THE ABSOLUTE ARITHMETIC CONTINUUM AND THE UNIFICATION OF ALL NUMBERS GREAT AND SMALL

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Introduction

> *Bridging the gap between the domains of discreteness and of continuity*, or between arithmetic and geometry, is a central, presumably even *the* central problem of the foundations of mathematics.

So wrote Abraham Fraenkel, Yehoshua Bar-Hillel and Azrel Levy in their mathematico-philosophical classic *Foundations of Set Theory* (1973, 211). Cantor and Dedekind of course believed they had bridged the gap with the creation of their *arithmetico-set theoretic continuum*, ℝ, of real numbers, and for roughly a century now it has been one of the central tenants of standard mathematical philosophy that indeed they had. In accordance with this view the *geometric linear continuum* is assumed to be isomorphic with the arithmetic continuum, the axioms of geometry being so selected to ensure this would be the case. In honor of Cantor and Dedekind, who first proposed this mathematico-philosophical thesis, the transference of $\mathbb{R}$'s purported continuity to the continuity of the Euclidean straight line has come to be called the *Cantor-Dedekind axiom*. Given the Archimedean nature of the real number system, once this axiom is adopted we have the classic result of standard mathematical philosophy that infinitesimals are superfluous to the analysis of the structure of a continuous straight line.

More than twenty years ago, however, we began to suspect that while the Cantor-Dedekind theory succeeds in bridging the gap between the domains of arithmetic and of classical Euclidean geometry, it only reveals a glimpse of a far richer theory of continua which not only allows for infinitesimals but leads to a vast generalization of portions of Cantor's theory of the infinite, a generalization which also provides a setting for Abraham Robinson's infinitesimal approach to analysis as well as for the profound and all too often overlooked non-Cantorian theories of the infinite (and infinitesimal) pioneered by Giuseppe Veronese (1891), Tullio Levi-Civita (1892; 1898), David Hilbert (1899) and Hans Hahn (1907) in connection with their work on non-Archimedean ordered algebraic and geometric systems and by Paul du Bois-Reymond (1871-1882), Otto Stolz (1883), Felix Hausdorff (1907; 1909) and G. H. Hardy (1910; 1912) in connection with their work on the rate of growth of real functions. Central to the theory is J. H. Conway's theory of surreal numbers (1976) and the present author's amplifications and generalizations thereof and other contributions thereto.

In a number of earlier works (Ehrlich 1987; 1989; 1992; 1994; 2005), we suggested that *whereas the real number system should be regarded as constituting an arithmetic continuum modulo the Archimedean axiom, the system of surreal numbers may be regarded as a sort of absolute arithmetic continuum (modulo von Neumann-Bernays-Gödel set theory with global choice, henceforth NBG)*. In the present discussion we will outline some of the properties of the system of surreal numbers that we believe lend credence to this thesis, and draw attention to the unifying framework this system provides not only for the systems of real and ordinal numbers but for the various other sorts of systems of numbers great and small alluded to above.

1. All Numbers Great and Small

In his monograph *On Numbers and Games* (1976), J. H. Conway introduced a real-closed field containing the reals and the ordinals as well as a great many less familiar numbers including $-\omega$, $\omega/2$, $1/\omega$, $\sqrt{\omega}$ and $\omega - \pi$ to name only a few. Indeed, this particular real-closed field, which Conway calls *No*, is so remarkably inclusive that, subject to the proviso that numbers--construed here as members of ordered "number" fields--be individually definable in terms of sets of NBG, it may be said to contain "All Numbers Great and Small." In this respect, *No* bears much the same relation to ordered fields that the system of real numbers bears to Archimedean ordered fields. This can be made precise by saying that:

Whereas $\mathbb{R}$ is (up to isomorphism) the unique *homogeneous universal Archimedean ordered field*, *No* is (up to isomorphism) the unique *homogeneous universal ordered field* (Ehrlich 1988; 1992).

However, in addition to its distinguished structure as an ordered field, *No* has a rich hierarchical structure that emerges from the recursive clauses in terms of which it is defined.

From the standpoint of Conway's construction, this algebraico-tree-theoretic structure, or *simplicity hierarchy*, as we have called it [Ehrlich 1994], depends upon *No*'s *implicit* structure as a lexicographically ordered binary tree and arises from the fact that the sums and products of any two members of the tree are the simplest possible elements of the tree consistent with *No*'s structure as an ordered group and an ordered field, respectively, it being understood that x is *simpler than* y just in case x is a predecessor of y in the tree.

In [Ehrlich 1994], the just-described simplicity hierarchy was brought to the fore and made part of an algebraico-tree-theoretic definition of *No*, and in [Ehrlich 2002] we introduced a novel class of structures whose properties generalize those of *No* so construed and explored some of the relations that exist between *No* and this more general class of *s-hierarchical ordered structures* as we call them. We defined a number of types of s-hierarchical ordered structures--groups, fields, vector spaces-

-as well as a corresponding type of *s-hierarchical mapping,* identified *No* as a *complete* s-hierarchical ordered group (ordered field; ordered vector space), and showed that there is one and only one s-hierarchical mapping of an s-hierarchical ordered structure into *No* (or any *complete* s-hierarchical ordered structure, more generally). These mappings were found to be monomorphisms of their respective kinds whose images are initial subtrees of *No*, and this together with the completeness of *No* enabled us to characterize *No*, up to isomorphism, as the unique complete as well as the unique *nonextensible* and the unique *universal,* s-hierarchical ordered group (ordered field, etc.). Following this, we turned our attention to uncovering the spectrum of s-hierarchical ordered structures. Given the nature of *No* alluded to above, this reduced to revealing the spectrum of *s-hierarchical substructures* of *No*, i.e., the subgroups, subfields, subspaces of *No* that are initial subtrees of *No*. Among the striking results that emerged from the latter investigation is that much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite recursion that provides an unfolding of the entire spectrum of numbers great and small (modulo the aforementioned provisos), the recursive process of defining *No*'s arithmetic in turn provides an unfolding of the entire spectrum of ordered number fields in such a way that an isomorphic copy of each such system either emerges as an initial subtree of *No* or is contained in a theoretically distinguished instance of such a system that does. In particular, we showed that

Every real-closed ordered field is isomorphic to an *initial subfield* of *No*.

This result, as we shall later see, plays a significant role in the unification referred to above.

2. The Surreal Number Tree

In von Neumann's ordinal construction, an ordinal emerges as the set of all its predecessors in the 'long' though rather trivial binary tree $\langle Ord, \in \rangle$ of all ordinals. Inspired by von Neumann's construction, in the following construction each surreal number x emerges as an ordered pair (L_x, R_x) of sets of surreal numbers where L_x and R_x turn out to be the sets of all predecessors of x *less than* x and *greater than* x, respectively, in the lexicographically ordered full binary tree of surreal numbers (Ehrlich 1994; 2002).

Construction of Games

If L and R are any two sets of games, then there is a game (L, R). All games are constructed in this way.

Preliminary Definitions

A game x is said to be *simpler than* a game $y = \left(L_y, R_y\right)$, written $x <_s y$, if $x \in L_y$ or $x \in R_y$; a chain of games (ordered by $<_s$) is said to be *ancestral* if it is closed under the simpler than relation, i.e., x is a member of the chain whenever y is a member of the chain and $x <_s y$; and a partition L, R of an ancestral chain of games is said to be *orderly*, if $L \supseteq L_x$ and $R_x \subseteq R$ for each element $x = \left(L_x, R_x\right)$ of the chain.

Construction of Surreal Numbers

If L, R is an orderly partition of an ancestral chain of surreal numbers, then there is a surreal number $\left(L, R\right)$. All surreal numbers are constructed in this way.

At this point it is not difficult to show that $\langle No, <_s \rangle$ is a full binary tree where the definition of the simpler than relation for surreal numbers is inherited from the definition for games. For this purpose, however, it is convenient to have available the ordinals. If one wishes, one could avail oneself of the von Neumann ordinals, which are already at hand. On the other hand, if one wants to develop the theory of ordinals within the theory of surreal numbers, as we intend to, before proving the above theorem one must first identify "our" ordinals.

Isolation of the Ordinals

A surreal number (L, R) will be said to be an *ordinal* if $R = \varnothing$. By *On* we mean the class of ordinals so defined. For all ordinals $x = (L_x, \varnothing)$ and $y = (L_y, \varnothing)$, x will be said to be *less than* y, written $x <_{On} y$, if $L_x \subset L_y$.

Theorem. The ordered class of ordinals (so defined) has the requisite properties possessed by any of the more familiar constructs so called.

Theorem. $(\varnothing,\varnothing)$ is a surreal number; if $x=(L_x,R_x)$ is a surreal number, then $(L_x,\{x\}\cup R_x)$ and $(L_x\cup\{x\},R_x)$ are surreal numbers; moreover, if $\{x_\alpha\}_{\alpha<\beta}$ is a chain of surreal numbers of infinite limit length, then $\left(\bigcup_{\alpha<\beta}L_{x_\alpha},\bigcup_{\alpha<\beta}R_{x_\alpha}\right)$ is a surreal number. Nothing is a surreal number except in virtue of the above.

The Rule of Order

For all surreal numbers $x=(L_x,R_x)$ and $y=(L_y,R_y)$, $x<y$ if and only if $x\in L_y$ or $y\in R_x$ or $R_x\cap L_y\neq\varnothing$.

Corollary. For each surreal number x,

$$x = \left(L_{s(x)}, R_{s(x)} \right)$$

where

$$L_{s(x)} = \{ a \in No : a <_s x \ \& \ a < x \}$$

and

$$R_{s(x)} = \{ a \in No : a <_s x \ \& \ x < a \}.$$

Moreover, if x is an ordinal, then

$$x = \left(L_{s(x)}, \varnothing \right).$$

Theorem. Let No be the class of surreal numbers. $\langle No, <, <_s \rangle$ is isomorphic to the familiar lexicographically ordered full binary tree consisting of sequences of 0s and 1s.

3. The s-Hierarchical Ordered Field of Surreal Numbers

Convention. If L and R are subsets of an ordered class A where every member of L precedes every member of R, we will write $L < R$.

Proposition. If L and R are two subsets of No for which $L < R$, there is a simplest member of No lying between the members of L and the members of R. Henceforth, the simplest such element will be denoted '$\{L \mid R\}$'.

Proposition. For all $x \in No$, there are (possibly empty) subsets L and R of No such that $L < R$ and for which $x = \{L \mid R\}$; in particular, $x = \{L_{S(x)} \mid R_{S(x)}\}$.

Theorem (Conway 1976; Ehrlich 2001). $\langle No, <, <_s, +, -, . \rangle$ is a ordered field when $+, -$ and $\cdot$ are defined by recursion as follows where x^L, x^R, y^L and y^R are understood to range over the members of $L_{s(x)}, R_{s(x)}, L_{s(y)}$ and $R_{s(y)}$, respectively.

Definition of $x+y$.

$$x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}.$$

Definition of $-x$.

$$-x = \{-x^R \mid -x^L\}.$$

Definition of xy.

$$xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R \mid x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}.$$

In fact, $\langle No, <, <_S, +, -, . \rangle$ is (up to isomorphism) the unique real-closed ordered field that is an η_{On} –ordering (Ehrlich 1988).

4. Surreal Numbers Have Their Own Proper Names

A remarkable surreal feature is that ordinal exponentiation with base ω extends to an operation $x \mapsto \omega^x : No \rightarrow No$ in such a way that every surreal number a can be written uniquely as a generalized power series

$$\sum_{\alpha<\beta} \omega^{y_\alpha} . r_\alpha$$

in ω with real coefficients and surreal exponents (called the *Conway name* or *normal form* of a).

The following result sheds light on the relationship between surreal numbers and their Conway names.

Theorem. Every surreal number has a Conway name; distinct surreal numbers have distinct Conway names; furthermore, the formal expression

$$\sum_{\alpha<\beta} \omega^{y_\alpha}.r_\alpha$$

is the Conway name of some surreal number if and only if $\{y_\alpha : \alpha < \beta \in On\}$ is a (possibly empty) descending sequence of members of No and $\{r_\alpha : \alpha < \beta\}$ is a sequence of nonzero real numbers. *In addition, the Conway name of an ordinal is just its Cantor Normal Form*.

5. The Unification of Numbers Great and Small

(i) Non-Archimedean Ordered (Number) Fields Inspired by Non-Archimedean Geometry

Theorem (Hahn 1907**).** (i) Let $\mathbb{R}$ be the ordered field of real numbers and G be a nontrivial ordered Abelian group. The collection, $\mathbb{R}(G)$, of all series

$$\sum_{\alpha<\beta} e_{y_\alpha}.r_\alpha$$

where $\{y_\alpha : \alpha < \beta \in On\}$ is a (possibly empty) descending sequence of members of G and $\{r_\alpha : \alpha < \beta\}$ is a sequence of members of $\mathbb{R} - \{0\}$ is a non-Archimedean ordered field when the order is defined lexicographically and sums and products are defined termwise (it being understand that $e_x \cdot e_y = e_{x+y}$).

(ii) The restricted structure, $\mathbb{R}(G)_{\alpha}$, that results by limiting the above construction to those series where β is less than a given infinite cardinal $\aleph_{\alpha}$ is likewise a non-Archimedean ordered field.

Theorem (Ehrlich 1988). There is isomorphism from *No* onto $\mathbb{R}(No)_{On}$ that sends the surreal number a having Conway name "$\sum_{\alpha<\beta} \omega^{y_{\alpha}}.r_{\alpha}$" to $\sum_{\alpha<\beta} e_{y_{\alpha}}.r_{\alpha}$.

(ii) **Paul du Bois-Reymond's *Infinitärcalcül* (calculus of infinities)**

Du Bois-Reymond (1870-1882) erects his calculus primarily on families of increasing functions from $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ to $\mathbb{R}$ such that for each function f of a given family, $\lim_{x\to\infty} f(x) = +\infty$, and for each pair of functions f and g of the family, $0 \le \lim_{x\to\infty} f(x)/g(x) \le +\infty$. He assigns to each such function f a so-called *infinity*, and defines an ordering on the infinities of such functions by stipulating that for each pair of such functions f and g:

$f(x)$ has an infinity *greater than* that of $g(x)$, if $\lim_{x\to\infty} f(x)/g(x) = \infty$;
$f(x)$ has an infinity *equal to* that of $g(x)$, if $\lim_{x\to\infty} f(x)/g(x) = a \in \mathbb{R}^+$;
$f(x)$ has an infinity *less than* that of $g(x)$, if $\lim_{x\to\infty} f(x)/g(x) = 0$.

Otto Stolz (1883)

"Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes", *Mathematische Annalen* 22, 504-519.

Stolz considers the set of all functions $f:\mathbb{R}^{+}\to\mathbb{R}^{+}$ formed by means of finite combinations of the operations +, -, ·, and ÷ from positive rational powers of the functions $x, \ln x, \ln(\ln x), \ldots; e^{x}, e^{e^{x}}, e^{e^{e^{x}}}, \ldots$ where $\ln x$ is the natural logarithm of x and e is the base of the natural logarithm. Following du Bois-Reymond, Stolz assigns to each such function f an *infinity* -- which he denotes by "$\mathfrak{U}(f)$" -- and defines an ordering on the infinities of such functions in the manner specified above. To complete the construction, Stolz defines addition and subtraction of the infinities by the rules:

$$\mathfrak{U}(f)+\mathfrak{U}(g)=\mathfrak{U}(f\cdot g)$$
$$\mathfrak{U}(f)-\mathfrak{U}(g)=\mathfrak{U}(f/g), \text{ if } \mathfrak{U}(f)>\mathfrak{U}(g).$$

Hausdorff (1907), working in a more general setting, calls a maximal set of such functions totally order by the “final order” a *pantachie* and he establishes the following results for an arbitrary pantachie $\mathbb{H}_P$.

Hausdorff (1907). $\mathbb{H}_P$ is an η_1-ordering of power $2^{\aleph_0}$. Moreover, $\mathbb{H}_P$ is (up to isomorphism) the unique η_1-ordering of power $\aleph_1$, assuming the Continuum Hypothesis.

Hausdorff (1909); Boshernitzan (1981). $\mathbb{H}_P$ (with sums and products defined in the manner familiar from the theory of Hardy fields) is a real-closed ordered field.

Let $No(\omega_1)$ be the subset of No consisting of all surreal numbers having tree rank $< \omega_1$.

Theorem (Ehrlich). $\mathbb{H}_P$ (considered as an ordered field) is isomorphic with an initial subfield of No extending $No(\omega_1)$; assuming the Continuum Hypothesis, $\mathbb{H}_P$ is in fact isomorphic to $No(\omega_1)$. Moreover, the orders of infinity of the members of $\mathbb{H}_P$ is isomorphic to the value group (i.e. the ordered Abelian group of Archimedean classes) of $No(\omega_1)$.

(i) Nonstandard models of analysis

H. J. Keisler's Axioms For Hyperreal Number Systems

Axiom A. $\mathbb{R}$ is a complete ordered field.

Axiom B. $\mathbb{R}^*$ is a proper ordered field extension of $\mathbb{R}$.

Axiom C. (Function Axiom). For each function f of n variables there is a corresponding hyperreal function f^* of n variables, called the *natural extension* of f. The field operations of $\mathbb{R}^*$ are the natural extensions of the field operations of $\mathbb{R}$.

Axiom D. (Solution Axiom). If two systems of formulas [i.e., finite sets of equations or inequalities between terms] have exactly the same real solutions, they have exactly the same hyperreal solutions.

“The real numbers are the unique complete ordered field. By analogy, we would like to uniquely characterize the hyperreal structure $\langle \mathbb{R}, \mathbb{R}^*, * \rangle$ by some sort of completeness property. However, we run into a set-theoretic difficulty; there are structures $\mathbb{R}^*$ of arbitrary large cardinal number which satisfy Axioms A-D, so there cannot be a largest one. Two ways around this difficulty are to make $\mathbb{R}^*$ a proper class rather than a set, or to put a restriction on the cardinal number of $\mathbb{R}^*$. We use the second method because it is simpler.” (Keisler 1976, p. 59)

Saturation Axiom

Axiom E. Let S be a set of equations and inequalities involving real functions, hyperreal constants, and variables, such that S has a smaller cardinality than $\mathbb{R}^*$. If every finite subset of S has a hyperreal solution, then S has a hyperreal solution.

Theorem (Keisler 1976). There is up to isomorphism a unique structure $\langle \mathbb{R}, \mathbb{R}^*, * \rangle$ such that Axioms A-E are satisfied and the cardinality of $\mathbb{R}^*$ is the first uncountable inaccessible cardinal.

Theorem (Keisler, Ehrlich). In NBG there is up to isomorphism a unique structure $\langle \mathbb{R}, \mathbb{R}^*, * \rangle$ such that Axioms A-E are satisfied and the cardinality of $\mathbb{R}^*$ is a proper class. Moreover, $\mathbb{R}^*$ is isomorphic to *No*.

H. J. Keisler to Ehrlich
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“What I had in mind in getting around the uniquess problem for the hyperreals in "Foundations of Infinitesimal Calculus" was to work in NBG with global choice (i.e. a class of ordered pairs that well orders the universe). This is a conservative extension of ZFC. I was not thinking of doing it within a superstructure, but just getting four objects R, R*, <*, * which satisfy Axioms A-E. R is a set, R* is a proper class, <* is a proper class of ordered pairs of elements of R*, and * is a proper class of ordered triples (f,x,y) of sets, where f is an n-ary real function for some n, x is an n-tuple of elements of R*, and y is in R*. In this setup, f*(x)=y means that (f,x,y) is in the class *. There should be no problem with * being a legitimate entity in NBG with global choice. Since each ordered triple of sets is again a set, * is just a class of sets. I believe that this can be done in an explicit way so that R, R*, <*, and * are definable in NBG with an extra symbol for a well ordering of V."