TIME-DEVELOPMENT OF EXPLOSIONS AND A PATH-SPACE MEASURE FOR DIFFUSION PROCESS WITH REPULSIVE HIGHER ORDER DRIFT

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We consider the stochastic differential equation,

$$dX(t) = f\big(X(t)\big)\,dt + dB(t),$$

for particle momentum $X(t) \in \mathsf{R}$ with a drift $f(x)$ and a random force $dB(t)/dt$, where $B(t)$ is the one-dimensional Brownian motion with diffusion coefficient D. We assume

(i) $f(x)$ is repulsive *i.e.* $xf(x) > 0$ for large $|x|$

and

(ii) $f(x)$ grows faster than linear.

Then, the process explodes to infinity with positive probability in finite times as Feller showed.

We wish to obtain in this case the time-development of the explosion by evaluating the probability of survival by time t, $P(t) = \int_{-\infty}^{\infty} \phi(t,x)dx$. Here, $\phi(t,x)$ is the probability density of the particle momentum x at time t, which satisfies the forward Fokker-Planck equation (FP-eq.),

$$\frac{\partial}{\partial t}\phi(t,x) = D\frac{\partial^2}{\partial x^2}\phi(t,x) - \frac{\partial}{\partial x}\big\{f(x)\phi(t,x)\big\}.$$

By transforming the FP-eq. into an imaginary-time Schrödinger equation, we obtain that $P(t)$ decreases to zero exponentially as time passes.

A probability measure over a set of paths including the exploding ones is constructed by making use of nonstandard analysis to give the solution $\phi(t,x)$ as a path integral with respect to the measure. In the construction, we introduce a momentum cut-off at an infinite magnitude using nonstandard analysis so that probabilities should be distributed properly also to the exploding paths.

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