On a Minkowski geometric flow in the plane*

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We consider a planar geometric flow in which the normal velocity is a nonlocal variant of the curvature. The flow is not scaling invariant and in fact has different behaviors at different spatial scales, thus producing phenomena that are different with respect to both the classical mean curvature flow and the fractional mean curvature flow.

In particular, we give examples of neckpinch singularity formation, we show that sets with "sufficiently small interior" remain convex under the flow, but, on the other hand, in general the flow does not preserve convexity.

We also take into account traveling waves for this geometric flow, showing that a new family of C^2 and convex traveling sets arises in this setting.

1 Introduction

In this paper we consider a planar geometric flow and we discuss its basic properties, such as singularity formation, convexity preserving and loss of convexity and existence of traveling waves. This geometric flow is the gradient flow of a nonlocal perimeter which is not invariant under scaling, therefore the evolution of a set presents different properties at different scales (in this, the flow has natural applications in image digitalization, especially when tiny details have to be preserved after denoising, as in the case of fingerprints storage).

The mathematical framework in which we work is the following. For any set $E \subset \mathbb{R}^2$ with C^2 boundary and any $x \in \partial E$ we denote by $B^{\rm ext}_{r,x}$ the ball of radius $r > 0$ which is locally externally tangent to ∂E at x (that is, $B^{\rm ext}_{r,x} := B_r(x + r\nu_E(x))$, where ν_E is the external unit normal). Similarly, we denote by $B^{\rm int}_{r,x}$ the ball of radius r which is locally internally tangent to ∂E at x (that is, $B^{\rm int}_{r,x} := B_r(x - r\nu_E(x))$).

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We also denote by $\kappa(E, x)$ the curvature of ∂E at the point x. Then we define the r-curvature of ∂E at x as

$$\begin{aligned} &\kappa_r(E,x) := \kappa_r^+(E,x) + \kappa_r^-(E,x), \\ \text{where} \quad &\kappa_r^+(E,x) := \begin{cases} \dfrac{\kappa(E,x)}{2} + \dfrac{1}{2r} & \text{if } B_{r,x}^{\text{ext}} \subseteq \mathbb{R}^2 \setminus E, \\ 0 & \text{otherwise,} \end{cases} \\ \text{and} \quad &\kappa_r^-(E,x) := \begin{cases} \dfrac{\kappa(E,x)}{2} - \dfrac{1}{2r} & \text{if } B_{r,x}^{\text{int}} \subseteq E, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{1.1}$$

see formulas (2.10), (2.11) and (2.12) and Lemma 2.1 in [CMP12].

Fixed $r \in (0, 1]$, we consider here the geometric flow with normal velocity equal to κ_r, namely if E_t denotes the evolution of a set $E \subset \mathbb{R}^2$ and $x_t \in \partial E_t$, we study the equation

$$\partial_t x_t \cdot \nu_{E_t}(x_t) = -\kappa_r(E_t, x_t). \tag{1.2}$$

The existence of a solution, in the viscosity sense, of such nonlocal geometric problem has been established in [CMP12, CMP13, CMP15][1]. The geometric equation in (1.2) can be seen as the gradient flow of a nonlocal functional built by the approximated Minkowski content, see [CGL10, CDNV17].

Of course, the quantity r in (1.2) plays a special role, producing a discontinuity in the velocity field of the geometric flow and detecting special features "at a small scale". Therefore, to perform our analysis, we consider a special class of sets, which are "slim" (or "pudgy") with respect to such scale.

Definition 1.1. *A set $E \subseteq \mathbb{R}^2$ is called "r-pudgy" if it contains a ball of radius r. Otherwise, it is called "r-slim".*

A particular case of r-slim sets is given by those which have the "diameter in one direction" that is less than r:

Definition 1.2. *A set $E \subseteq \mathbb{R}^2$ is called "r-thin" if, after a rigid motion, it holds that $E \subseteq \mathbb{R} \times (-\rho, \rho)$, with $\rho \in (0, r)$.*

The first problem that we take into account is the possible formation of neckpinch singularities in the flow defined by (1.2). We recall that in the classical mean curvature flow (or curve shortening flow), Grayson's Theorem [Gra87] gives that no singularity occurs in the plane, and, in fact, the initial set becomes convex and then shrinks smoothly towards a point. Interestingly, such result is not true for the planar nonlocal geometric flow in (1.2) and neckpinch singularities occur.

We construct two families of counterexamples, one for r-thin and one for r-pudgy sets. The first result is the following:

[1] As a technical remark, we point out that in [CMP12, CMP13, CMP15] a smoothed version of κ_r (which can be considered as a nonlocal curvature κ_f depending on a given function f) is taken into account, and an existence and uniqueness result is established for such flow. By approximating $\chi_{(0,r)}$ with a smooth function f and taking limits, one could deduce from this the existence of a viscosity solution for the flow driven by κ_r. For additional details on this, see the forthcoming Section 2.

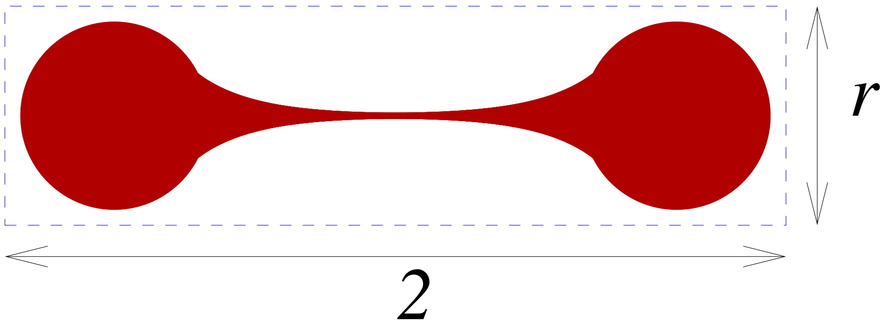

Figure 1: *The set in Theorem 1.3.*

Theorem 1.3 (Neckpinch singularity formation for r-thin sets)**.** *Assume that r is sufficiently small. Then, there exists an r-thin connected set $E \subset (-1,1) \times \left(-\frac{r}{2}, \frac{r}{2}\right)$, with C^∞ boundary and such that the viscosity solution of the r-mean curvature flow* (1.2) *starting from E does not shrink to a point (and the viscosity evolution of E becomes disconnected).*

Such E has a "narrow dumbbell shape" in the sense that it is obtained by gluing two balls of radius $r/4$ with a neck contained in $(-1,1) \times \left(-\frac{r}{100}, \frac{r}{100}\right)$.

The idea of the set constructed in Theorem 1.3 is depicted in Figure 1. Roughly speaking, the vertical trapping of the set will force the inner r-curvatures κ_r^- in (1.1) to vanish, while the outer r-curvatures κ_r^+ will make the neck of the set shrink faster than the two balls on the side, thus producing the singularity.

In a sense, the example in Theorem 1.3 is quite "pathological" since the singularity is produced by all the sets lying in a very small slab. Next result provides instead an example of a dumbbell in which the two initial balls have radius of order one, and still the evolution produces a singularity:

Theorem 1.4 (Neckpinch singularity formation for r-pudgy sets)**.** *Assume that r is sufficiently small. Then, there exists an r-pudgy connected set $E \subset (-10,10) \times (-10,10)$, with C^∞ boundary and such that the viscosity solution of the r-mean curvature flow* (1.2) *starting from E does not shrink to a point (and the viscosity evolution of E becomes disconnected).*

Such E has a "fat dumbbell shape" in the sense that it is obtained by gluing two balls of radius R, with $R > 0$ independent of r, and with a neck contained in $(-10,10) \times \left(-\frac{r}{100}, \frac{r}{100}\right)$.

A picture of the set E in Theorem 1.4 is sketched in Figure 6 on page 13.

We notice that the formation of neckpinch singularities also in low dimension is a treat shared by other nonlocal geometric flows, see [CSV16]. Nevertheless, the case in (1.2) is conceptually quite different than that in [CSV16], since the latter is scaling invariant and the nonlocal aspect of the curvature involves the global geometry of the set (while (1.2)

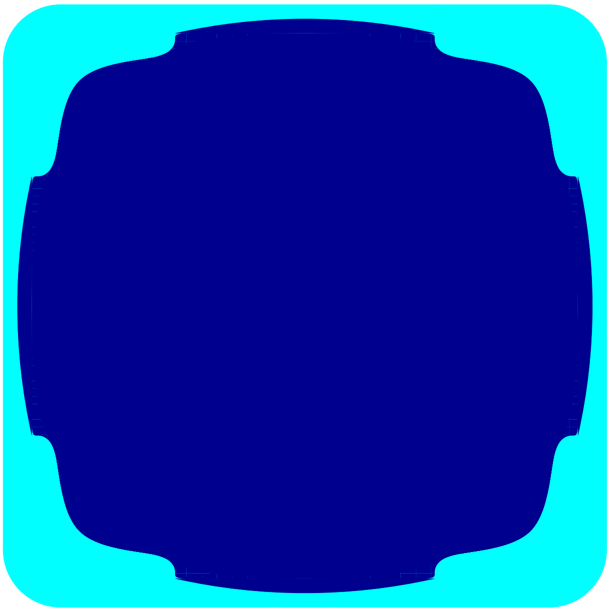

Figure 2: *The loss of convexity in Theorem 1.6.*

is not scaling invariant and the calculation of κ_r only involves a neighborhood of fixed side of a given point).

Interestingly, the examples considered in Theorems 1.3 and 1.4 have initial sets possessing a "large curvature" at some points. We think that it is interesting to investigate whether or not singularities may also emerge when the initial set has curvatures that are controlled uniformly when r is small.

Now, we study the convexity preservation for the flow in (1.2). This is a classical topic in the setting of the mean curvature flow, see e.g. [GH86], and, once again, the behavior of the solutions of (1.2) turns out to be very different from the classical case. Indeed, the geometric flow in (1.2), differently from the mean curvature flow, does not preserve convexity in general, but so it does for r-thin sets. In details, the results that we have are the following:

Theorem 1.5 (Convexity preserving for r-thin sets)**.** *Let E_t be a smooth evolution of a set E according to the flow in* (1.2)*. Suppose that E is r-thin and convex. Then so is E_t (till the extinction time).*

Theorem 1.6 (Convexity loss for r-pudgy sets)**.** *There exists a smooth convex set E which cannot have a $C^{1,1}$-evolution E_t for short times $t \in [0, T)$ according to* (1.2) *which preserves its convexity for $t \in (0, T)$.*

We have sketched in Figure 2 a possible loss of convexity for the geometric flow in (1.2) (a quantitative version of this picture will ground the rigorous analysis performed in Section 6).

We refer to [SV15, CNR17] for results on the preservation of a nonlocal mean curvature and of the convex structure of a set under a fractional mean curvature evolution (but the situation of (1.2) here is very different and indeed Theorem 1.6 says that no convexity preservation holds in this case, underlying a different behavior with respect to the classical mean curvature flow and also with respect to the fractional mean curvature flow).

It is interesting to observe that the example constructed in Theorem 1.6 starts from an initial set possessing a "large curvature" at some points. We think that it is interesting to investigate whether or not convexity is preserved if the initial set has curvatures that are positive and bounded uniformly when r is small.

Now, we consider traveling waves for the flow in (1.2), i.e. solutions of (1.2) in which E_t is of the form

$$y > h(x) + ct, \tag{1.3}$$

for some real function h, and $c \in \mathbb{R}$. In this setting, we have that the geometric flow in (1.2) presents a new class of traveling waves, which are obtained by gluing together a convex function depending on r near the origin and the "standard grim reaper" at infinity:

Theorem 1.7. *For any*

$$c \in (0, r) \tag{1.4}$$

there exists a traveling wave for the geometric flow in (1.2) *with speed equal to* c. *The corresponding traveling set is* C^2 *and convex.*

A more precise description of the shape of this traveling wave will be given in the forthcoming formula (7.16). See also Figure 8 for a picture of such traveling wave.

The rest of the paper is organized as follows. In Section 2, we first discuss some "pathologies" of the geometric flow in (1.2) and recall an approximation scheme exploited in [BKL+10, CMP12, CMP15] (such approximation is not explicitly used here, but it provides a conceptual framework for the flow in (1.2) from a viscosity perspective). In Section 3 we consider the neckpinch formation for r-thin sets and we prove Theorem 1.3. Then, in Section 4 we consider the neckpinch formation for r-pudgy sets and we prove Theorem 1.4. The convexity preservation for r-thin sets is discussed in Section 5, where we present the proof of Theorem 1.5. The loss of convexity and the proof of Theorem 1.6 are presented in Section 6, and the traveling waves, with the proof of Theorem 1.7, are discussed in Section 7.

2 A viscosity approximation of (1.2)

The geometric flow in (1.2) is rather special, given its lack of invariance and different behaviors at different scales. Also, the velocity field is discontinuous (even for convex sets) at points where tangent balls possess two or more projections along the boundary. This lack of regularity in the velocity produces some instability properties in the set evolution of (1.2) (corresponding to a "fattening" of the associated evolution by level sets). For instance, if one considers the initial set $E := \mathbb{R} \times (-\ell, \ell)$, from (1.1) it holds that $\kappa_r = 0$ if $\ell > r$ and $\kappa_r = \frac{1}{2r}$ if $\ell \in (0, r)$. Therefore this set stays put under the geometric flow in (1.2) if $\ell > r$, but it shrinks to a line in finite time if $\ell \in (0, r)$.

Due to phenomena of this sort, to compensate the lack of continuity of the velocity field in (1.2), it is desirable to approximate such flow with a more regular one. For this, we recall a procedure discussed in Section 6.4 of [CMP15]. We consider a smooth function $f : \mathbb{R} \to [0, 1]$, which is even, supported in $[-r, r]$ and such that $f(x) = 1$ for any $x \in \left[-\frac{r}{2}, \frac{r}{2}\right]$, and $f'(x) \leqslant 0$ for any $x \geqslant 0$. Recalling (1.1), for any $x \in \partial E$ one defines

$$\kappa_f(x, E) := \kappa_f^+(x, E) + \kappa_f^-(x, E),$$

where

$$\begin{aligned} \kappa_f^+(x, E) &:= -\int_0^r \frac{\sigma}{r} f'(\sigma)\, \kappa_\sigma^+(x, E)\, d\sigma \\ \text{and} \quad \kappa_f^-(x, E) &:= -\int_0^r \frac{\sigma}{r} f'(\sigma)\, \kappa_\sigma^-(x, E)\, d\sigma. \end{aligned} \tag{2.1}$$

Then, one can consider the geometric flow associated to κ_f, that is, the flow in (1.2) with κ_r replaced by κ_f,

$$\partial_t x_t \cdot \nu_{E_t}(x_t) = -\kappa_f(E_t, x_t). \tag{2.2}$$

This flow can be seen as an approximation of that in (1.2) and it is used in [CMP12] to establish uniqueness results and in [BKL$^+$10] for numerical purposes.

In this article, we will not make use explicitly of the flow in (2.2) (though similar arguments as the ones exploited here may be used in such framework as well), but we expect that solutions of (1.2) emerge from an appropriate limit of the viscosity solutions of (2.2) as the function f approaches the characteristic function of $(0, r)$. To rigorously perform such limit procedure, one has to check uniform regularity properties of the viscosity solutions of (2.2).

3 Proof of Theorem 1.3

3.1 Geometric barriers

We start with the construction of a narrow barrier. To this aim, we fix $\eta \in \left(0, \frac{r}{64\pi^2}\right)$ and we define

$$G_\eta := \left\{ (x, y) \in \mathbb{R}^2 \text{ s.t. } |y| \leqslant \eta + \frac{r}{32\pi^2}\big(1 - \cos(4\pi x)\big) \right\}.$$

Then, we have:

Lemma 3.1. *If r is sufficiently small then*

$$\kappa_r(G_\eta, p) \geqslant \frac{1}{4r} \qquad \textit{for any } p = (p_1, p_2) \in \partial G_\eta. \tag{3.1}$$

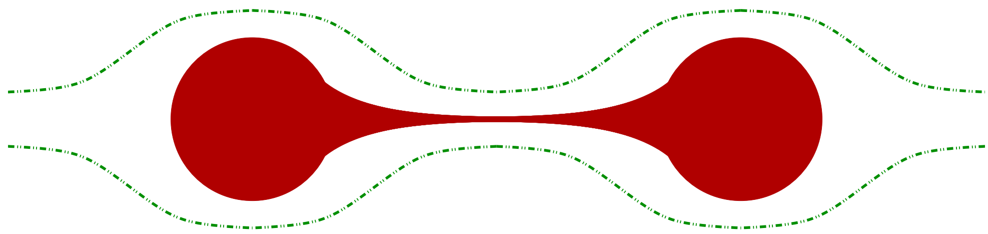

Figure 3: *The sets E and G_{η_0} in the proof of Theorem 1.3.*

Proof. Let

$$g_\eta(x) := \eta + \frac{r}{32\pi^2}\big(1 - \cos(4\pi x)\big).$$

We have that

$$|g_\eta'(x)| = \left|\frac{r}{8\pi}\sin(4\pi x)\right| \leqslant \frac{r}{8\pi}$$
$$\text{and} \qquad |g_\eta''(x)| = \left|\frac{r}{2}\cos(4\pi x)\right| \leqslant \frac{r}{2}.$$

This implies that the curvature of the graph of g_η is bounded in absolute value by $\frac{r}{2}$, provided that r is sufficiently small, and therefore G_η can always be touched from outside by a ball of radius r. Consequently, by (1.1), it holds that

$$\kappa_r^+(G_\eta, x) = \frac{\kappa(G_\eta, x)}{2} + \frac{1}{2r} \geqslant -\frac{r}{2} + \frac{1}{2r}. \tag{3.2}$$

On the other hand, the vertical diameter of G_η is $\frac{r}{16\pi^2}$ and so no ball of radius r can be contained inside G_η. Therefore, by (1.1), we conclude that $\kappa_r^-(G_\eta, x) = 0$. This and (3.2) give that

$$\kappa_r(G_\eta, x) \geqslant -\frac{r}{2} + \frac{1}{2r},$$

from which the desired result follows. □

3.2 Completion of the proof of Theorem 1.3

With Lemma 3.1, we can complete the proof of Theorem 1.3. For this, we take $t_r > 0$ to be the extinction time of the ball $B_{r/10^6}$. We define

$$\eta_0 := \min\left\{\frac{t_r}{8r}, \frac{r}{128\pi^2}\right\} \qquad \text{and} \qquad \eta(t) := \eta_0 - \frac{t}{4r}.$$

We take $q_\pm := \left(\pm\frac{1}{2}, 0\right)$ and

$$N := [-q_-, q_+] \times \left[-\frac{\eta_0}{2}, \frac{\eta_0}{2}\right].$$

We consider a connected and smooth set $E \subset (-1,1) \times \left(-\frac{r}{2}, \frac{r}{2}\right)$ such that

$$G_{\eta_0} \supseteq E \supseteq B_{r/10^6}(q_-) \cup B_{r/10^6}(q_+) \cup N,$$

with $E \cap \{|x| \leqslant 1/10\} = N$, see Figure 3. Confronting with halfplanes, the comparison principle for (1.2) (see e.g. Section 3.3 in [CMP12]) gives that also the evolution E_t lies in $(-1,1) \times \left(-\frac{r}{2}, \frac{r}{2}\right)$. In addition, the velocity of $G_{\eta(t)}$ in modulus coincides with $\frac{1}{4r}$, which is less than the r-curvature of $G_{\eta(t)}$, thanks to (3.1). As a consequence of this and of the comparison principle, we obtain that $E_t \subseteq G_{\eta(t)}$. Since $t_\star := 4r\eta_0 < t_r$ and $\eta(t_\star) = 0$, we have that $G_{\eta(t_\star)}$ develops a neck singularity, and so does E_t for some $t \in (0, t_\star)$. This completes the proof of Theorem 1.3.

4 Proof of Theorem 1.4

4.1 Geometric barriers

This section is devoted to the construction of an explicit barrier for the geometric flow in (1.2). Roughly speaking, this barrier is constructed by taking the region trapped between a graph and its reflection along the horizontal axis. Such graph is constructed by interpolating a parabola with curvature comparable to r near the origin with a uniformly concave function. The interpolation will occur when the values on the abscissa are of order r and the functions are also of order r, but the gradients are of order 1. This quantitative construction is needed to compute efficiently the r-curvatures in (1.1) and the example that we provide may turn out to be useful also in other cases.

We fix $M \geqslant 1$, to be taken appropriately large in the sequel. We also consider a bump function $\varphi \in C_0^\infty([-2,2],[0,1])$, with $\varphi = 1$ in $[-1,1]$, $|\varphi'| \leqslant 2$ and $|\varphi''| \leqslant 2$. We also define $\rho := Mr$ and, for any $x \in \mathbb{R}$,

$$g(x) := \frac{x^2}{2M^2\rho}\,\varphi\left(\frac{x}{\rho}\right) + \left(1 - \varphi\left(\frac{x}{\rho}\right)\right) \frac{|x|}{M^2\,(1+|x|)}.$$

For any $\varepsilon > 0$ we set $g_\varepsilon(x) := \varepsilon + g(x)$ and

$$F_\varepsilon := \Big\{(x,y) \in \mathbb{R}^2 \text{ s.t. } |y| \leqslant g_\varepsilon(x)\Big\}.$$

The graph of g_ε is depicted in Figure 4. The set F_ε is a useful barrier for the r-geometric flow, according to the following calculation:

Lemma 4.1. *There exist $M > 1$ and $c_0 \in (0,1)$ such that if $r \in \left(0, \frac{1}{M}\right)$ and $\varepsilon \in \left(0, \frac{r}{M}\right)$ then*

$$\kappa_r(F_\varepsilon, p) \geqslant c_0 \qquad \text{for any } p = (p_1,p_2) \in \partial F_\varepsilon \text{ with } |p_1| \leqslant 10. \tag{4.1}$$

Proof. By symmetry, we can reduce our analysis to the first quadrant, i.e. prove (4.1) for $p = (p_1,p_2) \in \partial F_\varepsilon$, with $p_1 \in [0,10]$ and $p_2 \in [0,+\infty)$. For any $x \in [0,10]$, it holds

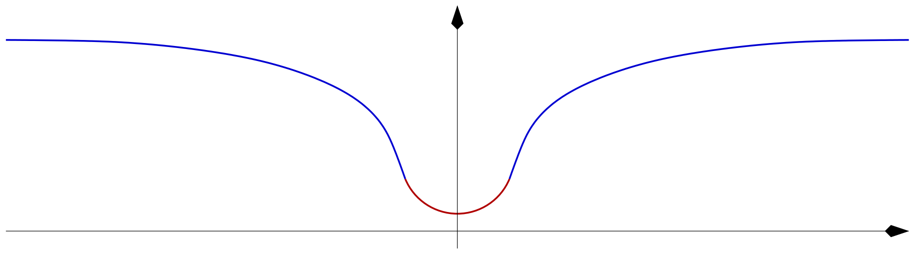

Figure 4: *The function* g_ε.

that

$$g_\varepsilon'(x) = \frac{x}{M^2\rho}\,\varphi\left(\frac{x}{\rho}\right) + \frac{x^2}{2M^2\rho^2}\,\varphi'\left(\frac{x}{\rho}\right) - \varphi'\left(\frac{x}{\rho}\right)\,\frac{x}{M^2\,\rho\,(1+x)} + \left(1-\varphi\left(\frac{x}{\rho}\right)\right)\,\frac{1}{M^2\,(1+x)^2}$$

and

$$g_\varepsilon''(x) = \frac{1}{M^2\rho}\,\varphi\left(\frac{x}{\rho}\right) + \frac{2x}{M^2\rho^2}\,\varphi'\left(\frac{x}{\rho}\right) + \frac{x^2}{2M^2\rho^3}\,\varphi''\left(\frac{x}{\rho}\right) - \varphi''\left(\frac{x}{\rho}\right)\,\frac{x}{M^2\,\rho^2\,(1+x)} - \varphi'\left(\frac{x}{\rho}\right)\,\frac{2}{M^2\,\rho\,(1+x)^2} - \left(1-\varphi\left(\frac{x}{\rho}\right)\right)\,\frac{2}{M^2\,(1+x)^3}.$$

Consequently, for any $x \in [0, 10]$,

$$\begin{aligned} |g_\varepsilon''(x)| &\leqslant \frac{1}{M^2\rho} + \frac{8\rho}{M^2\rho^2} + \frac{8\rho^2}{2M^2\rho^3} + \frac{4\rho}{M^2\,\rho^2} + \frac{4}{M^2\,\rho} + \frac{2}{M^2} \\ &= \frac{(21+2\rho)}{M^2\,\rho} \\ &= \frac{(21+2Mr)}{M^3\,r} \\ &\leqslant \frac{23}{M^3\,r}. \end{aligned} \tag{4.2}$$

Also,

$$\text{for any } x \in [2\rho, 10], \qquad |g_\varepsilon'(x)| = \frac{1}{M^2\,(1+x)^2} \leqslant \frac{1}{M^2}. \tag{4.3}$$

Moreover,

$$\text{for any } x \in [2\rho, 10], \qquad g_\varepsilon''(x) = -\frac{2}{M^2\,(1+x)^3} \in \left[-\frac{2}{M^2}, -\frac{2}{M^2\,11^3}\right]. \tag{4.4}$$

On the other hand,

$$\text{for any } x\in[0,2\rho], \qquad \begin{aligned} g_\varepsilon(x) &\leqslant \varepsilon + \frac{4\rho^2}{2M^2\rho} + \frac{2\rho}{M^2} \\ &= \varepsilon + \frac{4\rho}{M^2} \\ &\leqslant \frac{5r}{M}. \end{aligned} \tag{4.5}$$

In the same way, we see that

$$\text{for any } x\in[0,2\rho], \qquad \begin{aligned} |g_\varepsilon'(x)| &\leqslant \frac{2\rho}{M^2\rho} + \frac{8\rho^2}{2M^2\rho^2} + \frac{4\rho}{M^2\,\rho} + \frac{1}{M^2} \\ &\leqslant \frac{11}{M^2}. \end{aligned} \tag{4.6}$$

Now, since the curvature of the graph is given by

$$-\left(\frac{g_\varepsilon'}{\sqrt{1+(g_\varepsilon')^2}}\right)' = -\frac{g_\varepsilon''}{(1+(g_\varepsilon')^2)^{3/2}},$$

it follows from (4.2) that

$$\text{the curvature of the graph is bounded everywhere in absolute value by } \frac{23}{M^3\,r}, \tag{4.7}$$

which is less than $\frac{1}{r}$ if M is sufficiently large. Hence, the set F_ε can always be touched from outside by balls of radius r, i.e. $B^{\text{ext}}_{r,p}\subseteq\mathbb{R}^2\setminus F_\varepsilon$ and so, by (1.1),

$$\kappa_r^+(F_\varepsilon,p) = \frac{\kappa(F_\varepsilon,p)}{2} + \frac{1}{2r}. \tag{4.8}$$

Similarly, from (4.3) and (4.4), we infer that

$$\begin{aligned} &\text{the curvature of the graph with abscissa in } [2\rho,10] \\ &\text{is bounded from below by } \frac{1}{M^2\,11^3}. \end{aligned} \tag{4.9}$$

Now, to compute $\kappa_r^-(F_\varepsilon,p)$ we distinguish the two cases $p_1\in[0,2\rho]$ and $p_1\in[2\rho,10]$. If $p_1\in[0,2\rho]$ we claim that

$$B^{\text{int}}_{r,p}\cap\{y<-g_\varepsilon(x)\}\neq\emptyset. \tag{4.10}$$

To check this, we use Figure 5 and we notice that the exterior normal at p is given by

$$\nu_{F_\varepsilon}(p) = \frac{(-g_\varepsilon'(p_1),1)}{\sqrt{1+(g_\varepsilon'(p_1))^2}}$$

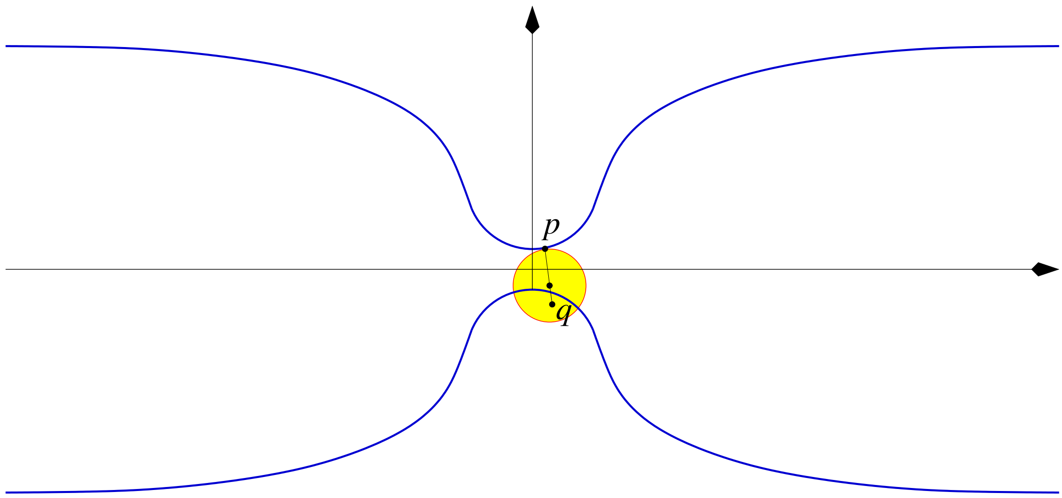

Figure 5: *The geometry involved in the proof of* (4.10).

and

$$q = (q_1, q_2) := p - r\nu_{F_\varepsilon}(p) - \frac{r\nu_{F_\varepsilon}(p)}{2} \in B^{\rm int}_{r,p}.$$

So, to prove (4.10), it is enough to check that

$$q_2 < -g_\varepsilon(q_1). \tag{4.11}$$

As a matter of fact, since $p_2 = g_\varepsilon(p_1)$,

$$\begin{aligned}
q_2 + g_\varepsilon(q_1) &= \left(p - \frac{3r\nu_{F_\varepsilon}(p)}{2}\right)_2 + g_\varepsilon\left(\left(p - \frac{3r\nu_{F_\varepsilon}(p)}{2}\right)_1\right) \\
&= g_\varepsilon(p_1) - \frac{3r}{2\sqrt{1+(g'_\varepsilon(p_1))^2}} + g_\varepsilon\left(p_1 + \frac{3r\, g'_\varepsilon(p_1)}{2\sqrt{1+(g'_\varepsilon(p_1))^2}}\right) \\
\text{by (4.6)} \qquad &\leqslant 2g_\varepsilon(p_1) - \frac{3r}{2\sqrt{1+(11/M^2)^2}} + \sup_{x\in[0,10]} |g'_\varepsilon(x)| \frac{3r\,|g'_\varepsilon(p_1)|}{2\sqrt{1+(g'_\varepsilon(p_1))^2}} \\
\text{by (4.3), (4.5) and (4.6)} \qquad &\leqslant \frac{10\,r}{M} - \frac{3r}{2\sqrt{1+(11/M^2)^2}} + \left(\frac{11}{M^2}\right)^2 \frac{3r}{2} \\
&\leqslant \frac{r}{2} - r,
\end{aligned}$$

as long as M is sufficiently large. This proves (4.11), and so (4.10).

Then, from (1.1) and (4.10), we obtain that

$$\text{if } p_2 \in [0, 2\rho], \text{ then } \kappa_r^-(F_\varepsilon, p) = 0.$$

From this and (4.8), we conclude that

$$\text{if } p_2 \in [0, 2\rho], \text{ then } \kappa_r(F_\varepsilon, p) = \frac{\kappa(F_\varepsilon, p)}{2} + \frac{1}{2r}.$$

This and (4.7) imply that

$$\text{if } p_2 \in [0, 2\rho], \text{ then } \kappa_r(F_\varepsilon, p) \geqslant -\frac{23}{2\,M^3\,r} + \frac{1}{2r} \geqslant \frac{1}{3r}, \tag{4.12}$$

as long as M is large enough.

On the other hand, by (4.9), we know that

$$\text{if } p_2 \in [2\rho, 10], \text{ then } \kappa(F_\varepsilon, p) \geqslant \frac{1}{M^2\,11^3}$$

and therefore, recalling (1.1) and (4.8), we have that

$$\begin{aligned}
\text{if } p_2 \in [2\rho, 10], \text{ then } \kappa_r(F_\varepsilon, p) &= \kappa_r^+(F_\varepsilon, p) + \kappa_r^-(F_\varepsilon, p)\\
&\geqslant \frac{\kappa(F_\varepsilon, p)}{2} + \frac{1}{2r} + \min\left\{0,\ \frac{\kappa(F_\varepsilon, p)}{2} - \frac{1}{2r}\right\}\\
&= \min\left\{\frac{\kappa(F_\varepsilon, p)}{2} + \frac{1}{2r},\ \kappa(F_\varepsilon, p)\right\}\\
&\geqslant \frac{1}{2\,M^2\,11^3}.
\end{aligned}$$

The desired result now follows plainly from this inequality and (4.12). □

4.2 Completion of the proof of Theorem 1.4

With the construction in Lemma 4.1, the proof of Theorem 1.4 follows by a comparison principle, with an argument similar to that in Section 3.2. We give the full argument for the facility of the reader. We take M and c_0 as in Lemma 4.1 and we define, for any $t \geqslant 0$,

$$\varepsilon(t) := \frac{r}{2M} - \frac{c_0\,t}{2}.$$

The set $F_{\varepsilon(t)}$ falls under the assumption of Lemma 4.1, so, by (4.1),

$$\kappa_r(F_{\varepsilon(t)}, p) \geqslant c_0 \qquad \text{for any } p = (p_1, p_2) \in \partial F_{\varepsilon(t)} \text{ with } |p_1| \leqslant 10. \tag{4.13}$$

Now we set $q_\pm := (\pm 3, 0)$ and

$$N := [-3, 3] \times \left[-\frac{\varepsilon(0)}{2}, \frac{\varepsilon(0)}{2}\right],$$

and we take $R > 0$ and a connected and smooth set $E \subset F_{\varepsilon(0)}$ such that

$$E \supseteq B_R(q_-) \cup B_R(q_+) \cup N.$$

The geometric situation of this proof is depicted in Figure 6. Notice that we can take such R independent of r and the modulus of the velocity of $F_{\varepsilon(t)}$ is $\frac{c_0}{2}$, which is less than the normal velocity of the flow (1.2), thanks to (4.13). Therefore, by comparison

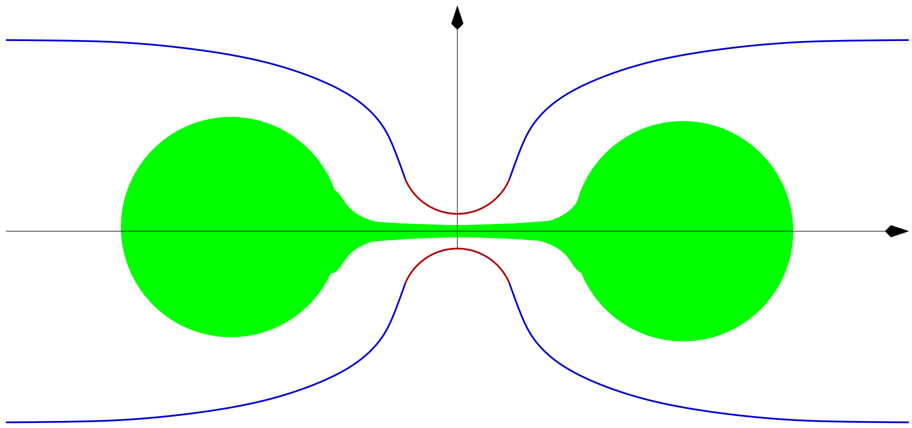

Figure 6: *The sets E and $F_{\varepsilon(0)}$.*

principle (see e.g. Section 3.3 in [CMP12]), we conclude that $E_t \subset \{|x| \leqslant 10\}$ and, more importantly, $E_t \subset F_{\varepsilon(t)}$ for all $t \in [0, T]$, being T the extinction time. Since the extinction time T is bounded from below by the one of the ball B_R, which is independent of r, we can take r sufficiently small and suppose that $\frac{r}{c_0 M} < T$. But, since $F_{\varepsilon(t)}$ develops a neckpinch at time $t = \frac{r}{c_0 M}$, also the viscosity evolution of the set E_t develops a singularity before such time, and gets disconnected. This completes the proof of Theorem 1.4.

5 Proof of Theorem 1.5

We compute the evolution of the curvature of a geometric flow with normal velocity v. For this, we denote by s the arclength variable and we recall (see e.g. formula (9) in [CNV11]) that if a set F_t is a solution of $\partial_t x_t \cdot \nu_{F_t}(x_t) = -v(x_t)$ then

$$\partial_t \kappa(F_t, x_t) = \partial^2_{ss} v(x_t) + \kappa^2(F_t, x_t)\, v(x_t). \tag{5.1}$$

Furthermore, comparing with halfplanes, we see that the evolution of E_t is r-thin for any time t (till extinction). Therefore, E_t does not contain balls of radius r and then, in view of (1.1), it holds that

$$\kappa^-_r(E_t, x_t) = 0 \quad \text{for any } x_t \in \partial E_t. \tag{5.2}$$

Now, suppose that E_t is convex: it follows from (1.1) that

$$\kappa^+_r(E_t, x_t) = \frac{\kappa(E_t, x_t)}{2} + \frac{1}{2r}.$$

This and (5.2) give that

$$\kappa_r(E_t, x_t) = \frac{\kappa(E_t, x_t)}{2} + \frac{1}{2r}.$$

As a consequence,

$$\kappa_r(E_t,x_t) \geqslant \frac{1}{2r}$$
$$\text{and} \qquad \partial^2_{ss}\kappa_r(E_t,x_t) = \frac{\partial^2_{ss}\kappa(E_t,x_t)}{2}.$$

In particular, by (5.1), we have that, if E_t is a solution of the geometric flow in (1.2), till it is convex it holds that

$$\begin{aligned}\partial_t\kappa(E_t,x_t) &= \partial^2_{ss}\kappa_r(E_t,x_t) + \kappa^2(E_t,x_t)\,\kappa_r(E_t,x_t)\\ &= \frac{\partial^2_{ss}\kappa(E_t,x_t)}{2} + \kappa^2(E_t,x_t)\,\kappa_r(E_t,x_t).\end{aligned}$$

Hence, if $x_t^\star$ is minimal for $\kappa(E_t,\cdot)$, we have that

$$\partial^2_{ss}\kappa(E_t,x_t^\star) \geqslant 0$$

and

$$\partial_t\kappa(E_t,x_t) \geqslant \kappa^2(E_t,x_t)\kappa_r(E_t,x_t) \geqslant 0.$$

This gives that $\kappa(E_t,\cdot)$ is nondecreasing at the minimal points, and thus nonnegative, which completes the proof of Theorem 1.5.

6 Proof of Theorem 1.6

We construct a convex set E as depicted in Figure 7. Namely, we consider the smoothing of a squared of side $1 >> r$ in which the corners are rounded by a curve with curvature of order $\varepsilon << r$. Notice that the curvature of ∂E vanishes along the dashed and solid line on the bottom of Figure 7, and it is of the order of $1/\varepsilon$ along the solid arc. As for the r-curvature, from (1.1) we see that κ_r is equal to κ along the solid line and to $\frac{\kappa}{2}+\frac{1}{2r}$ along the dashed line and the solid arc. That is, κ_r is equal to 0 along the solid line, equal to $\frac{1}{2r}$ along the dashed line and of order $\frac{1}{\varepsilon}$ along[2] the solid arc.

Consequently, if we put Cartesian axes as in Figure 7, we can describe κ_r along the bottom of the set E by a function φ, and, considering intervals $(a,b) \supset (c,d)$ as in Figure 7, it holds that $\varphi = \frac{1}{2r}$ on (c,d) and $\varphi = 0$ on (d,b). In particular,

$$\varphi \text{ is not convex.} \tag{6.1}$$

Now, to establish Theorem 1.6 we argue by contradiction and suppose that the set E evolves in time $t \in [0,T)$ in a $C^{1,1}$-fashion into a convex set E_t. Hence, the boundary

[2] It is interesting to remark that Figure 7 well explains the difference between the r-curvature and the classical one for convex sets. Namely, along the solid line, we are on a "large scale" and the r-curvature coincides with the classical one. Then, on a scale of order r, given by the dashed line, the r-curvature becomes greater than the classical one. But on a very small scale, the r-curvature may become smaller than the classical one, since, on the solid arc, we have that $\kappa_r = \frac{\kappa}{2}+\frac{1}{2r} < \kappa$, since $\kappa \sim \frac{1}{\varepsilon} >> \frac{1}{r}$.

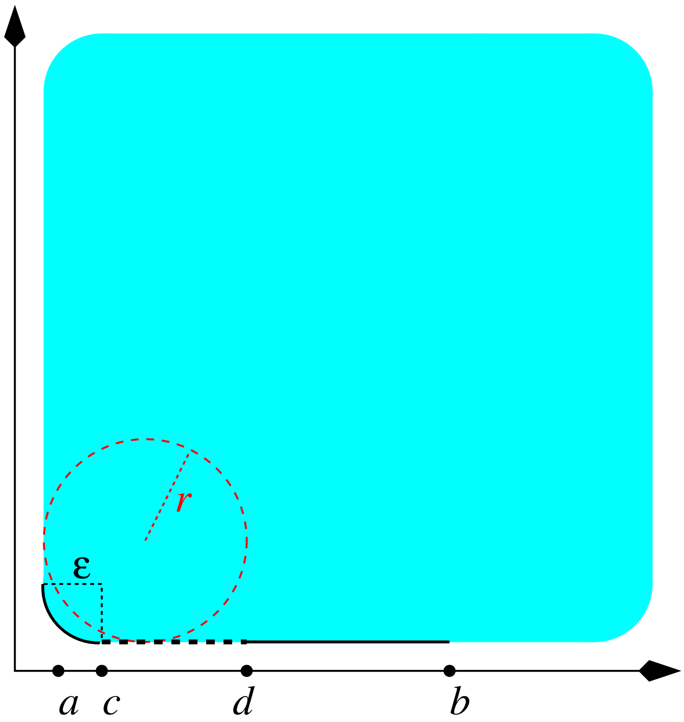

Figure 7: *The set in the proof of Theorem 1.6.*

of the set E_t can be locally described, in the vicinity of the interval (a, b) by a convex function $\Psi = \Psi(x, t)$. By (1.2) and the fact that

$$\Psi(x, 0) \text{ is constant for all } x \in (c, b), \tag{6.2}$$

we know that the velocity of the flow at time 0 coincides with $\partial_t \Psi(x, 0)$, hence

$$\partial_t \Psi(x, 0) = \varphi(x) \quad \text{ for all } x \in (c, b). \tag{6.3}$$

Also, since Ψ is convex, for any x, $y \in (c, b)$, any $\vartheta \in [0, 1]$ and any $t \in (0, T)$ we have that

$$\Psi\big((1 - \vartheta)x + \vartheta y, t\big) \leqslant (1 - \vartheta)\Psi(x, t) + \vartheta \Psi(y, t),$$

and therefore, in view of (6.2) and (6.3),

$$\begin{aligned}
\varphi((1-\vartheta)x+\vartheta y) &= \partial_t\Psi\big((1-\vartheta)x+\vartheta y,0\big)\\
&= \lim_{t\searrow 0}\frac{\Psi\big((1-\vartheta)x+\vartheta y,t\big)-\Psi\big((1-\vartheta)x+\vartheta y,0\big)}{t}\\
&\leqslant \lim_{t\searrow 0}\frac{(1-\vartheta)\Psi(x,t)+\vartheta\Psi(y,t)}{t}-\frac{\Psi\big((1-\vartheta)x+\vartheta y,0\big)}{t}\\
&= \lim_{t\searrow 0}\frac{(1-\vartheta)\Psi(x,t)+\vartheta\Psi(y,t)}{t}-\frac{(1-\vartheta)\Psi\big(x,0\big)+\vartheta\Psi\big(y,0\big)}{t}\\
&= \lim_{t\searrow 0}\frac{(1-\vartheta)\big(\Psi(x,t)-\Psi(x,0)\big)+\vartheta\big(\Psi(y,t)-\Psi(y,0)\big)}{t}\\
&= (1-\vartheta)\partial_t\Psi(x,0)+\vartheta\partial_t\Psi(y,0)\\
&= (1-\vartheta)\varphi(x)+\vartheta\varphi(y).
\end{aligned}$$

This gives that φ is convex in (c,b), which is a contradiction with (6.1) and so Theorem 1.6 is proved.

7 Proof of Theorem 1.7

Without loss of generality, we can normalize the speed c to be equal to 1. Such dilation, in the new coordinate frame, transforms condition (1.4) into

$$r > 1. \tag{7.1}$$

Then, if h is a traveling wave as in (1.3) with $c = 1$ for a geometric flow with inner normal velocity v, one sees that h is a solution of

$$v(x)\sqrt{1+|h'(x)|^2} = 1. \tag{7.2}$$

In particular, for the classical mean curvature flow, we have that

$$v(x) = \frac{h''(x)}{(1+|h'(x)|^2)^{3/2}},$$

and so (7.2) becomes

$$\frac{h''(x)}{1+|h'(x)|^2} = 1. \tag{7.3}$$

Similarly, in the regime in which $\kappa_r = \frac{\kappa}{2} + \frac{1}{2r}$, the flow in (1.2) and (7.2) yield the equation

$$\frac{h''(x)}{1+|h'(x)|^2} = 2 - \frac{\sqrt{1+|h'(x)|^2}}{r}. \tag{7.4}$$

Our objective is now to consider a (suitable translation of a) solution h_∞ of (7.3) (far from the origin) and glue it to a solution h_0 of (7.4) (near the origin). The joint will be done in a C^1 fashion and this will ensure in fact that the final curve is C^2 (roughly speaking, the building arcs will share the tangent line and the curvature at the matching point).

To implement this construction, we consider the Cauchy problem

$$\begin{cases} \phi'(x) = 2(1+|\phi(x)|^2) - \frac{1}{r}\,(1+|\phi(x)|^2)^{3/2}, \\ \phi(0) = 0. \end{cases} \tag{7.5}$$

The solution to this problem exists (and it is unique) for small values of x, and we extend it to its largest existence interval (x_-, x_+), with $-\infty \leqslant x_- < 0 < x_+ \leqslant +\infty$. We claim that

$$\text{for all } x \in (x_-, x_+), \text{ we have that } \phi(x) \leqslant \sqrt{4r^2-1}. \tag{7.6}$$

Indeed, suppose, by contradiction, that there exists $\bar{x} \in (x_-, x_+)$ such that $\phi(\bar{x}) > \sqrt{4r^2-1}$. Then, for any $\varepsilon > 0$ sufficiently small, there exists $\bar{x}_\varepsilon$ on the segment joining 0 to $\bar{x}$ such that $\phi(\bar{x}_\varepsilon) = \sqrt{4r^2(1+\varepsilon)-1}$ with $\phi'(\bar{x}_\varepsilon) \geqslant 0$. Then, we have that

$$\begin{aligned} 0 &\leqslant \phi'(\bar{x}_\varepsilon) \\ &= 2(1+|\phi(\bar{x}_\varepsilon)|^2) - \frac{1}{r}\,(1+|\phi(\bar{x}_\varepsilon)|^2)^{3/2} \\ &= 2(4r^2(1+\varepsilon)) - \frac{1}{r}\,(4r^2(1+\varepsilon))^{3/2} \\ &= 8r^2(1+\varepsilon) - 8r^2(1+\varepsilon)^{3/2} \\ &= 8r^2(1+\varepsilon)(1-\sqrt{1+\varepsilon}) \\ &< 0, \end{aligned}$$

which is a contradiction, proving (7.6).

We also have that

$$\text{for all } x \in (x_-, x_+), \text{ it holds that } \phi(x) \geqslant 0. \tag{7.7}$$

Indeed, suppose, by contradiction, that for any $\varepsilon > 0$ small enough there exists $\tilde{x}_\varepsilon \in (x_-, x_+)$ such that $\phi(\tilde{x}_\varepsilon) = -\varepsilon$ with $\phi'(\tilde{x}_\varepsilon) \leqslant 0$. Then, it holds that

$$\begin{aligned} 0 &\geqslant \phi'(\tilde{x}_\varepsilon) \\ &= 2(1+|\phi(\tilde{x}_\varepsilon)|^2) - \frac{1}{r}\,(1+|\phi(\tilde{x}_\varepsilon)|^2)^{3/2} \\ &= 2(1+\varepsilon^2) - \frac{1}{r}\,(1+\varepsilon^2)^{3/2}. \end{aligned}$$

Hence, taking ε arbitrarily small, we obtain that $0 \geqslant 2 - \frac{1}{r}$, which gives a contradiction with (7.1), thus proving (7.7).

As a consequence of (7.6) and (7.7), we obtain that $x_+ = +\infty$ and $x_- = -\infty$, and so ϕ is a global solution of (7.5). In addition, since $x \mapsto -\phi(-x)$ is also a solution of (7.5), by the uniqueness result of the Cauchy problem we obtain that $\phi(x) = -\phi(-x)$, i.e.

$$\phi \text{ is even.} \tag{7.8}$$

Moreover, from (7.6), we have that

$$2-\frac{1}{r}\sqrt{1+|\phi(x)|^2}\geqslant 0$$

and so, by (7.5),

$$\phi'(x)=(1+|\phi(x)|^2)\left(2-\frac{1}{r}\sqrt{1+|\phi(x)|^2}\right)\geqslant 0.$$

Accordingly,

$$\phi \text{ is monotone nondecreasing} \tag{7.9}$$

and so the following limit exists

$$\ell:=\lim_{x\to+\infty}\phi(x).$$

Also, by (7.5), (7.6) and (7.9), it holds that

$$0=\lim_{x\to+\infty}\phi'(x)=2(1+\ell^2)-\frac{1}{r}(1+\ell^2)^{3/2},$$

and therefore

$$\lim_{x\to+\infty}\phi(x)=\ell=\sqrt{4r^2-1}.$$

Consequently, for any $\lambda\in(0,\sqrt{4r^2-1})$ there exists $x(\lambda)$ such that $\phi(x(\lambda))=\lambda$.

Now we claim that

$$\text{such } x(\lambda) \text{ is unique.} \tag{7.10}$$

Indeed, if $\phi(a)=\phi(b)=\lambda\in(0,\sqrt{4r^2-1})$ with $a<b$, the monotonicity of ϕ implies that ϕ is constantly equal to λ in $[a,b]$ and then $\phi'(x)=0$ for any $x\in(a,b)$. As a consequence, by (7.5), for any $x\in(a,b)$,

$$0=\phi'(x)=2(1+\lambda^2)-\frac{1}{r}(1+\lambda^2)^{3/2}=(1+\lambda^2)\left(2-\frac{1}{r}\sqrt{1+\lambda^2}\right)>0,$$

which proves (7.10).

Now, recalling (7.1), we consider $x(\lambda)$ with $\lambda:=\sqrt{r^2-1}$. That is, we take $x_r:=x(\sqrt{r^2-1})>0$ to be the unique solution of

$$\phi(x_r)=\sqrt{r^2-1}. \tag{7.11}$$

We also define

$$h_0(x):=\int_0^x\phi(\xi)\,d\xi. \tag{7.12}$$

Notice that h_0 is convex, in view of the monotonicity of ϕ, and it is even, due to (7.8). Also, by (7.5), we know that

$$\begin{cases} h_0''(x)=2(1+|h_0'(x)|^2)-\dfrac{1}{r}(1+|h_0'(x)|^2)^{3/2},\\ h_0(0)=0=h_0'(0),\end{cases}$$

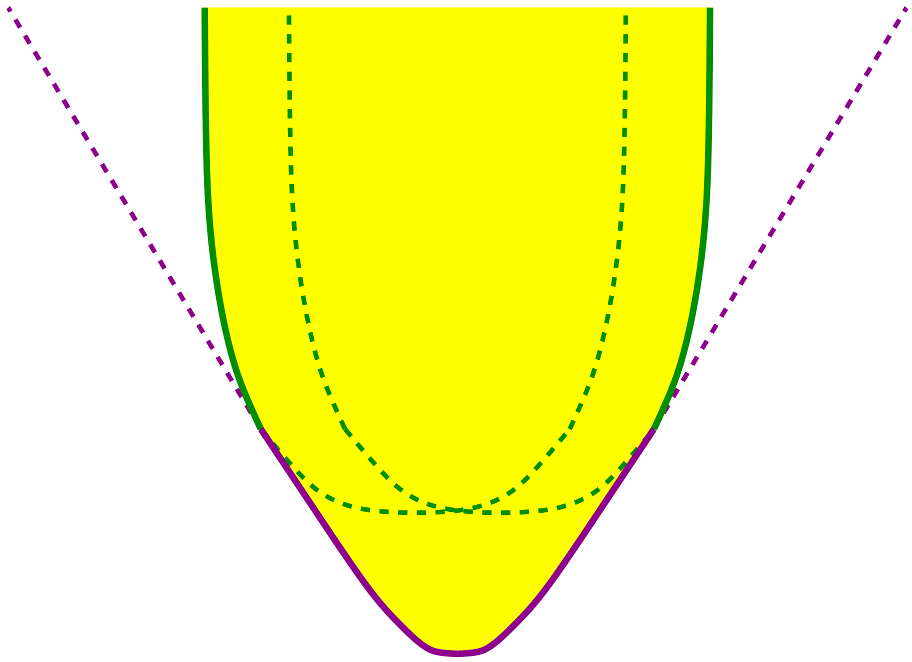

Figure 8: *The new traveling wave in* (7.16).

and so h_0 is a solution of (7.4). Furthermore, by (7.11),

$$h_0'(x_r) = \sqrt{r^2 - 1}. \tag{7.13}$$

Now we consider the "standard grim reaper"

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \ni x \mapsto h_\infty(x) = -\log(\cos x), \tag{7.14}$$

and we define

$$\tilde{x}_r := \arctan \sqrt{r^2 - 1} \in \left(0, \frac{\pi}{2}\right), \tag{7.15}$$

due to (7.1), and

$$h_\star(x) := \begin{cases} h_0(x) & \text{if } |x| \leqslant x_r, \\ h_\infty(x + \tilde{x}_r - x_r) - h_\infty(\tilde{x}_r) + h_0(x_r) & \text{if } x \in \left(x_r, \frac{\pi}{2} + x_r - \tilde{x}_r\right), \\ h_\infty(x - \tilde{x}_r + x_r) - h_\infty(\tilde{x}_r) + h_0(x_r) & \text{if } x \in \left(-\frac{\pi}{2} + \tilde{x}_r - x_r, -x_r\right). \end{cases} \tag{7.16}$$

The function $h_\star$ is depicted in Figure 8. By inspection, we see that $h_\star$ is continuous and even. Also, by (7.13), (7.14) and (7.15), we have that

$$h_\infty'(\tilde{x}_r) = \tan \tilde{x}_r = \sqrt{r^2 - 1} = h_0'(x_r),$$

and so $h_\star \in C^1(\mathbb{R})$.

Moreover, we have that

$$\cos(\arctan \vartheta) = \frac{1}{\sqrt{1 + \vartheta^2}}.$$

Using this identity with $\vartheta := \sqrt{r^2-1}$, we find that

$$h''_\infty(\tilde{x}_r) = \frac{1}{\cos^2 \tilde{x}_r} = \frac{1}{\cos^2(\arctan\sqrt{r^2-1})} = 1 + (r^2-1) = r^2. \tag{7.17}$$

Furthermore, recalling (7.5) and (7.11),

$$h''_0(x_r) = \phi'(x_r) = 2(1+|\phi(x_r)|^2) - \frac{1}{r}(1+|\phi(x_r)|^2)^{3/2} = 2r^2 - \frac{1}{r}(r^2)^{3/2} = r^2.$$

This and (7.17) say that $h_\star \in C^2(\mathbb{R})$.

To complete the proof of Theorem 1.7, we now check that the translating supgraph of $h_\star$ is a solution of the geometric flow in (1.2). For this, since such supgraph is convex, and h_∞ and h_0 are solutions of (7.3) and (7.4), respectively, it is enough to check that

$$\begin{aligned}&\text{the curvature of the subgraph is greater than } 1/r \text{ when } |x|<x_r\\ &\text{and smaller than } 1/r \text{ when } |x| \in \Big(x_r,\, \frac{\pi}{2}+x_r-\tilde{x}_r\Big).\end{aligned} \tag{7.18}$$

To this aim, we recall (7.4), (7.9) (7.11) and (7.12), and we remark that, if $|x|<x_r$,

$$\begin{aligned}\frac{h''_\star(x)}{(1+|h'_\star(x)|^2)^{3/2}} &= \frac{h''_0(x)}{(1+|h'_0(x)|^2)^{3/2}} = \frac{2}{\sqrt{1+|h'_0(x)|^2}} - \frac{1}{r} = \frac{2}{\sqrt{1+|\phi(x)|^2}} - \frac{1}{r}\\ &\geqslant \frac{2}{\sqrt{1+|\phi(x_r)|^2}} - \frac{1}{r} = \frac{2}{r} - \frac{1}{r} = \frac{1}{r}.\end{aligned}$$

This shows (7.18) when $|x|<x_r$. On the other hand, when $x \in \big(x_r,\, \frac{\pi}{2}+x_r-\tilde{x}_r\big)$, we have that $x+\tilde{x}_r-x_r \in \big(\tilde{x}_r,\, \frac{\pi}{2}\big) \subseteq \big(0,\, \frac{\pi}{2}\big)$, and thus we deduce from (7.14) and (7.15) that

$$\begin{aligned}\frac{h''_\star(x)}{(1+|h'_\star(x)|^2)^{3/2}} &= \frac{h''_\infty(x+\tilde{x}_r-x_r)}{(1+|h'_\infty(x+\tilde{x}_r-x_r)|^2)^{3/2}} = |\cos(x+\tilde{x}_r-x_r)|\\ &= \cos(x+\tilde{x}_r-x_r) \leqslant \cos(\tilde{x}_r) = \cos(\arctan\sqrt{r^2-1}) = \frac{1}{r}.\end{aligned}$$

This establishes (7.18) when $x \in \big(x_r,\, \frac{\pi}{2}+x_r-\tilde{x}_r\big)$, and the case $x \in \big(-\frac{\pi}{2}+\tilde{x}_r-x_r,\, -x_r\big)$ is symmetric. Hence, the proof of (7.18), and so of Theorem 1.7, is complete.

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