minimal spectrum generalises the real spectrum for real closed fields to the o-minimal situation.

2-1

The spectral topology is coarser than the Stone topology, therefore $\widetilde{A}$ is quasi-compact. However, it is not T_1: not all points are closed.

The subspace $\underset{\sim}{A}$ of closed points of $\widetilde{A}$ is Hausdorff and compact and dense in $\widetilde{A}$. The map

$$\pi : \widetilde{A} \to \underset{\sim}{A}$$

sending $q \in \widetilde{A}$ to the unique closed point in the closure of q is a continuous and closed surjection.

Example: $M = \mathbb{R}$, $q(x) = 0^+ := \{ 0 < x < r : r \in \mathbb{R} \}$. q is not a closed point. $\pi(q) = 0$.

Čech cohomology

X is a topological space, $\mathscr{A}$ is a collection of nonempty subsets of X. The nerve of $\mathscr{A}$, denoted by $\mathcal{N}(\mathscr{A})$, is the simplicial complex whose simplices are the finite nonempty subsets of $\mathscr{A}$ with nonempty intersection. The vertices of $\mathcal{N}(\mathscr{A})$ are the elements of $\mathscr{A}$.
$\mathcal{H}^\star(\mathscr{A})$ is the simplicial cohomology of $\mathcal{N}(\mathscr{A})$. If $\mathscr{B}$ is refinement of $\mathscr{A}$, the simplicial map $\mathscr{B} \to \mathscr{A}$ induces a map in cohomology $\mathcal{H}^\star(\mathscr{A}) \to \mathcal{H}^\star(\mathscr{B})$.

The Čech cohomology of X is given by

$$\check{\mathcal{H}}^\star(X) := \varinjlim \mathcal{H}^\star(\mathscr{A}),$$

taking finer and finer coverings $\mathscr{A}$ of X.

For every covering $\mathscr{A}$ of X there is a canonical map

$$\iota^X_{\mathscr{A}} : \mathcal{H}^\star(\mathscr{A}) \to \check{\mathcal{H}}^\star(X)$$

Sites

A site $\mathfrak{F}$ on X is given by a collection $\mathrm{Obj}(\mathfrak{F})$ of open subsets of X (the admissible open sets) and, for every $A \in \mathrm{Obj}(\mathfrak{F})$, a collection of coverings of A by admissible open sets (the admissible open coverings of A), satisfying certain axioms.

The Čech cohomology of X with respect to $\mathfrak{F}$ is

$$\check{\mathcal{H}}^\star_{\mathfrak{F}}(X) := \varinjlim_{\mathscr{A}\ \text{admissible}} \mathcal{H}^\star(\mathscr{A}),$$

taking finer and finer admissible open coverings $\mathscr{A}$ of X.

O-minimal site

$A \subseteq M^n$ definable.

O-minimal site $\mathfrak{F}_A$ on A : admissible open sets := definable open sets

admissible coverings := finite coverings.

Every admissible covering $\mathscr{A}$ of A induces open coverings $\widetilde{\mathscr{A}}$ of $\widetilde{A}$ and $\underset{\sim}{\mathscr{A}}$ of $\underset{\sim}{A}$.

$$\widetilde{\mathscr{A}} := \{\, \widetilde{U} : U \in \mathscr{A} \,\} \quad \underset{\sim}{\mathscr{A}} := \{\, \widetilde{U} \cap \underset{\sim}{A} : U \in \mathscr{A} \,\}.$$

The nerves $\mathcal{N}(\mathscr{A})$, $\mathcal{N}(\widetilde{\mathscr{A}})$ and $\mathcal{N}(\underset{\sim}{\mathscr{A}})$ are isomorphic.

Conversely, every open covering $\mathscr{B}$ of $\widetilde{A}$ can be refined to a covering of the form $\widetilde{\mathscr{A}}$, with $\mathscr{A}$ admissible covering of A, and similarly for $\underset{\sim}{A}$.

Čech Cohomology

$\check{\mathcal{H}}^\star_{\mathfrak{F}}(A)$:= Čech cohomology of A w.r.t. $\mathfrak{F}_A$.

$\check{\mathcal{H}}^\star_{\mathfrak{F}}(A)$ is canonically isomorphic to $\check{\mathcal{H}}^\star(\widetilde{A})$ and to $\check{\mathcal{H}}^\star(\underset{\sim}{A})$.

Real model

Hypothesis. *M has an Archimedean prime submodel M_0.*

Remark 1. It is possible to extend uniquely the structure on M_0 to the real line $\mathbb{R}$, in such a way that M_0 is an elementary submodel of $\mathbb{R}$.

Let $A \subseteq M^n$ be $\emptyset$-definable. $A(\mathbb{R})$ is the subset of $\mathbb{R}^n$ defined by the same formula defining A. Define

$$\check{\mathcal{H}}^\star_{\mathbb{R}}(A) := \check{\mathcal{H}}^\star\big(A(\mathbb{R})\big).$$

We will prove that

$$\check{\mathcal{H}}^\star_{\mathfrak{F}}(A) \cong \check{\mathcal{H}}^\star_{\mathbb{R}}(A).$$

The extension from M_0 to $\mathbb{R}$ is built trough Cauchy sequences.
If $x \in \mathbb{R} \setminus M_0$, and $f : M_0 \to M_0$ is definable, define

$$f(x) := \lim_{y \to x, y \in M_0} f(y).$$

The fact that $\check{\mathcal{H}}^\star_{\mathfrak{F}}(A) \cong \check{\mathcal{H}}^\star_{\mathbb{R}}(A)$ is non-trivial even for $M = \mathbb{R}$, because $\underset{\sim}{A}$ is compact, while $A = A(\mathbb{R})$ might not be compact.

Leray's theorem

Let X be a topological space. $U \subseteq X$ is acyclic iff

$$\forall n > 0 \; \check{\mathcal{H}}^n(U) = 0.$$

$\mathscr{A}$ is an acyclic covering of X iff every finite intersection of elements in $\mathscr{A}$ is acyclic.

Theorem 1 (Leray). *Let X be a compact Hausdorff space, and $\mathscr{A}$ an open covering of X. If $\mathscr{A}$ is acyclic, then the canonical map $\mathcal{H}^\star(\mathscr{A}) \to \check{\mathcal{H}}^\star(X)$ is an isomorphism.*

Cohomology of a simplex

Lemma 1. *Let Δ be a standard $(n-1)$-simplex in M^n. Then,*

$$\check{\mathcal{H}}^m_{\mathfrak{F}}(A) = \begin{cases} G & \text{if } m = 0, \\ 0 & \text{if } m > 0, \end{cases}$$

where G is the coefficient group for the cohomology.

Let D be a simplicial complex. D is linearly contractible iff some (and hence all) realisation $|D|$ of D which is $\emptyset$-definable is contractible via a $\emptyset$-definable map in the semi-linear language.

Give a vertex v of D, the star of v, denoted by $v^\star$, is the set of simplices in D having v as a vertex.

Remark 2. If $v_1, \ldots, v_n$ are vertices of D, then , $v_1^\star \cap \cdots \cap v_n^\star$ is linearly contractible (if it is non-empty).

Remark 3. The set $\{\, |v^\star| : v \in D \,\}$ is an admissible open covering of $|D|$.

page 9

D is an abstract complex, while $|D|$ is the "concrete" realisation of D in some M^n.

We will apply Leray's theorem to coverings of the form $\{\, |v^\star| : v \in D \,\}$.

Let $\mathscr{A}$ be an admissible covering of the definable set A. (C, g) is a triangulation of A compatible with $\mathscr{A}$, and C' is the barycentric subdivision of A.

$$\mathscr{B} := \{ g(|v^\star|) : v \text{ vertex of } C' \}$$

is the good refinement of $\mathscr{A}$ given by (C', g).

More in general, a good covering of A is of the form

$$\{ g(|v^\star|) : v \in C \},$$

for some (C, g) triangulation of A.

Proof of Lemma 1

$\mathscr{A}$ is an admissible covering of Δ. It suffices to prove that $\mathscr{A}$ has a refinement $\mathscr{B}$ such that $\mathcal{H}^{\#}(\mathscr{B}) = 0$. Choose $\mathscr{B}$ a good refinement of $\mathscr{A}$ given by (C, g), and

$$\mathscr{C} := g(\mathscr{B}) = \{ |v^\star| : v \in C \}.$$

$\mathcal{N}(\mathscr{C}) \cong \mathcal{N}(\mathscr{B})$: it suffice to prove that $\mathcal{H}^{\#}(\mathscr{C}) = 0$.

The map g is a definable homeomorphism between $|C|$ and Δ. By transfer, we can assume that g is $\emptyset$-definable. Thus,

$$\check{\mathcal{H}}^\star_{\mathbb{R}}(g) : \check{\mathcal{H}}^\star_{\mathbb{R}}(\Delta) \to \check{\mathcal{H}}^\star_{\mathbb{R}}(|C|)$$

is an isomorphism. Since $\check{\mathcal{H}}^{\#}_{\mathbb{R}}(\Delta) = 0$, we have $\check{\mathcal{H}}^{\#}_{\mathbb{R}}(|C|) = 0$.

Fact 1. *$\mathscr{C}(\mathbb{R})$ is an acyclic covering of $|C|(\mathbb{R})$.*

Lemma 1 says that if Δ is a simplex, then

$$\check{\mathcal{H}}^{\star}_{\mathfrak{F}}(\Delta) = \check{\mathcal{H}}^{\star}_{\mathbb{R}}(\Delta).$$

Use of transfer: if $g = g_a$ is definable with parameters a, then it is true in M that $\exists x$ such that g_x is an homeomorphism between $|C|$ and Δ. Then, such an x must exist also in M_0, and g_{x_0} is the $\emptyset$-definable homeomorphism we were looking for.

Since g is a $\emptyset$-definable homeomorphism, also $g(\mathbb{R})$ is a homeomorphism. $\check{\mathcal{H}}^{\star}_{\mathbb{R}}(g)$ is the morphism in cohomology induced by $g(\mathbb{R})$.

11-1

Proof of Lemma 1

By Leray's theorem,

$$\mathcal{H}^{\#}(\mathscr{C}(\mathbb{R})) \cong \check{\mathcal{H}}^{\#}_{\mathbb{R}}(|C|) = 0.$$

Finally, $\mathcal{N}(\mathscr{C}(\mathbb{R})) \cong \mathcal{N}(\mathscr{C})$. □

We used only that $\check{\mathcal{H}}^{\#}_{\mathbb{R}}(\Delta) = 0$. We proved that if A is $\emptyset$-definable and $\check{\mathcal{H}}^{\#}_{\mathbb{R}}(A) = 0$, then $\check{\mathcal{H}}^{\#}_{\mathfrak{F}}(A) = 0$.

Corollary 2. *If B is definable and contractible, then there exists A which is $\emptyset$-definable, contractible via a $\emptyset$-definable map, and that is definably homeomorphic to B. Therefore, $\check{\mathcal{H}}^{\#}_{\mathfrak{F}}(B) \cong \check{\mathcal{H}}^{\#}_{\mathfrak{F}}(A) = 0$.*

The A in the corollary exists by transfer. In fact, if $B = B_c$ is defined using parameters c, there exists a $\emptyset$-definable cell U containing c such that B_d is definably homeomorphic to B_c for every $d \in U$. Since U is $\emptyset$-definable and nonempty, there exists $c_0 \in U \cap M_0^k$. Define $A := B_{c_0}$.

Functors

Given $N \equiv M$, let $\mathbf{Def}_N$ be the category of sets and continuous maps definable in N. Call $\mathbf{Def}_0 := \mathbf{Def}_{M_0}$.

$\widetilde{\square}$, $\underset{\sim}{\square}$, $\check{\mathcal{H}}^\star_{\mathfrak{F}}$ and $\check{\mathcal{H}}^\star_{\mathbb{R}}$ are functors, with domain $\mathbf{Def}_M$ (or $\mathbf{Def}_0$).

Covariant functors

Let $A \subseteq M^n$ and $B \subseteq M^m$ be definable sets, and $f : A \to B$ be a definable continuous map.

Given $q(x) \in \widetilde{A}$, define

$$\widetilde{f}(q)(y) := \left\{\, \psi(y) : \psi \in \mathcal{L} \;\&\; \psi(f(x)) \in q(x) \,\right\} \in \widetilde{B}.$$

If $q \in \underset{\sim}{A}$, define

$$\underset{\sim}{f}(q) := \pi\big(\widetilde{f}(q)\big) \in \underset{\sim}{B}.$$

If A is definably compact and q is closed, then $\widetilde{f}(q)$ is already closed. However, if A is not definably compact, $\widetilde{f}(q)$ might not be closed.

E.g.: let $A := M^2$, $B := M$, $f(y_1, y_2) = y_1$,

$$q(y_1, y_2) := \big(y_1 \models 0^+, y_1 \cdot y_2 = 1\big).$$

Contra-variant functors

Given an admissible covering $\mathscr{B}$ of B,

$$f^{-1}(\mathscr{B}) := \left\{ f^{-1}(U) : U \in \mathscr{B} \right\}$$

is an admissible covering of A. There is a map $\theta_{\mathscr{A}} : \mathcal{H}^\star(\mathscr{B}) \to \mathcal{H}^\star(\mathscr{A})$.
Define $\check{\mathcal{H}}^\star_{\mathfrak{F}}(f) : \check{\mathcal{H}}^\star_{\mathfrak{F}}(B) \to \check{\mathcal{H}}^\star_{\mathfrak{F}}(A)$ as the limit of the maps $\theta_{\mathscr{A}}$.
Under the identification of $\check{\mathcal{H}}^\star_{\mathfrak{F}}(A)$ with $\check{\mathcal{H}}^\star(\widetilde{A})$, we have that $\check{\mathcal{H}}^\star_{\mathfrak{F}}(f) = \check{\mathcal{H}}^\star(\widetilde{f})$, and similarly for $\underset{\sim}{A}$.

If moreover A, B and f are $\emptyset$-definable, then $f(\mathbb{R}) : A(\mathbb{R}) \to B(\mathbb{R})$ is continuous, and therefore it induces a map

$$\check{\mathcal{H}}^\star_{\mathbb{R}}(f) : \check{\mathcal{H}}^\star_{\mathbb{R}}(B) \to \check{\mathcal{H}}^\star_{\mathbb{R}}(A).$$

Natural isomorphism

We will prove that $\check{\mathcal{H}}^{\star}_{\mathfrak{F}_0}$ and $\check{\mathcal{H}}^{\star}_{\mathbb{R}}$ are naturally isomorphic, where $\mathfrak{F}_0$ is the site on M_0.
Namely, if $A \subseteq M_0^n$ and $B \subseteq M_0^m$ are definable sets, and $f : A \to B$ is a definable continuous map, then there exists a pair of isomorphisms such that the following diagram commutes:

$$\begin{array}{ccc}
\check{\mathcal{H}}^{\star}_{\mathfrak{F}}(A) & = & \check{\mathcal{H}}^{\star}_{\mathbb{R}}(A) \\
{\scriptstyle \check{\mathcal{H}}^{\star}_{\mathfrak{F}}(f)}\uparrow & & \uparrow{\scriptstyle \check{\mathcal{H}}^{\star}_{\mathbb{R}}(f)} \\
\check{\mathcal{H}}^{\star}_{\mathfrak{F}}(B) & = & \check{\mathcal{H}}^{\star}_{\mathbb{R}}(B)
\end{array}$$

Given $N \succeq M$, let $\mathfrak{F}(N)$ be the o-minimal site on N, and $\check{\mathcal{H}}^{\star}_{\mathfrak{F}(N)}$ be the corresponding functor.

Let ι^M_N be the immersion of $\mathbf{Def}_M$ in $\mathbf{Def}_N$. Then, $\check{\mathcal{H}}^{\star}_{\mathfrak{F}}$ and $\check{\mathcal{H}}^{\star}_{\mathfrak{F}(N)} \circ \iota^M_N$ are naturally isomorphic. Namely, if $f : A \to B$ is definable with parameters in M, then there exists a pair of isomorphisms such that the following diagram commutes:

$$\begin{array}{ccc}
\check{\mathcal{H}}^{\star}_{\mathfrak{F}}(A) & = & \check{\mathcal{H}}^{\star}_{\mathfrak{F}(N)}(A) \\
{\scriptstyle \check{\mathcal{H}}^{\star}_{\mathfrak{F}}(f)}\uparrow & & \uparrow{\scriptstyle \check{\mathcal{H}}^{\star}_{\mathfrak{F}(N)}(f)} \\
\check{\mathcal{H}}^{\star}_{\mathfrak{F}}(B) & = & \check{\mathcal{H}}^{\star}_{\mathfrak{F}(N)}(B)
\end{array}$$

First theorem

Theorem 2. *Let $N \succeq M$. Then, $\check{\mathcal{H}}^\star_{\mathfrak{F}}$ and $\check{\mathcal{H}}^\star_{\mathfrak{F}(N)} \circ \iota^M_N$ are naturally isomorphic.*

Proof. Let A be a definable subset of M^n, and $\mathscr{C}$ be a good covering of A. By Corollary 2, $\mathscr{C}$ is acyclic, hence (by Leray's theorem) the canonical map $\mathcal{H}^\star(\underset{\sim}{\mathscr{C}}) \to \check{\mathcal{H}}^\star(\underset{\sim}{A})$ is an isomorphism. Remember that $\mathcal{N}(\underset{\sim}{\mathscr{C}})$ and $\mathcal{N}(\mathscr{C})$ are canonically isomorphic. Similarly, $\mathscr{C}(N)$ is acyclic, therefore the map $\mathcal{H}^\star(\mathscr{C}(N)) \to \check{\mathcal{H}}^\star_{\mathfrak{F}(N)}(A)$ is also acyclic. Therefore,

$$\check{\mathcal{H}}^\star_{\mathfrak{F}}(A) \cong \mathcal{H}^\star(\mathscr{C}) \cong \mathcal{H}^\star(\mathscr{C}(N)) \cong \check{\mathcal{H}}^\star_{\mathfrak{F}(N)}(A).$$

The isomorphism is natural, because everything is canonical. □

More precisely, we have proved that the natural transformation $\lambda_A : \check{\mathcal{H}}^\star_{\mathfrak{F}(N)}(A) \to \check{\mathcal{H}}^\star_{\mathfrak{F}}(A)$ is an isomorphism, where λ_A is the map induced by the surjection $\widetilde{A(N)} \to \widetilde{A}$, given by the restriction of N-types to M-types.

page 17

$\mathscr{C}$ is acyclic, because it is good, and therefore for every $U_1, \dots U_n \in \mathscr{C}$, $U_1 \cap \dots \cap U_n$ is (homeomorphic to a linearly) contractible.

Complete formula:

$$\check{\mathcal{H}}^\star_{\mathfrak{F}}(A) \cong \check{\mathcal{H}}^\star(\widetilde{A}) \cong \mathcal{H}^\star(\widetilde{\mathscr{C}}) \cong \mathcal{H}^\star(\mathscr{C}) \cong$$
$$\cong \mathcal{H}^\star(\mathscr{C}(N)) \cong \mathcal{H}^\star(\widetilde{\mathscr{C}(N)}) \cong \check{\mathcal{H}}^\star(\widetilde{A(N)}) \cong \check{\mathcal{H}}^\star_{\mathfrak{F}(N)}(A).$$

Second theorem

Theorem 3. *$\check{\mathcal{H}}^\star_{\mathfrak{F}_0}$ and $\check{\mathcal{H}}^\star_{\mathbb{R}}$ are naturally isomorphic.*

Proof. We will assume that $M = M_0$. Let A be a definable subset of M^n and $\mathscr{C}$ be a good covering of A. By Corollary 2, $\mathscr{C}$ is acyclic, hence the canonical map $\mathcal{H}^\star(\underset{\sim}{\mathscr{C}}) \to \check{\mathcal{H}}^\star(\underset{\sim}{A})$ is an isomorphism. For the same reason, $\mathcal{H}^\star(\mathscr{C}(\mathbb{R})) \to \check{\mathcal{H}}^\star_{\mathbb{R}}(A)$ is an isomorphism. Therefore,

$$\check{\mathcal{H}}^\star_{\mathfrak{F}}(A) \cong \mathcal{H}^\star(\mathscr{C}) \cong \mathcal{H}^\star(\mathscr{C}(\mathbb{R})) \cong \check{\mathcal{H}}^\star_{\mathbb{R}}(A).$$

The isomorphism is natural, because everything is canonical. □

Proof details

Let $\mathscr{C}$ and $\mathscr{C}'$ be good coverings of A, with $\mathscr{C}$ a refinement of $\mathscr{C}'$.

We have the following diagram:

$$\begin{array}{ccccc}
\check{\mathcal{H}}^\star_{\mathfrak{F}}(A) & \longleftarrow & \mathcal{H}^\star(\mathscr{C}) & \longleftarrow & \mathcal{H}^\star(\mathscr{C}') \\
\downarrow & & \downarrow & & \downarrow \\
\check{\mathcal{H}}^\star_{\mathbb{R}}(A) & \longleftarrow & \check{\mathcal{H}}^\star(\mathscr{C}(\mathbb{R})) & \longleftarrow & \check{\mathcal{H}}^\star_{\mathbb{R}}(\mathscr{C}'(\mathbb{R}))
\end{array}$$

The horizontal arrows are canonical maps, and are isomorphisms by Leray's theorem and Corollary 2. The rightmost vertical arrow is the map induced by the the canonical isomorphism between the nerve of $\mathscr{C}'$ and of $\mathscr{C}'(\mathbb{R})$, and similarly for the central arrow.

The rightmost square commutes, because the corresponding maps on the nerves of the coverings commutes, and hence there is a well defined leftmost isomorphism between $\check{\mathcal{H}}^\star_{\mathfrak{F}}(A)$ and $\check{\mathcal{H}}^\star_{\mathbb{R}}(A)$ that makes the whole diagram commute. Namely, the isomorphism in cohomology induced by $\mathscr{C}$ and the one induced by $\mathscr{C}'$ are the same.

If $\mathscr{C}'$ and $\mathscr{C}''$ are any good coverings of A, there exists a common refinement $\mathscr{C}$ which is also good. Therefore, can apply the above observation to the pairs $(\mathscr{C}, \mathscr{C}')$ and $(\mathscr{C}, \mathscr{C}'')$.

Consider now a definable map $g : A \to B$. Let $\mathscr{B}$ be a good covering of B, and $\mathscr{A}$ a good refinement of $g^{-1}(\mathscr{B})$. Hence, we have the following diagram:

$$\begin{array}{ccccccc}
\check{\mathcal{H}}^\star_{\mathfrak{F}}(B) & \longleftarrow & \mathcal{H}^\star(\mathscr{B}) & = & \mathcal{H}^\star(\mathscr{B}(\mathbb{R})) & \longrightarrow & \check{\mathcal{H}}^\star_{\mathbb{R}}(B) \\
{\scriptstyle \check{\mathcal{H}}^\star_{\mathfrak{F}}(g)}\downarrow & & {\scriptstyle g^\star_{\mathscr{B}}}\downarrow & & \downarrow{\scriptstyle g^\star_{\mathscr{B}}(\mathbb{R})} & & \downarrow{\scriptstyle \check{\mathcal{H}}^\star_{\mathbb{R}}(g)} \\
\check{\mathcal{H}}^\star_{\mathfrak{F}}(A) & \longleftarrow & \mathcal{H}^\star(\mathscr{A}) & = & \mathcal{H}^\star(\mathscr{A}(\mathbb{R})) & \longrightarrow & \check{\mathcal{H}}^\star_{\mathbb{R}}(A)
\end{array}$$

The horizontal arrows are the canonical maps. They are isomorphisms: the leftmost ones and the rightmost ones by Leray's theorem and Corollary 2, the central ones because the corresponding maps on the nerves are isomorphisms.

We deduce that the isomorphisms in cohomology induced by $\mathscr{C}'$ and $\mathscr{C}''$ are the same.

21-1

The vertical arrows are the maps induced by g. The leftmost square commutes by definition of $\check{\mathcal{H}}^\star_{\mathfrak{F}}(g)$, and similarly for the rightmost. The central square also commutes, because the corresponding maps on the nerves commute. Therefore, the whole diagram commutes, and hence the following one does:

$$\begin{array}{ccc} \check{\mathcal{H}}^\star_{\mathfrak{F}}(B) & = & \check{\mathcal{H}}^\star_{\mathbb{R}}(B) \\ {\scriptstyle \check{\mathcal{H}}^\star_{\mathfrak{F}}(g)}\downarrow & & \downarrow{\scriptstyle \check{\mathcal{H}}^\star_{\mathbb{R}}(g)} \\ \check{\mathcal{H}}^\star_{\mathfrak{F}}(A) & = & \check{\mathcal{H}}^\star_{\mathbb{R}}(A) \end{array}$$

References

[1] Mário J. Edmundo, Gareth O. Jones, and Nicholas J. Peatfield. Sheaf cohomology in o-minimal structures. *Preprint*, 2005.

[2] Mário J. Edmundo and Nicholas J. Peatfield. O-minimal Čech cohomology. *Preprint*, 2006.

Available at `http://www.ciul.ul.pt/~edmundo/`

[3] Michel Carral and Michel Coste. Normal spectral spaces and their dimensions. *J. Pure Appl. Alg.*, 30(3):227–235, 1983.

[4] Anand Pillay. Sheaves of continuous definable functions. *J. Symbolic Logic*, 53(4):1165–1169, 1988

List of Slides