# Existence of periodic orbits near heteroclinic connections

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**Abstract**

We consider a potential $W : \mathbb{R}^m \to \mathbb{R}$ with two different global minima $a_-, a_+$ and, under a symmetry assumption, we use a variational approach to show that the Hamiltonian system

$$\ddot{u} = W_u(u), \tag{0.1}$$

has a family of $T$-periodic solutions $u^T$ which, along a sequence $T_j \to +\infty$, converges locally to a heteroclinic solution that connects $a_-$ to $a_+$. We then focus on the elliptic system

$$\Delta u = W_u(u), \quad u : \mathbb{R}^2 \to \mathbb{R}^m, \tag{0.2}$$

that we interpret as an infinite dimensional analogous of (0.1), where $x$ plays the role of time and $W$ is replaced by the action functional $J_{\mathbb{R}}(u) = \int_{\mathbb{R}}(\frac{1}{2}|u_y|^2 + W(u))dy$. We assume that $J_{\mathbb{R}}$ has two different global minimizers $\bar{u}_-, \bar{u}_+ : \mathbb{R} \to \mathbb{R}^m$ in the set of maps that connect $a_-$ to $a_+$. We work in a symmetric context and prove, via a minimization procedure, that (0.2) has a family of solutions $u^L : \mathbb{R}^2 \to \mathbb{R}^m$, which is $L$-periodic in $x$, converges to $a_\pm$ as $y \to \pm\infty$ and, along a sequence $L_j \to +\infty$, converges locally to a heteroclinic solution that connects $\bar{u}_-$ to $\bar{u}_+$.

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## 1 Introduction

The dynamics of the Newton equation

$$\ddot{u} = W'(u), \qquad W(u) = \frac{1}{4}(1-u^2)^2, \tag{1.1}$$

includes a heteroclinic solution $u^H : \mathbb{R} \to \mathbb{R}$ that connects $-1$ to 1:

$$\lim_{t\to\pm\infty} u^H(t) = \pm 1,$$

and a family of $T$-periodic solutions $u^T$ that, along a sequence $T_j \to +\infty$, converges to $u^H$

$$\lim_{T\to+\infty} u^T(t) = u^H(t),$$

uniformly in compact intervals.

Each map $t \to u^T(t)$ satisfies

$$u^T\left(\frac{T}{4} - t\right) = u^T\left(\frac{T}{4} + t\right), \quad t \in \mathbb{R},$$

and therefore oscillates twice for period on the same trajectory with extremes at $u^T(\pm\frac{T}{4})$ where the speed $\dot{u}^T(\pm\frac{T}{4})$ vanishes and for this reason is called a *brake orbit*. There is a large literature on brake orbits [17], [16], [8], [21].

We can ask whether a similar picture holds true in the vector case where $W : \mathbb{R}^m \to \mathbb{R}$, $m > 1$ satisfies

$$0 = W(a_\pm) < W(u), \;\; u \neq a_\pm, \tag{1.2}$$

for some $a_- \neq a_+ \in \mathbb{R}^m$, or even in the infinite dimensional case where the potential $W$ is replaced by a functional $J : \mathcal{H} \to \mathbb{R}$, where $\mathcal{H}$ is a suitable function space, with two distinct global minima $\bar{u}_\pm \in \mathcal{H}$ that correspond to the zeros $a_\pm$ of $W$ in the finite dimensional case.

If we assume that $W$ is of class $C^2$ and that $a_\pm$ are non degenerate in the sense that the Hessian matrix $W_{uu}(a_\pm)$ is positive definite, the existence of a family of $T$-periodic brake maps that, as $T \to +\infty$, converges to a heteroclinic connection between $a_-$ and $a_+$ can be established by direct minimization of the action functional

$$J_{(t_1^u, t_2^u)}(u) = \int_{t_1^u}^{t_2^u} \Big(\frac{1}{2}|\dot{u}|^2 + W(u)\Big) ds, \qquad -\infty < t_1^u < t_2^u < +\infty,$$

on a suitable set of admissible maps $u \in H^1((t_1^u, t_2^u); \mathbb{R}^m)$. Indeed the non degeneracy of $a_\pm$ implies that, for small $\delta > 0$, the boundary of the set $\{u \in \mathbb{R}^m : W(u) > \delta\}$ is partitioned into two compact connected subsets $\Gamma_-$ and $\Gamma_+$ that satisfy the condition

$$W_u(u) \neq 0, \;\; u \in \Gamma_\pm. \tag{1.3}$$

Then Theorem 5.5 in [1] or Corollary 1.5 in [12] yields the existence of a brake orbit $u^\delta$ that oscillates between $\Gamma_-$ and $\Gamma_+$ and whose period $T_\delta$ diverges to $+\infty$ as $\delta \to 0^+$. Even though the condition (1.3) can be relaxed by allowing $\Gamma_\pm$ to contain hyperbolic critical points of $W$ [12], the extension of this approach to the infinite dimensional setting requires new ideas to overcome the difficulties related to the formulation of a condition analogous to

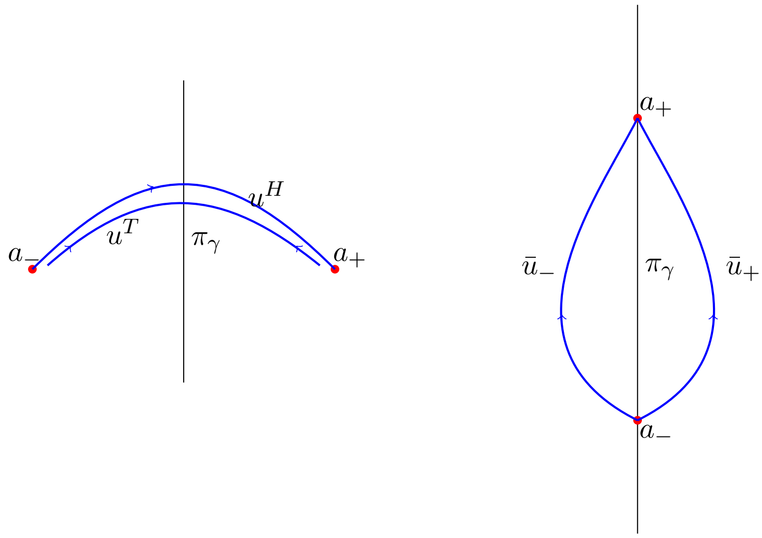


Figure 1: The symmetry of $W$: finite dimension (left); infinite dimension (right)

(1.3) and to the non compactness of the boundary of the sets $\{u \in \mathcal{H} : J(u) - J(\bar{u}_\pm) > \delta\}$. To avoid these pathologies the idea is to minimize on a set of $T$-periodic maps. But we can not expect that $u^\delta$ is a minimizer in the class of maps of period $T = T_\delta$. Indeed, returning to the case $m = 1$, we note that, as a solution of (1.1), $u^T$ is a critical point of the action functional

$$J_{(0,T)}(u) := \int_0^T \Big(\frac{1}{2}|\dot{u}|^2 + W(u)\Big)dt, \tag{1.4}$$

in the set of $H^1$ $T$-periodic maps but is not a minimizer. In fact it is well known [9], [13], [7] that, in the dynamics of the scalar parabolic equation

$$u_\tau = u_{tt} - W'(u), \;\; u(t+T) = u(t),$$

nearest layers attract each other and therefore, for large $T$, $u^T$ has Morse index 1 in the context of periodic perturbations.

To mode out this instability we work in a symmetric context. We assume that $W$ is invariant under a reflection $\gamma : \mathbb{R}^m \to \mathbb{R}^m$, that is,

$$W(\gamma u) = W(u), \;\; u \in \mathbb{R}^m. \tag{1.5}$$

In the finite dimensional case we assume that $\gamma$ exchanges $a_-$ with $a_+$:

$$a_\pm = \gamma\, a_\mp, \tag{1.6}$$

and we restrict ourselves to equivariant maps:

$$u(-t) = \gamma u(t), \;\; t \in \mathbb{R}.$$

We show that, under these restrictions and minimal assumptions on $W$, the existence of periodic solutions to

$$\ddot{u} = W_u(u), \qquad W_u(u) = \Big(\frac{\partial W}{\partial u_1}(u), \dots, \frac{\partial W}{\partial u_m}(u)\Big)^\top, \tag{1.7}$$

can be established by minimizing $J_{(0,T)}$ on a suitable set of $T$-periodic maps.

In the infinite dimensional case our choice for the functional that replaces $W$ is the action functional

$$J_{\mathbb{R}}(u) = \int_{\mathbb{R}} \Big(\frac{1}{2}|u'|^2 + W(u)\Big)ds, \ \ u \in \bar{\mathrm{u}} + H^1(\mathbb{R};\mathbb{R}^m),$$

where $W$ satisfies (1.2) and $\bar{\mathrm{u}}$ is a smooth map such that $\lim_{s\to\pm\infty} \bar{\mathrm{u}}(s) = a_\pm$ with exponential convergence. We assume that (1.5) holds with $\gamma$ a reflection that, in analogy with the finite dimensional case, satisfies

$$\bar{u}_\pm(s) = \gamma \bar{u}_\mp(s), \ \ s \in \mathbb{R}, \tag{1.8}$$

with $\bar{u}_-$ and $\bar{u}_+$ distinct global minimizers of $J_{\mathbb{R}}$ on $\bar{\mathrm{u}} + H^1(\mathbb{R};\mathbb{R}^m)$. The maps $\bar{u}_-$ and $\bar{u}_+$ represent two distinct orbits that connect $a_-$ to $a_+$:

$$\lim_{s\to\pm\infty} \bar{u}_\pm(s) = a_\pm. \tag{1.9}$$

We assume that $\bar{u}_-$ and $\bar{u}_+$ are unique modulo translation. Note that (1.9) and (1.8) imply that $a_\pm = \gamma a_\pm$, that is $a_\pm$ belong to the plane $\pi_\gamma$ fixed by $\gamma$, see Figure 1. We restrict ourselves to symmetric maps and replace the dynamical equation (1.7) with

$$\ddot{u} = \nabla_{L^2} J_{\mathbb{R}}(u) = -u'' + W_u(u).$$

This is actually an elliptic system which, after setting $x = t$ and $y = s$ takes the form

$$u_{xx} + u_{yy} = \Delta u = W_u(u). \tag{1.10}$$

We prove that for all $L \geq L_0$, for some $L_0 > 0$, there is a classical solution $u^L : \mathbb{R}^2 \to \mathbb{R}^m$ of (1.10) which is equivariant:

$$u^L(-x, y) = \gamma u^L(x, y),$$

$L$-periodic in $x \in \mathbb{R}$ and such that, along a subsequence $L_j \to +\infty$, converges locally to a heteroclinic solution that connects $\bar{u}_-$ and $\bar{u}_+$. That is, to a map $u^H : \mathbb{R}^2 \to \mathbb{R}^m$ that satisfies (1.10) and

$$\begin{aligned} &\lim_{y\to\pm\infty} u^H(x, y) = a_\pm, \\ &\lim_{x\to\pm\infty} u^H(x, y) = \bar{u}_\pm(y). \end{aligned} \tag{1.11}$$

We remark that, in the proof of this, there is an extra difficulty which is not present in the finite dimensional case: $\bar{u}_-$ and $\bar{u}_+$ are not isolated but any translate $\bar{u}_-(\cdot - r)$ or $\bar{u}_+(\cdot - r)$, $r \in \mathbb{R}$, is again a global minimizer of $J_{\mathbb{R}}$. Therefore for each $x$ there is a $\bar{u} \in \{\bar{u}_-, \bar{u}_+\}$ and a translation $h(x)$ that determines the point $\bar{u}(\cdot - h(x))$ in the manifolds generated by $\bar{u}_-$ and $\bar{u}_+$ which is the closest to the *fiber* $u^L(x, \cdot)$ of $u^L$. The map $h$ depends on $L$ and to prove convergence to a heteroclinic solution one needs to control $h$ and show that can be bounded by a quantity that does not depend on $L$ and that, for $L_j \to +\infty$, converges to a limit map $h^\infty : \mathbb{R} \to \mathbb{R}$ with a definite limit for $x \to \pm\infty$.

The paper is organized as follows. After stating our main results, that is Theorem 1.1 in Section 1.1 and Theorem 1.2 in Section 1.2, we prove Theorem 1.1 and Theorem 1.2 in Sections 2 and 3 respectively. The approach used in Section 3 is inspired by [11]. We include an Appendix where we present an elementary proof of a property of the functional $J_{\mathbb{R}}$.

### 1.1 The finite dimensional case

We assume that $W : \mathbb{R}^m \to \mathbb{R}$ is a continuous function that satisfies (1.2), (1.5) and (1.6). We also assume that there is a non-negative function $\sigma : [0, +\infty) \to \mathbb{R}$ such that $\int_0^{+\infty} \sigma(r)dr = +\infty$ and[1]

$$\sqrt{W(z)} \geq \sigma(|z|), \quad z \in \mathbb{R}^m. \tag{1.12}$$

*Remark* 1. The assumptions on $W$ imply (see for example [14], [22] and [12]) the existence of a Lipschitz continuous map $u^H : \mathbb{R} \to \mathbb{R}^m$ that satisfies

$$\begin{aligned} &\lim_{t\to\pm\infty} u(t) = a_\pm, \\ &\frac{1}{2}|\dot{u}|^2 - W(u) = 0, \\ &u(-t) = \gamma u(t), \quad t \in \mathbb{R}. \end{aligned} \tag{1.13}$$

We refer to a map with these properties as a heteroclinic connection between $a_-$ and $a_+$.

Define

$$\mathcal{A}^T := \Big\{ u \in H^1_T(\mathbb{R}; \mathbb{R}^m),\ u\Big(\frac{T}{4} + t\Big) = u\Big(\frac{T}{4} - t\Big),\ u(-t) = \gamma u(t), \quad t \in \mathbb{R} \Big\}, \tag{1.14}$$

and observe that there exists $\tilde{u} \in \mathcal{A}^T$ and a constant $C_0 > 0$ independent of $T > 4$ such that

$$J_{(0,T)}(\tilde{u}) \leq C_0. \tag{1.15}$$

Indeed the map $\tilde{u}$ can be defined by

$$\begin{aligned} &\tilde{u}(t) = \frac{1}{2}(a_+ + a_- + t(a_+ - a_-)), \quad t \in [-1, 1], \\ &\tilde{u}(t) = a_+, \quad t \in [1, \frac{T}{2} - 1]. \end{aligned}$$

Since we are interested in periodic orbits near $u^H$ we restrict our search to orbits lying in a large ball. Fix $M$ as the solution of the equation

$$C_0 = \sqrt{2} \int_{2(|a_+| \vee |a_-|)}^{M} \sigma(s)ds. \tag{1.16}$$

We determine $T$-periodic maps near heteroclinic solutions by minimizing the action functional (1.4) on the set $\mathcal{A}^T \cap \{\|u\|_{L^\infty} \leq 2M\}$.

**Theorem 1.1.** *Assume that $W : \mathbb{R}^m \to \mathbb{R}$ is a continuous function that satisfies* (1.2), (1.5), (1.6) *and* (1.12). *Then, there exists $T_0$ such that for each $T \geq T_0$ there exists a $T$-periodic minimizer $u^T$ of the functional* (1.4) *in $\mathcal{A}^T \cap \{\|u\|_{L^\infty} \leq 2M\}$, which is Lipschitz continuous and satisfies*

*(i)* $J_{(0,T)}(u^T) \leq C_0, \qquad \|u^T\|_{L^\infty} \leq M,$

*(ii)* $u^T(-t) = \gamma u^T(t),$

[1]This condition was first introduced in [14]

*(iii)* $\frac{1}{2}|\dot{u}^T|^2 - W(u^T) = -W(u^T(\pm\frac{T}{4}))$, *a.e.*

*For each* $0 < q \leq q_0$, *for some* $q_0 > 0$, *there is a* $\tau_q > 0$ *such that for each* $T > 4\tau_q$

$$|u^T(t) - a_+| < q, \quad t \in \left[\tau_q, \frac{T}{2} - \tau_q\right], \quad q \in (0, q_0], \tag{1.17}$$

*and therefore*

$$\lim_{T\to+\infty} u^T\Big(\pm\frac{T}{4}\Big) = a_\pm.$$

*Moreover, there is a sequence* $T_j \to +\infty$ *and a heteroclinic connection between* $a_-$ *and* $a_+$ $u^H : \mathbb{R} \to \mathbb{R}^m$ *such that*

$$\lim_{j\to+\infty} u^{T_j}(t) = u^H(t),$$

*uniformly in compacts.*

*If* $W$ *is of class* $C^1$, *then* $u^T$ *is a classical* $T$*-periodic solution of* (1.7).

Note that, if $a_\pm$ is nondegenerate in the sense that the Hessian matrix $W_{uu}(a_\pm)$ is positive definite or, more generally, if

$$W_u(u) \cdot (u - a_\pm) \geq \mu|u - a_\pm|^2, \quad \text{for } |u - a_\pm| \leq r_0,$$

for some $\mu > 0$, $r_0 > 0$, then (1.17) can be strengthened to

$$|u^T(t) - a_+| \leq Ce^{-ct}, \quad t \in \Big[0, \frac{T}{4}\Big],$$

where $c, C$ are positive constants independent of $T$. This follows by

$$\frac{d^2}{dt^2}|u^T(t) - a_+|^2 \geq 2\ddot{u}^T \cdot (u^T - a_+) = 2W_u(u^T) \cdot (u^T - a_+) \geq 2\mu|u^T - a_+|^2,$$

and a comparison argument.

*Remark* 2. Depending on the behavior of $W$ in a neighborhood of $a_\pm$ it may happen that the map $u^H$ connects $a_-$ and $a_+$ in a finite time, that is, $\exists\, \tau_0 < +\infty : u^H((-\tau_0, \tau_0)) \cap \{a_-, a_+\} = \emptyset$, $u^H(\pm\tau_0) = a_\pm$. We do not exclude this case. A sufficient condition for $\tau_0 = +\infty$, is

$$W(u) \leq c|u - a_\pm|^2,$$

for $u$ in a neighborhood of $a_\pm$.

Note that, if $\tau_0 < +\infty$, one can immediately construct a $T$-periodic map $u^T$ $(T = 4\tau_0)$ that satisfies (1.13), by setting

$$u^T\Big(\frac{T}{4} + t\Big) = u^H\Big(\frac{T}{4} - t\Big), \quad \text{for } t \in \Big(0, \frac{T}{2}\Big).$$

## 1.2 The infinite dimensional case

We assume that $W:\mathbb{R}^m\to\mathbb{R}$ is of class $C^3$, that (1.2), (1.5) and (1.8) hold with $\bar{u}_\pm$ as before. Moreover we assume

**h**$_1$ $\liminf_{|u|\to+\infty} W(u)>0$ and there is $M>0$ such that

$$W(su)\geq W(u),\quad \text{for } |u|=M,\ s\geq 1. \tag{1.18}$$

**h**$_2$ $a_\pm$ are non degenerate in the sense that the Hessian matrix $W_{uu}(a_\pm)$ is definite positive.

For each $r\in\mathbb{R}$ $\bar{u}(\cdot-r)$, $\bar{u}\in\{\bar{u}_-,\bar{u}_+\}$, is a solution of (1.7). Therefore differentiating (1.7) with respect to $r$ yields $\bar{u}'''=W_{uu}(\bar{u})\bar{u}'$ that shows that 0 is an eigenvalue of the operator $T:H^2(\mathbb{R};\mathbb{R}^m)\to L^2(\mathbb{R};\mathbb{R}^m)$ defined by

$$Tv=-v''+W_{uu}(\bar{u})v,\ \ \bar{u}=\bar{u}_\pm,$$

and $\bar{u}'$ is a corresponding eigenvector.

We also assume

**h**$_3$ The maps $\bar{u}_\pm$ are non degenerate in the sense that 0 is a simple eigenvalue of $T$.

The above assumptions ensure the existence of a heteroclinic connection between $\bar{u}_-$ and $\bar{u}_+$. This was proved by Schatzman in [18] without restricting to equivariant maps (see also [11] and [15]). The first existence result for a heteroclinic that connects $\bar{u}_-$ to $\bar{u}_+$ was given in [2] under the assumption that $W$ is symmetric with respect to the reflection that exchanges $a_\pm$ with $a_\mp$ but without requiring (1.8).

*Remark* 3. It is well known that the non-degeneracy of $a_\pm$ implies

$$\begin{aligned}&|\bar{u}(y)-a_+|\leq Ke^{-ky},\ \ y>0,\quad |\bar{u}(y)-a_-|\leq Ke^{ky},\ \ y<0,\\&|\bar{u}'(y)|,|\bar{u}''(y)|\leq Ke^{-k|y|},\ \ y\in\mathbb{R},\end{aligned} \tag{1.19}$$

for some constants $k>0,K>0$.

Under the above assumptions we prove the following:

**Theorem 1.2.** *There is $L_0>0$ and positive constants $k,K,\ k',K'$ such that for each $L\geq L_0$ there exists a classical solution $u^L:\mathbb{R}^2\to\mathbb{R}^m$ of (1.10), with the following properties:*

*(i)* $|u^L(x,y)-a_-|\leq Ke^{ky},\ \ x\in\mathbb{R},\ y\leq 0,$

$|u^L(x,y)-a_+|\leq Ke^{-ky},\ \ x\in\mathbb{R},\ y\geq 0.$

*(ii)* $u^L$ *is $L$-periodic in $x\in\mathbb{R}$:* $u^L(x+L,y)=u^L(x,y),\ \ (x,y)\in\mathbb{R}^2.$

*(iii)* $u^L$ *is a brake orbit:* $u^L(\frac{L}{4}+x,y)=u^L(\frac{L}{4}-x,y),$

*(iv)* $u^L$ *is equivariant* $u^L(-x,y)=\gamma u^L(x,y)$

*(v)* $u^L$ *satisfies the identities:*

$\frac{1}{2}\|u^L_x(x,\cdot)\|^2_{L^2(\mathbb{R};\mathbb{R}^m)}-J_{\mathbb{R}}(u^L(x,\cdot))=-J_{\mathbb{R}}(u^L(\frac{L}{4},\cdot)),$

$\langle u^L_x(x,\cdot),u^L_y(x,\cdot)\rangle_{L^2(\mathbb{R};\mathbb{R}^m)}=0,\ x\in\mathbb{R}.$

*(vi)* $u^L$ *minimizes*

$$\mathcal{J}(u)=\int_{(0,L)\times\mathbb{R}}\Big(\frac{1}{2}|\nabla u|^2+W(u)\Big)dxdy$$

*on the set of the* $H^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^m)$ *maps that satisfy* $(ii)-(iv)$ *and* $\lim_{y\to\pm\infty}u(x,y)=a_\pm$.

*(vii)* $\min_{r\in\mathbb{R}}\|u^L(x,\cdot)-\bar{u}_+(\cdot-r)\|_{L^2(\mathbb{R};\mathbb{R}^m)}\le K'e^{-k'x},\ \ x\in[0,\frac{L}{4}]$.
*In particular, as* $L\to+\infty$, $u^L(\frac{L}{4},\cdot)$ *converges to the manifold of the translates of* $\bar{u}_+$.

*Moreover, there exist* $\eta\in\mathbb{R}$, *a sequence* $L_j\to+\infty$ *and a heteroclinic solution* $u^H:\mathbb{R}^2\to\mathbb{R}^m$ *connecting* $\bar{u}_-$ *to* $\bar{u}_+$ *that satisfy*

$$\lim_{j\to+\infty}u^{L_j}(x,y-\eta)=u^H(x,y),\ \ (x,y)\in\mathbb{R}^2,$$

*uniformly in* $C^2$ *in any strip of the form* $(-l,l)\times\mathbb{R}$, *for* $l>0$.

Note that a by product of this theorem is a new proof of the existence of a heteroclinic solution $u^H$ in the class of equivariant maps.

# 2 The proof of Theorem 1.1

From (1.15) we can restrict ourselves to consider maps in the subset

$$\mathcal{A}^T_{C_0,M}=\{u\in\mathcal{A}^T\cap\{\|u\|_{L^\infty}\le 2M\}:J_{(0,T)}(u)\le C_0\},\tag{2.1}$$

where $M$ is given by (1.16).
Step 1. $\quad u\in\mathcal{A}^T_{C_0,M}\quad\Rightarrow\quad\|u\|_{L^\infty}\le M.$
Define

$$W_m(s)=\min_{|u-a_\pm|\ge s,\,|u|<2M}W(u),\tag{2.2}$$

Since $u\in\mathcal{A}^T$ implies $u(0)=\gamma u(0)$ we have

$$|u(0)-a_\pm|\ge\frac{1}{2}|a_+-a_-|.$$

Therefore, given $p\in(0,\frac{1}{2}|a_+-a_-|)$, for $u\in\mathcal{A}^T_{C_0,M}$, there are $t_p\in\big(0,\frac{C_0}{W_m(p)}\big)$ and $a\in\{a_-,a_+\}$ such that, for $T>4t_p$, it results

$$\begin{aligned}&|u(t)-a_\pm|>p,\quad\text{for }\ t\in[0,t_p),\\&|u(t_p)-a|=p.\end{aligned}\tag{2.3}$$

Note, in passing, that since $u\in\mathcal{A}^T$ implies $u(\frac{T}{4}-t)=u(\frac{T}{4}+t)$ we also have

$$\Big|u\Big(\frac{T}{2}-t_p\Big)-a\Big|=p.\tag{2.4}$$

Let $\bar{t}$ be such that $|u(\bar{t})|=\|u\|_{L^\infty}$, then we have

$$\begin{aligned}\sqrt{2}\int_{2(|a_+|\vee|a_-|)}^{M}\sigma(s)ds&=C_0\ge J_{(t_p,\bar{t})}(u)\\&\ge\int_{t_p}^{\bar{t}}\sqrt{2W(u(t))}|\dot{u}(t)|dt\ge\sqrt{2}\int_{|a|+p}^{\|u\|_{L^\infty}}\sigma(s)ds\end{aligned}$$

that proves the claim.

It follows that the constraint $\|u\|_{L^\infty} \leq 2M$ imposed in the definition of the admissible set is inactive for any $u \in \mathcal{A}^T_{C_0,M}$.

Next we prove a key lemma which is a refinement of Lemma 3.4 in [3] based on an idea from [19].

**Lemma 2.1.** *Assume that $u \in H^1((\alpha,\beta);\mathbb{R}^m)$, $(\alpha,\beta) \subset \mathbb{R}$ a bounded interval, satisfies*

$$\begin{aligned} &J_{(\alpha,\beta)}(u) \leq C', \\ &\|u\|_{L^\infty} \leq M' \end{aligned}$$

*for some $C', M' > 0$. Let $q_0 = \frac{1}{2}|a_+ - a_-|$. Given $q \in (0, q_0]$, there is $q'(q) \in (0,q)$ such that, if*

$$\begin{aligned} &|u(t_i) - a_+| \leq q'(q), \ i = 1,2 \\ &|u(t^*) - a_+| \geq q, \quad \text{for some} \ \ t^* \in (t_1, t_2), \end{aligned}$$

*for some $\alpha \leq t_1 < t_2 \leq \beta$, then there exists $v$ which coincides with $u$ outside $(t_1, t_2)$ and is such that*

$$\begin{aligned} &|v(t) - a_+| < q, \quad \text{for} \ \ t \in [t_1, t_2], \\ &J_{(t_1,t_2)}(v) < J_{(t_1,t_2)}(u). \end{aligned}$$

*Proof.* For $t, t' \in \mathbb{R}$, we have

$$|u(t) - u(t')| \leq \Big|\int_t^{t'} |\dot{u}| ds\Big| \leq |t-t'|^{\frac{1}{2}} \Big(\int_t^{t'} |\dot{u}|^2 ds\Big)^{\frac{1}{2}} \leq \sqrt{C_0}|t-t'|^{\frac{1}{2}}. \tag{2.5}$$

Define the intervals $(\tilde{\tau}_1, \tilde{\tau}_2) \subset (\tau_1, \tau_2)$ by setting

$$\begin{aligned} &\tilde{\tau}_1 = \max\{t > t_1 : |u(s) - a_+| \leq q, \ \ \text{for} \ \ s \leq t\}, \\ &\tau_1 = \max\{t < \tilde{\tau}_1 : |u(t) - a_+| \leq q'\}, \\ \\ &\tilde{\tau}_2 = \min\{t < t_2 : |u(s) - a_+| \leq q, \ \ \text{for} \ \ s \geq t\}, \\ &\tau_2 = \min\{t > \tilde{\tau}_2 : |u(t) - a_+| \leq q'\}. \end{aligned}$$

From (2.5) we have

$$q - q' = |u(\tilde{\tau}_1) - a_+| - |u(\tau_1) - a_+| \leq |u(\tilde{\tau}_1) - u(\tau_1)| \leq \sqrt{C_0}|\tau_1 - \tilde{\tau}_1|^{\frac{1}{2}},$$

and therefore

$$\tilde{\tau}_1 - \tau_1 \geq \frac{1}{C_0}(q-q')^2,$$

and similarly for $\tau_2 - \tilde{\tau}_2$. Next we set $\delta_{q,q'} := \frac{1}{C_0}(q-q')^2$ and, see Figure 2, define $v$:

$$v = \begin{cases} u, \quad \text{for} \ \ t \notin (\tau_1, \tau_2), \\ a_+, \quad \text{for} \ \ t \in (\tau_1 + \delta_{q,q'}, \tau_2 - \delta_{q,q'}), \\ u(\tau_1) - (u(\tau_1) - a_+)\frac{t-\tau_1}{\delta_{q,q'}}, \quad \text{for} \ \ t \in (\tau_1, \tau_1 + \delta_{q,q'}), \\ u(\tau_2) - (u(\tau_2) - a_+)\frac{\tau_2 - t}{\delta_{q,q'}}, \quad \text{for} \ \ t \in (\tau_2 - \delta_{q,q'}, \tau_2). \end{cases}$$

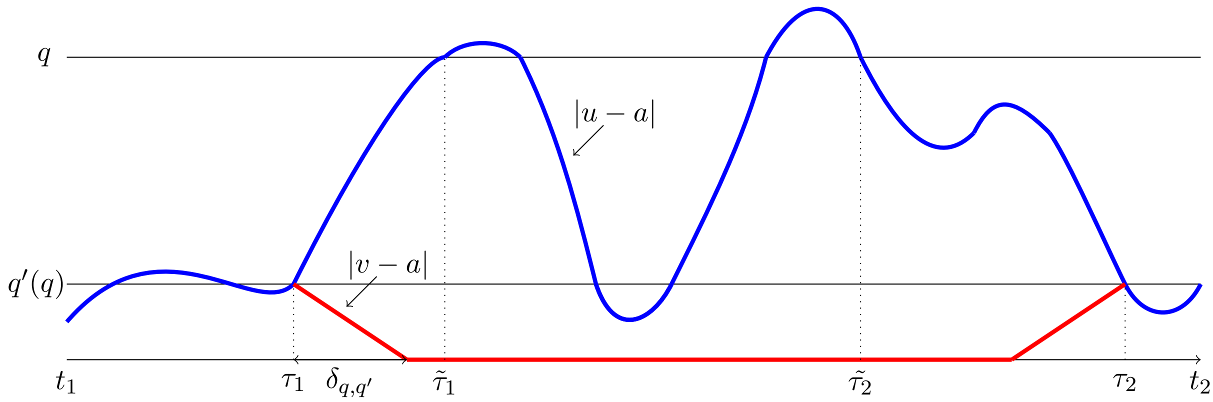


Figure 2: The construction of the map $v$ in Lemma 2.1

For each $s \in (0, q_0]$ define

$$W_M(s) = \max_{|u-a_+|\leq s} W(u).$$

We observe that $|u(\tau_i) - a_+| = q'$, $i = 1, 2$ and estimate

$$\begin{aligned}
&J_{(\tau_1,\tau_2)}(v) = J_{(\tau_1,\tau_1+\delta_{q,q'})}(v) + J_{(\tau_2-\delta_{q,q'},\tau_2)}(v) \leq 2\Big(\frac{1}{2}\frac{q'^2}{\delta_{q,q'}} + \delta_{q,q'} W_M(q')\Big),\\
&J_{(\tau_1,\tau_2)}(u) \geq J_{(\tau_1,\tilde{\tau}_1)}(u) + J_{(\tilde{\tau}_2,\tau_2)}(u)\\
&\geq \int_{\tau_1}^{\tilde{\tau}_1} \sqrt{2W(u)}|\dot{u}|dt + \int_{\tilde{\tau}_2}^{\tau_2} \sqrt{2W(u)}|\dot{u}|dt\\
&\geq 2\int_{q'}^{q} \sqrt{2W_m(s)}ds.
\end{aligned}$$

where $W_m(s)$ is defined as in (2.2) with $M'$ instead of $2M$.

Since $\delta_{q,q'} \leq \frac{q^2}{C_0}$ is a decreasing function of $q' \in (0, q)$ and $W_M(q')$ is infinitesimal with $q'$ we can fix a $q' = q'(q)$ so small that

$$\frac{1}{2}\frac{q'^2}{\delta_{q,q'}} + \delta_{q,q'} W_M(q') < \int_{q'}^{q} \sqrt{2W_m(s)}ds.$$

The proof is complete. □

Step 2. From Step 1 and Lemma 2.1 it follows that, if in (2.3) and (2.4) we take $p = q'(q)$ and set $\tau_q = t_{q'(q)}$, then, for $T > 4\tau_q$, in the minimization process we can restrict ourselves to the maps $u \in \mathcal{A}^T_{C_0,M}$ that satisfy

$$\begin{aligned}
&|u(t) - a_+| < q,\ t \in \Big[\tau_q, \frac{T}{2} - \tau_q\Big],\\
&|u(t) - a_-| < q,\ t \in \Big[\frac{T}{2} + \tau_q, T - \tau_q\Big].
\end{aligned}$$

Step 3. The existence of a minimizer $u^T \in \mathcal{A}^T_{C_0,M}$ is quite standard. From Step 1 and (2.5) $\mathcal{A}^T_{C_0,M}$ is an equibounded and equicontinuous family of maps. Therefore from Ascoli-Arzela theorem there exists a minimizing sequence $\{u_j\}_j \subset \mathcal{A}^T_{C_0,M}$ that converges uniformly to a map $u^T \in \mathcal{A}^T_{C_0,M}$. This and $J_{(0,T)}(u_j) \leq C_0$ imply that $\{u_j\}_j$ is bounded in $H^1((0,T);\mathbb{R}^m)$ and therefore, by passing to a subsequence if necessary, that $u_j$ converges weakly in $H^1$ to $u^T$. From the lower semicontinuity of the norm we have $\liminf_{j\to+\infty}\int_0^T |\dot{u}_j|^2 dt \geq \int_0^T |\dot{u}^T|^2 dt$ while uniform convergence implies $\lim_{j\to+\infty}\int_0^T W(u_j(t))dt \geq \int_0^T W(u^T(t))dt$.

Step 4. The minimizer $u^T$ is Lipschitz continuous and satisfies conservation of energy. Let $t_0 < t_1 < t_2 < t_3$ be numbers such that $t_3 - t_0 \leq T$. Given a small number $\xi \in \mathbb{R}$ let $\phi : \mathbb{R} \to \mathbb{R}$ be the $T$-periodic piecewise-linear map that satisfies $\phi(t_0) = t_0$, $\phi(t_1+\xi) = t_1$, $\phi(t_2+\xi) = t_2$, $\phi(t_3) = t_3$ and let $\psi$ be the inverse of $\phi$. Set

$$v_\xi(t) = u^T(\phi(t))$$

and

$$f(\xi) = J_{(0,T)}(v_\xi) - J_{(0,T)}(u^T).$$

The minimality of $u^T$ implies that $f'(0) = 0$. A simple computation yields

$$\begin{aligned}
f(\xi) &= \int_{(t_0,t_1)\cup(t_2,t_3)} \Big(\frac{1}{2}\Big(\frac{1}{\psi'(\tau)} - 1\Big)|\dot{u}^T(\tau)|^2 + (\psi'(\tau)-1)W\big(u^T(\tau)\big)\Big)d\tau \\
&= \int_{t_0}^{t_1}\Big(\frac{-\xi}{2(t_1-t_0+\xi)}|\dot{u}^T(\tau)|^2 + \frac{\xi}{t_1-t_0}W\big(u^T(\tau)\big)\Big)d\tau \\
&+ \int_{t_2}^{t_3}\Big(\frac{\xi}{2(t_3-t_2-\xi)}|\dot{u}^T(\tau)|^2 - \frac{\xi}{t_3-t_2}W\big(u^T(\tau)\big)\Big)d\tau,
\end{aligned}$$

and we obtain

$$\begin{aligned}
&0 = f'(0) \\
&\Leftrightarrow \frac{1}{t_1-t_0}\int_{t_0}^{t_1}\Big(\frac{1}{2}|\dot{u}^T(\tau)|^2 - W\big(u^T(\tau)\big)\Big)d\tau = \frac{1}{t_3-t_2}\int_{t_2}^{t_3}\Big(\frac{1}{2}|\dot{u}^T(\tau)|^2 - W\big(u^T(\tau)\big)\Big)d\tau.
\end{aligned}$$

This shows that there exists $C \in \mathbb{R}$ independent of $t$ such that

$$\lim_{t'\to t}\frac{1}{t'-t}\int_t^{t'}\Big(\frac{1}{2}|\dot{u}^T(\tau)|^2 - W\big(u^T(\tau)\big)\Big)d\tau = C.$$

Therefore we have

$$\frac{1}{2}|\dot{u}^T(t)|^2 - W\big(u^T(t)\big) = C$$

for each Lebesgue point $t \in \mathbb{R}$. From $u(\frac{T}{4}-t) = u(\frac{T}{4}+t)$ it follows that $\dot{u}(\frac{T}{4}) = 0$, which implies $C = W\big(u^T(\pm\frac{T}{4})\big)$.

Step 5. If $W$ is of class $C^1$, then $u^T$ is a classical solution of (1.7). Since $u^T$ is a minimizer, if $w : (t_1,t_2) \to \mathbb{R}^m$ is a smooth map that satisfies $w(t_i) = 0$, $i = 1,2$ we have

$$\begin{aligned}
0 &= \frac{d}{d\lambda}J_{(t_1,t_2)}(u^T + \lambda w)|_{\lambda=0} \\
&= \int_{t_1}^{t_2}(\dot{u}^T\cdot\dot{w} + W_u(u^T)\cdot w)dt = \int_{t_1}^{t_2}\Big(\dot{u}^T - \int_{t_1}^{t}W_u\big(u^T(s)\big)ds\Big)\cdot\dot{w}dt.
\end{aligned} \tag{2.6}$$

Since this is valid for all $0 < t_1 < t_2 < T$ and $\dot{w} : (t_1,t_2) \to \mathbb{R}^m$ is an arbitrary smooth map with zero average (2.6) implies

$$\dot{u}^T = \int_{t_1}^{t} W_u\big(u^T(s)\big)ds + const.$$

The continuity of $u^T$ and of $W_u$ implies that the right hand side of this equation is a map of class $C^1$. It follows that we can differentiate and obtain

$$\ddot{u}^T = W_u(u^T), \;\; t \in (0,T).$$

The proof of Theorem 1.1 is complete.

# 3 The proof of Theorem 1.2

In analogy with the finite dimensional case we define

$$\begin{aligned}\mathcal{A}^L = \Big\{ & u \in H^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^m) : u(x+L,\cdot) = u(x,\cdot), \lim_{y\to\pm\infty} u(x,y) = a_\pm, \\ & u\Big(\frac{L}{4}+x,y\Big) = u\Big(\frac{L}{4}-x,y\Big), \;\; u(-x,y) = \gamma u(x,y). \Big\}\end{aligned}$$

We will show that the solution of (1.10) in Theorem 1.2 can be determined as a minimizer of the energy

$$\mathcal{J}_{(0,L)\times\mathbb{R}}(u) = \int_{(0,L)\times\mathbb{R}} \Big(\frac{1}{2}|\nabla u|^2 + W(u)\Big) dxdy \tag{3.1}$$

on $\mathcal{A}^L$.

We can assume

$$\|u\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^m)} \le M, \tag{3.2}$$

where $M$ is the constant in $\mathbf{h}_1$ and

$$\mathcal{J}_{(0,L)\times\mathbb{R}}(u) \le C_0 + c_0 L, \tag{3.3}$$

where $C_0 > 0$ is a constant independent of $L > 4$ and

$$c_0 = J_{\mathbb{R}}(\bar{u}_\pm). \tag{3.4}$$

To prove (3.2) set $u_M = 0$ if $u = 0$ and $u_M = \min\{|u|, M\}u/|u|$ otherwise and note that (1.18) implies

$$W(u_M) \le W(u),$$

while

$$|\nabla u_M| \le |\nabla u|, \;\; \text{a.e.},$$

because the mapping $u \to u_M$ is a projection. It follows

$$\begin{aligned}&\mathcal{J}_{(0,L)\times\mathbb{R}}(u) - \mathcal{J}_{(0,L)\times\mathbb{R}}(u_M) \\ &= \int_{\{|u|\ge M\}} \Big(W(u) - W(u_M) + \frac{1}{2}(|\nabla u|^2 - |\nabla u_M|^2)\Big) dxdy \ge 0,\end{aligned}$$

that proves the claim. To prove (3.3) we define a map $\tilde{u} \in \mathcal{A}^L$ that satisfies (3.3) by setting:

$$\begin{aligned}&\tilde{u}(x,\cdot) = \frac{1}{2}(\bar{u}_+ + \bar{u}_- + x(\bar{u}_+ - \bar{u}_-)), \;\; x \in [-1,1], \\ &\tilde{u}(x,\cdot) = \bar{u}_+, \;\; x \in [1, \frac{L}{2} - 1].\end{aligned}$$

*Remark* 4. From (3.3) and the minimality of $\bar{u}_\pm$ it follows that

$$\int_0^L \int_{\mathbb{R}} \frac{1}{2}|u_x|^2 dxdy \leq C_0 + c_0 L - \int_0^L \int_{\mathbb{R}} \Big(\frac{1}{2}|u_y|^2 + W(u)\Big) dxdy \leq C_0.$$

Since $a_\pm$ are non degenerate zeros of $W \geq 0$, there exist positive constants $\gamma, \Gamma$ and $r_0 > 0$ such that

$$\begin{aligned} &W_{uu}(a_\pm + z)\zeta \cdot \zeta \geq \gamma^2 |\zeta|^2, \ \ \zeta \in \mathbb{R}^m, \ |z| \leq r_0, \\ &\frac{1}{2}\gamma^2 |z|^2 \leq W(a_\pm + z) \leq \frac{1}{2}\Gamma^2 |z|^2, \ \ |z| \leq r_0. \end{aligned} \tag{3.5}$$

For a map $v : \mathbb{R} \to \mathbb{R}^m$ we simply denote the norms $\|v\|_{L^2(\mathbb{R};\mathbb{R}^m)}$ and $\|v\|_{H^1(\mathbb{R};\mathbb{R}^m)}$ with $\|v\|$ and $\|v\|_1$ respectively.

One of the difficulties with the minimization on $\mathcal{A}^L$ is the fact that $\mathcal{J}_{(0,L)\times\mathbb{R}}$ is translation invariant on $\mathcal{A}^L$. This corresponds to a loss of compactness. We show in the next lemma that we can restrict ourselves to a subset of $\mathcal{A}^L$ of maps $u$ that, aside from a bounded interval independent of $u$, remain near to $a_-$ and $a_+$. This restores compactness.

**Lemma 3.1.** *There is $d_L > 0$ such that in the minimization of the functional (3.1) on $\mathcal{A}^L$ we can restrict ourselves to the subset of maps that satisfy*

$$\begin{aligned} &|u(x,y) - a_-| < \frac{r_0}{2}, \ \textit{for}\ x \in \mathbb{R},\ y < -d_L, \\ &|u(x,y) - a_+| < \frac{r_0}{2}, \ \textit{for}\ x \in \mathbb{R},\ y > d_L, \end{aligned} \tag{3.6}$$

*with $r_0$ as in* (3.5)

*Proof.* Set $\bar{y} = \frac{1}{k}\log\frac{4K}{r_0}$, then from (1.19) it follows

$$\begin{aligned} &|\bar{u}(y) - a_-| \leq \frac{r_0}{4}, \ \text{for}\ y \leq -\bar{y},\ \bar{u} \in \{\bar{u}_-, \bar{u}_+\}, \\ &|\bar{u}(y) - a_+| \leq \frac{r_0}{4}, \ \text{for}\ y \geq +\bar{y},\ \bar{u} \in \{\bar{u}_-, \bar{u}_+\}. \end{aligned} \tag{3.7}$$

Given $u \in \mathcal{A}^L$, define

$$X_0 = \Big\{x \in [0, L] : \|u(x,\cdot) - \bar{u}_\pm(\cdot - r)\|_1 \geq \frac{r_0}{8\sqrt{2}},\ r \in \mathbb{R}\Big\}.$$

If $u$ satisfies (3.3), then Lemma 3.6 or Proposition A.1 implies

$$|X_0| \leq \frac{C_0}{e_{\frac{r_0}{8\sqrt{2}}}}.$$

Therefore for all $L > \frac{C_0}{e_{\frac{r_0}{8\sqrt{2}}}}$ there exist $\bar{x} \in [0, L]$, $\bar{r} \in \mathbb{R}$ and $\bar{u} \in \{\bar{u}_-, \bar{u}_+\}$ such that

$$\|u(\bar{x},\cdot) - \bar{u}(\cdot - \bar{r})\|_1 < \frac{r_0}{8\sqrt{2}}. \tag{3.8}$$

Since we have $\mathcal{J}(u_r) = \mathcal{J}(u)$ for $u_r(x,y) = u(x, y + r), r \in \mathbb{R}$, (3.8), we can identify $u_r$ with $u$. Then (3.8) implies, via $\|v\|_{L^\infty} \leq \sqrt{2}\|v\|_1$, the estimate

$$\|u(\bar{x},\cdot) - \bar{u}\|_{L^\infty(\mathbb{R};\mathbb{R}^m)} < \frac{r_0}{8}. \tag{3.9}$$

Consider now the set

$$Y_0 = \Big\{ y \in \mathbb{R} : |u(x_y, y) - u(\bar{x}, y)| \geq \frac{r_0}{8}, \text{ for some } x_y \in (\bar{x}, \bar{x} + L) \Big\}.$$

For $y \in Y_0$ it results

$$\begin{aligned} \frac{r_0}{8} &\leq |u(x_y, y) - u(\bar{x}, y)| \leq |x_y - \bar{x}|^{\frac{1}{2}} \Big( \int_{\bar{x}}^{x_y} |u_x(x, y)|^2 dx \Big)^{\frac{1}{2}} \\ &\leq L^{\frac{1}{2}} \Big( \int_{\bar{x}}^{\bar{x}+L} |u_x(x, y)|^2 dx \Big)^{\frac{1}{2}}, \end{aligned}$$

so that

$$\frac{r_0^2}{64} |Y_0| \leq L \int_{\mathbb{R}} \int_{\bar{x}}^{\bar{x}+L} |u_x(x, y)|^2 dx \leq 2LC_0.$$

It follows

$$|Y_0| \leq 128 \frac{LC_0}{r_0^2},$$

therefore there exists an increasing sequence $\{y_j\} \in \mathbb{R} \setminus Y_0$ such that

$$\begin{aligned} &y_0 = \bar{y}, \quad y_j - y_{j-1} > |Y_0|, \; j = 1, 2, \ldots \\ &|u(x, y_j) - a_+| < \frac{r_0}{2}, \text{ for } x \in [\bar{x}, \bar{x} + L]. \end{aligned}$$

This follows from (3.7) and (3.9). From the proof of the cut-off lemma in [5] we infer that, if the measure of the set

$$\Big\{ (x, y) \in [\bar{x}, \bar{x} + L] \times [y_{j-1}, y_j] : |u(x, y) - a_+| > \frac{r_0}{2} \Big\}$$

is positive, then there exists a map $v_j : \mathbb{R} \times [y_j, y_{j+1}] \to \mathbb{R}^m$ which is L-periodic in $x \in \mathbb{R}$, coincides with $u$ on the boundary of the strip $\mathbb{R} \times (y_j, y_{j+1})$ and satisfies

$$\mathcal{J}_{\Omega_j}(v_j) < \mathcal{J}_{\Omega_j}(u), \tag{3.10}$$

where $\Omega_j = (\bar{x}, \bar{x} + L) \times (y_j, y_{j+1})$, $j = 1, 2, \ldots$. From this we see that to each map $u \in \mathcal{A}^L$ that satisfies (3.3) but not

$$|u(x, y) - a_+| < \frac{r_0}{2}, \text{ for } x \in \mathbb{R}, \; y > \bar{y} + |Y_0|.$$

we can associate a map $v$ that satisfies this inequality and (3.10). This and a similar argument concerning the other inequality in (3.6) establish the lemma with $d_L = \bar{y} + |Y_0|$. ☐

With Lemma 3.1 at hand the existence of a minimizer $u^L \in \mathcal{A}^L$ follows by standard variational arguments. The minimizer $u^L$ satisfies (3.2). From this, the assumed smoothness of $W$ and elliptic theory it follows

$$\|u^L\|_{C^{2,\beta}(\mathbb{R}^2;\mathbb{R}^m)} \leq C^*, \tag{3.11}$$

for some constants $C^* > 0$, $\beta \in (0, 1)$ independent of $L$ and $u^L$ is a classical solution of (1.10). Moreover, from the fact that $u^L$ satisfies (3.6) and a comparison argument we obtain

$$\begin{aligned} &|u(x, y) - a_-| \leq Ke^{-k(|y| - d_L)}, \text{ for } x \in \mathbb{R}, \; y < -d_L, \\ &|u(x, y) - a_+| \leq Ke^{-k(|y| - d_L)}, \text{ for } x \in \mathbb{R}, \; y > d_L. \end{aligned} \tag{3.12}$$

and, for $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i = 1, 2$, $|\alpha| = 1, 2$

$$|D^\alpha u^L(x, y)| \leq Ke^{-k(|y| - d_L)}, \text{ for } |y| > d_L.$$

### 3.1 Basic lemmas

To show that the minimizer $u^L$ has the properties listed in Theorem 1.2, in particular *(i)*, *(vii)* and *(viii)*, we need point-wise estimates on $u^L$ that do not depend on $L$. For example to prove *(i)* we need to show that $d_L$ in (3.12) can be taken independent of $L$. For *(vii)* and *(viii)* a detailed analysis of the behavior of the trace $u^L(x,\cdot)$ as a function of $x\in(0,L)$ is necessary. To complete this program we use several ingredients: a decomposition of $u^L(x,\cdot)$ that we discuss next; two Hamiltonian identities that, together with the decomposition of $u^L(x,\cdot)$, allow a representation of the energy $\mathcal{J}_{(0,L)\times\mathbb{R}}(u^L)$ with a one dimensional integral in $x$ (see Lemma 3.3 and Lemma 3.4) and an analysis of the behavior of the *effective potential* $J_{\mathbb{R}}(\bar{u}+v)-J_{\mathbb{R}}(\bar{u})$, $\bar{u}\in\{\bar{u}_-,\bar{u}_+\}$ as a function of $v\in H^1(\mathbb{R};\mathbb{R}^m)$ that we present in Lemma 3.5 and in Lemma 3.6.

Let $\bar{\mathrm{u}}:\mathbb{R}\to\mathbb{R}^m$ be a smooth map with the same asymptotic behavior as $\bar{u}_\pm$. Set $H^0(\mathbb{R};\mathbb{R}^m)=L^2(\mathbb{R};\mathbb{R}^m)$ and let $H^1(\mathbb{R};\mathbb{R}^m)$ be the standard Sobolev space. For $j=0,1$ let $\langle\cdot,\cdot\rangle_j$ be the inner product in $H^j(\mathbb{R};\mathbb{R}^m)$ and $\|\cdot\|_j$ the associated norm. If there is no risk of confusion, for $j=0$ we simply write $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ instead of $\langle\cdot,\cdot\rangle_0$ and $\|\cdot\|_0$. Set

$$\mathcal{H}^j=\bar{\mathrm{u}}+H^j(\mathbb{R};\mathbb{R}^m),$$

Define

$$q_j^u=\inf_{r\in\mathbb{R},\pm}\|u-\bar{u}_\pm(\cdot-r)\|_j,\quad u\in\mathcal{H}^j.$$

Note that for large $|r|$ we have

$$\|u-\bar{u}_\pm(\cdot-r)\|_j\geq\frac{1}{2}|a_+-a_-|\sqrt{|r|}.$$

This and the fact that $\|u-\bar{u}_\pm(\cdot-r)\|_j$ is continuous in $r$ imply the existence of $h_j\in\mathbb{R}$ and $\bar{u}_j\in\{\bar{u}_-,\bar{u}_+\}$ such that

$$q_j^u=\|u-\bar{u}_j(\cdot-h_j)\|_j.$$

$q_j^u$ is a continuous function of $u\in\mathcal{H}^j$ and a standard argument implies that

$$\langle u-\bar{u}_j(\cdot-h_j),\bar{u}_j'(\cdot-h_j)\rangle_j=0. \tag{3.13}$$

Note that $\bar{u}_j$ remains equal to some fixed $\bar{u}\in\{\bar{u}_-,\bar{u}_+\}$ while $u$ changes continuously in the subset of $\mathcal{H}^j$ where

$$q_j^u<\frac{1}{2}\inf_{r\in\mathbb{R}}\|\bar{u}_+-\bar{u}_-(\cdot-r)\|_0.$$

We quote from Section 2 in [18]

**Lemma 3.2.** *There exists $\bar{q}>0$ such that $q_j^u<\bar{q}$ implies that $u_j$ and $h_j$ are uniquely determined. Moreover $h_j$ is a function of class $C^{3-j}$ of $u\in\mathcal{H}^j$ and*

$$(D_uh_j)w=-\frac{\langle w,\bar{u}'(\cdot-h_j)\rangle_j}{\|\bar{u}'\|_j^2-\langle u-\bar{u}(\cdot-h_j),\bar{u}''(\cdot-h_j)\rangle_j}. \tag{3.14}$$

*There are constants $C,\tilde{C}>0$ such that, for $q_1^u<\bar{q}$,*

$$\begin{aligned}&|h_0-h_1|\leq Cq_1^u,\\&\|u-\bar{u}(\cdot-h_0)\|_1\leq\tilde{C}q_1^u.\end{aligned} \tag{3.15}$$

In the following we drop the subscript 0 and write simply $q^u$, $\|\cdot\|$, etc. instead of $q_0^u$, $\|\cdot\|_0$, etc.

From Lemma 3.2 and (3.13) it follows that $u \in \mathcal{H}$ can be decomposed in the form

$$\begin{aligned} &u = \bar{u}(\cdot - h) + v(\cdot - h), \\ &\langle v, \bar{u}' \rangle = 0, \end{aligned} \tag{3.16}$$

for some $h \in \mathbb{R}$ and $\bar{u} \in \{\bar{u}_-, \bar{u}_+\}$ and that, provided $q^u < \bar{q}$, $h \in \mathbb{R}$ and $\bar{u}$ are uniquely determined. Note that from (3.16) we have

$$v(s) = u(s + h) - \bar{u}(s)$$

and

$$\|v\| = q^u.$$

In particular the decomposition (3.16) applies to the minimizer $u^L \in \mathcal{A}^L$:

$$\begin{aligned} &u^L(x, \cdot) = \bar{u}(\cdot - h^L(x)) + v^L(x, \cdot - h^L(x)), \\ &\langle v^L(x, \cdot), \bar{u}' \rangle = 0, \end{aligned} \tag{3.17}$$

for some $\bar{u} \in \{\bar{u}_-, \bar{u}+\}$. Given $x \in \mathbb{R}$ we set $q^L(x) = q^{u^L(x,\cdot)}$ and $q_1^L(x) = q_1^{u^L(x,\cdot)}$ and recall that

$$q^L(x) = \|v^L(x, \cdot)\| = \|u^L(x, \cdot) - \bar{u}(\cdot - h^L(x))\|.$$

In general $h^L(x)$ is not uniquely determined if $q^L(x)$ is not sufficiently small. In the following, if there is no risk of confusion, we drop the superscript $L$ and write simply $q(x)$, $v(x, y)$, $h(x)$, etc.. instead of $q^L(x)$, $v^L(x, y)$, $h^L(x)$, etc..

From the minimality of $u = u^L$ and its smoothness properties established in (3.11) and (3.12) it follows that $u^L$ satisfies two Hamiltonian identities. This is the content of the following lemma, where $c_0$ is defined in (3.4).

**Lemma 3.3.** *Set $u = u^L$. Then there exist constants $\omega$ and $\tilde{\omega}$ such that, for $x \in \mathbb{R}$, it results*

$$\int_{\mathbb{R}} \frac{1}{2}|u_x(x, y)|^2 dy = \int_{\mathbb{R}} \Big(W(u(x, y)) + \frac{1}{2}|u_y(x, y)|^2\Big) dy - c_0 - \omega \tag{3.18}$$

*and*

$$\int_{\mathbb{R}} u_x(x, y) \cdot u_y(x, y) dy = \tilde{\omega}, \qquad \text{for } x \in \mathbb{R}. \tag{3.19}$$

*Moreover it results*

$$\begin{aligned} &\omega = \int_{\mathbb{R}} \Big(W(u(\frac{L}{4}, y)) + \frac{1}{2}|u_y(\frac{L}{4}, y)|^2)\Big) dy - c_0 \geq 0, \\ &\tilde{\omega} = 0. \end{aligned} \tag{3.20}$$

*Proof.* The identities (3.18) and (3.19) are well known, see for instance [18] or [11]. To prove (3.20) we observe that $u(\frac{L}{4} - x, y) = u(\frac{L}{4} + x, y)$ implies $u_x(\frac{L}{4}, y) = 0$. $\square$

**Lemma 3.4.** *The constant $\bar{q}$ in Lemma 3.2 can be chosen such that, if*

$$0 < q(x) \leq q_1(x) \leq \bar{q}, \quad x \in I, \tag{3.21}$$

*for some interval $I \subset \mathbb{R}$, then, for $x \in I$ the maps $h(x) = h^L(x)$, $v(x,y) = v^L(x,y)$ $\bar{u} \in \{\bar{u}_-, \bar{u}_+\}$ in the decomposition (3.17) are uniquely determined and are smooth functions of $x \in I$. With $\nu(x,\cdot) = \nu^L(x,\cdot)$ defined by $v(x,\cdot) = q(x)\nu(x,\cdot)$, it results*

$$h'(x) = \frac{\langle v_x(x,\cdot), v_y(x,\cdot)\rangle}{\|\bar{u}' + v_y(x,\cdot)\|^2} = \frac{q^2(x)\langle \nu_x(x,\cdot), \nu_y(x,\cdot)\rangle}{\|\bar{u}' + q(x)\nu_y(x,\cdot)\|^2}, \tag{3.22}$$

*and*

$$\begin{aligned} \|u_x(x,\cdot)\|^2 &= \|v_x(x,\cdot)\|^2 - \frac{\langle v_x(x,\cdot), v_y(x,\cdot)\rangle^2}{\|\bar{u}' + v_y(x,\cdot)\|^2} \\ &= q'(x)^2 + q^2(x)\|\nu_x(x,\cdot)\|^2 - q^4(x)\frac{\langle \nu_x(x,\cdot), \nu_y(x,\cdot)\rangle^2}{\|\bar{u}' + q(x)\nu_y(x,\cdot)\|^2}. \end{aligned} \tag{3.23}$$

*Moreover the map*

$$(0, q(x)] \ni p \to f(p,x)\|\nu_x(x,\cdot)\|^2 := p^2\|\nu_x(x,\cdot)\|^2 - p^4\frac{\langle \nu_x(x,\cdot), \nu_y(x,\cdot)\rangle^2}{\|\bar{u}' + p\nu_y(x,\cdot)\|^2}$$

*is non-negative and non-decreasing for each fixed $x \in I$.*

*Proof.* From (3.17) with $u = u^L$, $v = v^L$ we obtain

$$\begin{aligned} u_x(x,\cdot) &= -h'(x)\Big(\bar{u}'(\cdot - h(x)) + v_y(x, \cdot - h(x))\Big) + v_x(x, \cdot - h(x)), \\ u_y(x,\cdot) &= \bar{u}'(\cdot - h(x)) + v_y(x, \cdot - h(x)). \end{aligned}$$

and therefore Lemma 3.3 and (3.17) that implies

$$\langle v_x(x,\cdot), \bar{u}'\rangle = 0, \quad x \in I,$$

yield

$$0 = \langle u_x(x,\cdot), u_y(x,\cdot)\rangle = -h'(x)\|\bar{u}' + v_y(x,\cdot)\|^2 + \langle v_x(x,\cdot), v_y(x,\cdot)\rangle. \tag{3.24}$$

From assumption (3.21) and (3.15) we have $\|v_y(x,\cdot)\| \leq \|v\|_1 \leq \tilde{C}q_1(x) \leq \tilde{C}\bar{q}$ and $\bar{q} \leq \frac{\|\bar{u}'\|}{2\tilde{C}}$ implies

$$\frac{1}{2}\|\bar{u}'\| \leq \|\bar{u}' + v_y(x,\cdot)\| \leq \frac{3}{2}\|\bar{u}'\|.$$

Therefore (3.24) can be solved for $h'(x)$ and the first expression of $h'(x)$ in (3.22) is established. For the other expression we observe that $\langle v_x, v_y\rangle = \langle q_x\nu + q\nu_x, q\nu_y\rangle = q^2\langle \nu_x, \nu_y\rangle$ that follows from $\langle \nu(x, \cdot - r), \nu(x, \cdot - r)\rangle = 1$, for $r \in \mathbb{R}$ which implies $\langle \nu_y(x,\cdot), \nu(x,\cdot)\rangle = 0$. A similar computation that also uses (3.22) yields (3.23).

It remains to prove the monotonicity of $p \to f(p,x)\|\nu_x(x,\cdot)\|^2$. We can assume $\|\nu_x\| > 0$ otherwise there is nothing to be proved. We have

$$p\|\nu_y(x,\cdot)\| \leq q(x)\|\nu_y(x,\cdot)\| = \|v_y(x,\cdot)\| \leq \tilde{C}\bar{q},$$

and therefore

$$\begin{aligned}D_pf(p,\cdot) &= 2p - 4p^3\frac{\langle\frac{\nu_x}{\|\nu_x\|},\nu_y\rangle^2}{\|\bar{u}'+p\nu_y\|^2} + 2p^4\frac{\langle\frac{\nu_x}{\|\nu_x\|},\nu_y\rangle^2\langle\bar{u}'+p\nu_y,\nu_y\rangle}{\|\bar{u}'+p\nu_y\|^4}\\ &\geq 2p\Big(1-2\frac{(\tilde{C}\bar{q})^2}{(\|\bar{u}'\|-\tilde{C}\bar{q})^2} - \frac{(\tilde{C}\bar{q})^3}{(\|\bar{u}'\|-\tilde{C}\bar{q})^3}\Big).\end{aligned}$$

This proves $D_pf(p,\cdot)>0$ for $\bar{q}\leq\frac{\|\bar{u}'\|}{3\tilde{C}}$. The proof is complete. □

Next we list some properties of the *effective potential* $J_{\mathbb{R}}(u)-c_0$ that depend on the decomposition (3.16) of $u$. Define

$$\mathcal{W}(v) = J_{\mathbb{R}}(\bar{u}+v) - J_{\mathbb{R}}(\bar{u}).$$

where $v$ is as in (3.16) and $u\in\mathcal{H}^1$. If we set $v=q\nu$, with $q=\|v\|\neq 0$, $\mathcal{W}$ can be considered as a function of $q\in\mathbb{R}$ and $\nu\in H^1(\mathbb{R};\mathbb{R}^m)$, $\|\nu\|=1$. We have (see [11])

**Lemma 3.5.** *Assume that $|v'|\leq C$ for some $C>0$. Then*

$$\|v\|_{L^\infty(\mathbb{R};\mathbb{R}^m)} \leq C_1\|v\|^{\frac{2}{3}}, \tag{3.25}$$

*for some $C_1>0$. The constant $\bar{q}>0$ in Lemma 3.2 can be chosen such that the effective potential $\mathcal{W}(q\nu)$ is increasing in $q$ for $q\in[0,\bar{q}]$ and there is $\mu>0$ such that*

$$\frac{\partial^2}{\partial q^2}\mathcal{W}(q\nu) \geq \mu(1+\|\nu'\|^2), \quad q\in(0,\bar{q}], \tag{3.26}$$

*and*

$$\begin{aligned}&\mathcal{W}(q\nu)\geq\frac{1}{2}\mu q^2(1+\|\nu'\|^2), \quad q\in(0,\bar{q}]\\ &\Leftrightarrow\\ &\mathcal{W}(v)\geq\frac{1}{2}\mu\|v\|_1^2, \quad \|v\|\in(0,\bar{q}].\end{aligned}$$

Lemma 3.5 describes the properties of the effective potential $\mathcal{W}$ in a neighborhood of one of the connections $\bar{u}_\pm$. We also need a lower bound for the effective potential away from a neighborhood of the connections. We have the following result, see Corollary 3.2 in [18]) or Proposition A.1 in the Appendix, where we give an elementary proof.

**Lemma 3.6.** *For each $p>0$ there exists $e_p>0$ such that, if $u\in\mathcal{H}^1$ satisfies*

$$q_1^u\geq p,$$

*then*

$$J_{\mathbb{R}}(u)-c_0\geq e_p.$$

*Moreover $e_p$ is continuous in $p$ and for $p\leq\|v\|_1$, $\|v\|_1$ small, it results*

$$e_p\leq J_{\mathbb{R}}(\bar{u}+v)-c_0\leq C^1\|v\|_1^2,\ \ v\in H^1(\mathbb{R};\mathbb{R}^m),\ \bar{u}\in\{\bar{u}_-,\bar{u}_+\}, \tag{3.27}$$

*with $C^1>0$ a constant.*

Set $u = u^L$ and let

$$p \in (0, \bar{q}),$$

be a number to be chosen later. From (3.27) there is $p^* < p$ such that $e_{p^*} < e_p$. Let $S_{p^*} \subset [0, L]$ be the complement of the set

$$\tilde{S}_{p^*} = \{x \in [0, L] : J_{\mathbb{R}}(u(x, \cdot)) - c_0 > e_{p^*}\}.$$

From (3.3) we have

$$e_{p^*} |\tilde{S}_{p^*}| \leq \int_0^L \Big( J_{\mathbb{R}}(u(x, \cdot)) - c_0 \Big) dx \leq C_0,$$

which implies

$$|\tilde{S}_{p^*}| \leq \frac{C_0}{e_{p^*}}, \;\; |S_{p^*}| \geq L - \frac{C_0}{e_{p^*}}.$$

For $x \in S_{p^*}$ we have $J_{\mathbb{R}}(u(x, \cdot)) - c_0 \leq e_{p^*} < e_p$ and therefore Lemma 3.6 implies $q_1(x) < p$. It follows $q(x) \leq q_1(x) \leq \bar{q}$ and Lemma 3.4 implies the uniqueness of the decomposition (3.17). On the other hand Lemma 3.5 yields

$$\|v_y(x, \cdot)\|^2 \leq \frac{2}{\mu} \Big( J_{\mathbb{R}}(u(x, \cdot)) - c_0 \Big) \leq \frac{2e_p}{\mu}, \;\; x \in S_{p^*}. \tag{3.28}$$

We fix $p$ so that

$$\frac{2e_p}{\mu} \leq \frac{1}{(1 + \sqrt{2})^2} \|\bar{u}'\|^2.$$

With this choice of $p$ we have

$$\|v_y(x, \cdot)\|^2 \leq \frac{1}{(1 + \sqrt{2})^2} \|\bar{u}'\|^2, \;\; x \in S_{p^*}. \tag{3.29}$$

We also have

$$\|v_x(x, \cdot)\|^2 \leq 4 \Big( J_{\mathbb{R}}(u(x, \cdot)) - c_0 \Big), \;\; x \in S_{p^*}. \tag{3.30}$$

To see this, note that from (3.18) and $\omega \geq 0$ it follows

$$\frac{1}{2} \|u_x(x, \cdot)\|^2 \leq J_{\mathbb{R}}(u(x, \cdot)) - c_0, \;\; x \in [0, L], \tag{3.31}$$

and that from (3.23) and (3.29) it follows

$$\|v_x(x, \cdot)\|^2 \leq 2 \|u_x(x, \cdot)\|^2, \;\; x \in S_{p^*}.$$

From (3.22), (3.28), (3.29) and (3.30) we obtain

$$\int_{S_{p^*}} |h'(x)| dx \leq \frac{\sqrt{2}(1 + \sqrt{2})^2}{\sqrt{\mu} \|\bar{u}'\|^2} \int_{S_{p^*}} \Big( J_{\mathbb{R}}(u(x, \cdot)) - c_0 \Big) dx \leq \frac{\sqrt{2}(1 + \sqrt{2})^2}{\sqrt{\mu} \|\bar{u}'\|^2} C_0. \tag{3.32}$$

**Lemma 3.7.** *There is a constant $C_h > 0$ independent of $L$ such that*

$$|h(x) - h(x')| \leq C_h, \;\; x, x' \in S_{p^*}.$$

*Proof.* $\tilde{S}_{p^*}$ is the union of a countable family of intervals $\tilde{S}_{p^*} = \cup_j(\alpha_j, \beta_j)$. Therefore, for each $x, x' \in S_{p^*}$ we have

$$|h(x) - h(x')| \leq \int_{S_{p^*}} |h'(x)|dx + \sum_j |h(\beta_j) - h(\alpha_j)|.$$

Since we have already estimated the first term, see (3.32), to complete the proof it remains to evaluate the sum on the right hand side of this inequality. Set $\lambda = \frac{\bar{q}^2}{8C_0}$ and let $I_\lambda = \{j : \beta_j - \alpha_j \leq \lambda\}$ and $\tilde{I}_\lambda = \{j : \beta_j - \alpha_j > \lambda\}$. Note that $\tilde{I}_\lambda$ contains at most $\frac{C_0}{e_{p^*}\lambda}$ intervals. For $j \in I_\lambda$ and $x \in (\alpha_j, \beta_j)$ we have

$$\begin{aligned}
&|u(x, y) - u(\alpha_j, y)| \leq |x - \alpha_j|^{\frac{1}{2}} \Big( \int_{\alpha_j}^x |u_x(s, y)|^2 ds \Big)^{\frac{1}{2}}, \\
&\|u(x, \cdot) - u(\alpha_j, \cdot)\|^2 \leq 2\lambda C_0 \leq \frac{\bar{q}^2}{4}.
\end{aligned}$$

From this and $\alpha_j \in S_{p^*}$, that implies

$$q(\alpha_j) = \|u(\alpha_j, \cdot) - \bar{u}(\cdot - h(\alpha_j))\| < p \leq \frac{\bar{q}}{2},$$

we conclude that

$$q(x) \leq \|u(x, \cdot) - \bar{u}(\cdot - h(\alpha_j)\| \leq \|u(x, \cdot) - u(\alpha_j, \cdot)\| + q(\alpha_j) \leq \bar{q}.$$

This and Lemma 3.2 imply that, for $x \in (\alpha_j, \beta_j)$, with $j \in I_\lambda$, $u = u^L$ can be decomposed as in (3.17) and that $h'(x) = (D_u h)u_x(x, \cdot)$. Therefore from (3.14) and assuming as we can $\bar{q} \leq \frac{\|\bar{u}'\|^2}{2\|\bar{u}''\|}$ we have

$$|h'(x)| \leq 2\frac{\|u_x(x, \cdot)\|}{\|\bar{u}'\|}, \;\; x \in (\alpha_j, \beta_j), \; j \in I_\lambda.$$

It follows

$$\begin{aligned}
&\sum_{j \in I_\lambda} |h(\beta_j) - h(\alpha_j)| \leq \int_{\cup_{j\in I_\lambda}(\alpha_j,\beta_j)} |h'(x)|dx \leq \frac{2}{\|\bar{u}'\|} \int_{\cup_{j\in I_\lambda}(\alpha_j,\beta_j)} \|u_x(x, \cdot)\|dx \\
&\leq \frac{2}{\|\bar{u}'\|} |\tilde{S}_{p^*}|^{\frac{1}{2}} \Big( \int_0^L \|u_x\|^2 dx \Big)^{\frac{1}{2}} \leq (2C_0)^{\frac{1}{2}} \frac{2}{\|\bar{u}'\|} |\tilde{S}_{p^*}|^{\frac{1}{2}}.
\end{aligned}$$

Assume now $j \in \tilde{I}_\lambda$ and observe that there is a number $\bar{y} > 0$ such that, if $r \geq 2\bar{y}$ and $y \in [\bar{y}, r - \bar{y}]$ or if $r \leq -2\bar{y}$ and $y \in [r + \bar{y}, -\bar{y}]$, it results for $\mathrm{sg}, \tilde{\mathrm{sg}} \in \{-, +\}$

$$|\bar{u}_{\mathrm{sg}}(y) - \bar{u}_{\tilde{\mathrm{sg}}}(y - r)| \geq \frac{1}{2}|a_+ - a_-|. \tag{3.33}$$

Consider first the indices $j \in \tilde{I}_\lambda$ such that $|h(\beta_j) - h(\alpha_j)| \leq 4\bar{y}$. We have

$$\sum_{j \in \tilde{I}_\lambda, |h(\beta_j) - h(\alpha_j)| \leq 4\bar{y}} |h(\beta_j) - h(\alpha_j)| \leq 4\bar{y} \frac{C_0}{e_{p^*}\lambda} = \frac{32C_0^2}{e_{p^*}\bar{q}^2}\bar{y}.$$

Let $(\alpha, \beta)$ be one of the intervals $(\alpha_j, \beta_j)$ corresponding to $j \in \tilde{I}_\lambda$ with $|h(\beta_j) - h(\alpha_j)| > 4\bar{y}$. If $r > 4\bar{y}$ the interval $(\bar{y}, r - \bar{y})$ (if $r < -4\bar{y}$ the interval $(r + \bar{y}, -\bar{y})$) has measure larger then $\frac{|r|}{2}$. This and the assumptions on $(\alpha,\beta)$ imply that there exist $y^0, y^1 \in (\alpha,\beta)$, that satisfy $y^1 - y^0 = |h(\beta_j) - h(\alpha_j)|/2$ and are such that

$$|u(\beta,y) - u(\alpha,y)| \geq \frac{1}{4}|a_+ - a_-|, \quad \text{for} \quad y \in (y^0, y^1). \tag{3.34}$$

This, provided $p > 0$ is sufficiently small, follows from (3.33). Indeed we have

$$\begin{aligned}
&|u(\beta,y) - u(\alpha,y)| \geq |\bar{u}_{\mathrm{sg}(\beta)}(y - h(\beta)) - \bar{u}_{\mathrm{sg}(\alpha)}(y - h(\alpha))| \\
&- |u(\beta,y) - \bar{u}_{\mathrm{sg}(\beta)}(y - h(\beta))| - |u(\alpha,y) - \bar{u}_{\mathrm{sg}(\alpha)}(y - h(\alpha))| \\
&\geq \frac{1}{2}|a_+ - a_-| - |v(\beta, y - h(\beta))| - |v(\alpha, y - h(\alpha))| \\
&\geq \frac{1}{2}|a_+ - a_-| - C_1(q(\alpha)^{\frac{2}{3}} + q(\beta)^{\frac{2}{3}}) \\
&\geq \frac{1}{2}|a_+ - a_-| - 2C_1 p^{\frac{2}{3}} \geq \frac{1}{4}|a_+ - a_-|, \quad \text{for} \quad y \in (y^0, y^1),
\end{aligned}$$

where we denoted by $\bar{u}_{\mathrm{sg}(x)}$ the map $\bar{u} \in \{\bar{u}_-, \bar{u}_+\}$ corresponding to $x \in S_{e_{p^*}}$ and we used (3.25) based on

$$|v_y(x,y)| \leq C,$$

that follows from (3.11) and (1.19). Integrating (3.34) in $(y^0, y^1)$ yields

$$\begin{aligned}
&\frac{|a_+ - a_-|}{8}|h(\beta) - h(\alpha)| \leq \int_{y^0}^{y^1} |u(\beta,y) - u(\alpha,y)| dy \leq \int_{y^0}^{y^1} \int_{\alpha}^{\beta} |u_x| dx dy \\
&\leq \frac{1}{\sqrt{2}}|h(\beta) - h(\alpha)|^{\frac{1}{2}} (\beta - \alpha)^{\frac{1}{2}} \Big( \int_{y^0}^{y^1} \int_{\alpha}^{\beta} |u_x|^2 dx dy \Big)^{\frac{1}{2}} \\
&\leq |h(\beta) - h(\alpha)|^{\frac{1}{2}} (\beta - \alpha)^{\frac{1}{2}} C_0^{\frac{1}{2}}.
\end{aligned}$$

It follows $|h(\beta) - h(\alpha)| \leq \frac{64 C_0}{|a_+ - a_-|^2}(\beta - \alpha)$ and in turn

$$\begin{aligned}
&\sum_{j \in \tilde{I}_\lambda, |h(\beta_j) - h(\alpha_j)| > 4\bar{y}} |h(\beta_j) - h(\alpha_j)| \\
&\leq \frac{64 C_0}{|a_+ - a_-|^2} \sum_{j \in \tilde{I}_\lambda, |h(\beta_j) - h(\alpha_j)| > 4\bar{y}} (\beta_j - \alpha_j) \leq \frac{64 C_0}{|a_+ - a_-|^2} |\tilde{S}_{p^*}|.
\end{aligned}$$

The proof is complete. □

With Lemma 3.7 at hand we can show that $d_L$ in (3.6) can be taken independent of $L$ and that $u = u^L$ converges to $a_\pm$ as $y \to \pm\infty$ uniformly in $x \in \mathbb{R}$.

Next we prove that the restriction $x, x' \in S_{p^*}$ in Lemma 3.7 can be removed. We have indeed

**Lemma 3.8.** *There is a constant $C_h > 0$ independent of $L$ such that*

$$|h(x) - h(x')| \leq C_h, \quad x, x' \in [0, L].$$

*Proof.* Assuming that $p > 0$ is sufficiently small, from (3.25) we have

$$|u(x,y) - \bar{u}_{\mathrm{sg}(x)}(y - h(x))| \leq C^1 p^{\frac{2}{3}} \leq \frac{r_0}{8}, \;\; x \in S_{p^*}, y \in \mathbb{R}, \tag{3.35}$$

where $r_0$ is defined in (3.5). By Lemma (3.7) there exist $h_+, h_-$ such that $2\delta_h := h_+ - h_-$ is bounded independently of $L$ and

$$\begin{aligned}
&|\bar{u}_{\mathrm{sg}(x)}(y - h(x)) - a_+| \leq \frac{r_0}{8}, \;\; y \geq h_+, \; x \in S_{p^*}, \\
&|\bar{u}_{\mathrm{sg}(x)}(y - h(x)) - a_-| \leq \frac{r_0}{8}, \;\; y \leq h_-, \; x \in S_{p^*}.
\end{aligned}$$

The first relation and (3.35) imply

$$|u(x,y) - a_+| \leq \frac{r_0}{4}, \;\; y \geq h_+, \; x \in S_{p^*}. \tag{3.36}$$

Now define $Y \subset \mathbb{R}$ by setting

$$Y = \{y > h_+ : \exists\, x_y \in [0,L] \text{ such that } |u(x_y, y) - a_+| \geq \frac{r_0}{2}\},$$

From (3.36) it follows that $y \in Y$ implies that $x_y$ belongs to $\tilde{S}_{p^*}$ and therefore to one of the intervals, say $(\alpha, \beta)$, that compose $\tilde{S}_{p^*}$. From (3.36) with $x = \alpha$ it follows $|u(x_y, y) - u(\alpha, y)| \geq \frac{r_0}{4}$ for $y \in Y$, and therefore we have

$$\frac{r_0}{4} \leq \int_\alpha^{x_y} |u_x(x,y)| dx \leq |\beta - \alpha|^{\frac{1}{2}} \Big( \int_\alpha^\beta |u_x(x,y)|^2 dx \Big)^{\frac{1}{2}}, \qquad y \in Y,$$

and in turn

$$|Y| \frac{r_0^2}{16} \leq |\tilde{S}_{p^*}| \int_{\tilde{S}_{p^*}} \int_\alpha^\beta |u_x(x,y)|^2 dx dy \leq 2C_0 |\tilde{S}_{p^*}|,$$

and we see that the measure of $Y$ is bounded independently of $L$. Then there exists an increasing sequence $y_j \to +\infty$ such that

$$\begin{aligned}
&y_1 \leq h_+ + 2|Y|, \\
&|u(x, y_j) - a_+| < \frac{r_0}{2}, \;\; x \in [0,L], \;\; j = 1, \ldots.
\end{aligned}$$

This and the cut-off lemma in [6] imply

$$|u(x,y) - a_+| \leq \frac{r_0}{2}, \;\; y \geq h_+ + 2|Y|, \; x \in [0,L].$$

A similar argument yields

$$|u(x,y) - a_-| \leq \frac{r_0}{2}, \;\; y \leq h_- - 2|Y|, \; x \in [0,L].$$

The lemma follows from these relations and the fact that $h_+ - h_-$ and $|Y|$ do not depend on $L$.

□

**Corollary 3.9.** *We can assume that the minimizer $u^L$ satisfies*

$$|u^L(x,y) - a_+| \leq Ke^{-ky}, \;\; y > 0, \; x \in \mathbb{R}, \qquad (3.37)$$
$$|u^L(x,y) - a_-| \leq Ke^{ky}, \;\; y < 0, \; x \in \mathbb{R}.$$

*and*

$$|D^\alpha u^L(x,y)| \leq Ke^{-k|y|}, \;\; y \in \mathbb{R}, \; x \in \mathbb{R}, \qquad (3.38)$$

*for $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = 1, 2$, with constants $k, K > 0$ independent of $L$.*

*Proof.* Using again the translation invariance of the energy $\mathcal{J}$, by identifying $u(x,y)$ with $u_{\delta_h}(x,y) = u(x, y+\delta_h)$, we can assume that the minimizer $u$ satisfy

$$|u(x,y) - a_+| \leq \frac{r_0}{2}, \;\; y \geq \delta_h + 2|Y|, \; x \in [0,L]$$
$$|u(x,y) - a_-| \leq \frac{r_0}{2}, \;\; y \leq -\delta_h - 2|Y|, \; x \in [0,L].$$

These inequalities and a standard argument, based on the non-degeneracy of $a_+, a_-$, imply (3.37). Inequality (3.38) follows from (3.37) and elliptic regularity. The proof is complete. □

*Remark* 5. From (3.37) it follows that we have $|h(x)| \leq C_h$, for $x \in [0,L]$ with $C_h$ independent of $L$. Note that this is true in spite of the fact that $h(x)$, if $q(x)$ is large, may be discontinuous.

The bound on $h(x)$ together with (3.37), (3.38) and (1.19) imply that

$$v(x,y) = u(x, y+h(x)) - \bar{u}_{\mathrm{sg}(x)}(y), \qquad (3.39)$$

and its first and second derivative with respect to $y$ satisfy exponential estimates of the form

$$|D_y^i v(x,y)| \leq Ke^{-k|y|}, \;\; y \in \mathbb{R}, \; x \in \mathbb{R}, \; i = 0, 1, 2. \qquad (3.40)$$

with constants $k, K > 0$ independent of $L$. From this and the identity $\|v_y\|^2 + \langle v, v_{yy} \rangle = 0$ it follows

$$\|v_y(x, \cdot)\| \leq C_2 q(x)^{\frac{1}{2}}, \qquad (3.41)$$

with $C_2 > 0$ independent of $L$. This inequality implies that in each interval where $q(x) \leq q^*$, for some $q^* > 0$, we can use the expressions of $h'(x)$ and $\|u_x(x,\cdot)\|$ in Lemma 3.4 and we have the monotonicity of the function $p \mapsto f(p,x)$.

## 3.2 Conclusion of the proof of Theorem 1.2

As before we set $u = u^L$. Since $u \in \mathcal{A}^L$ we have in particular $u(0,y) = \gamma u(0,y)$ that means $u(0,y) \in \pi_\gamma$, $\pi_\gamma$ the plane fixed by $\gamma$. From this and $\bar{u}_- = \gamma \bar{u}_+$, $\bar{u}_- \neq \bar{u}_+$ it follows

$$q_1(0) = \inf_{r \in \mathbb{R}, \pm} \|u(0,\cdot) - \bar{u}_\pm(\cdot - r)\|_1 \geq \frac{1}{2}\|\bar{u}_+ - \bar{u}_-\|.$$

We assume that the constant $q^*$ introduced above satisfies $q^* < \frac{1}{2}\|\bar{u}_+ - \bar{u}_-\|$ and set $p = q^*/2$. Then, provided $L$ is sufficiently large, there exists $x_p > 0$ such that

$$\begin{aligned} &q_1(x) > p, \quad x \in [0, x_p), \\ &q_1(x_p) = p. \end{aligned} \tag{3.42}$$

Indeed, from Lemma 3.6 and (3.3) it follows $x_p e_p \leq \int_0^{x_p} (J_{\mathbb{R}}(u(x,\cdot) - c_0))dx \leq C_0$, so that

$$x_p \leq l_p := \frac{C_0}{e_p}. \tag{3.43}$$

From (3.42), (3.43) and the symmetry $u(\frac{L}{4} - x, y) = u(\frac{L}{4} + x, y)$ with $x = \frac{L}{4} - x_p$ we obtain

$$\begin{aligned} &q(x_p) = q(\frac{L}{2} - x_p) \leq q_1(x_p) = p = q^*/2, \\ &\bar{u}_{\mathrm{sg}(x_p)} = \bar{u}_{\mathrm{sg}(\frac{L}{2} - x_p)}, \\ &h(x_p) = h(\frac{L}{2} - x_p). \end{aligned} \tag{3.44}$$

We now show, see Lemma 3.10 below, that the minimality of $u = u^L$ and (3.44) imply

$$q(x) \leq p \quad , x \in [x_p, \frac{L}{2} - x_p].$$

In the proof of this fact, for $x$ in certain intervals, we use test maps of the form

$$\hat{u}(x, y) = \bar{u}(y - \hat{h}(x)) + \hat{q}(x)\nu(x, y - \hat{h}(x)) \tag{3.45}$$

for suitable choices of the functions $\hat{q} = \hat{q}(x)$ and $\hat{h} = \hat{h}(x)$. We always take $\hat{q}(x) \leq q(x) \leq p$. Note that in (3.45) the direction vector $\nu(x, \cdot)$ is the one associated to $v(x, \cdot) = q(x)\nu(x, \cdot)$ with $v(x, \cdot)$ defined in the decomposition (3.17) of $u$.
From (3.45) it follows

$$\int_{\mathbb{R}} |\hat{u}_x|^2 dy = (\hat{h}')^2 \|\bar{u}' + \hat{q}\nu_y\|^2 - 2\hat{h}'\hat{q}^2\langle \nu_x, \nu_y\rangle + (\hat{q}')^2 + \hat{q}^2\|\nu_x\|^2. \tag{3.46}$$

We choose the value of $\hat{h}'$ that minimizes (3.46) that is

$$\hat{h}' = \hat{q}^2 \frac{\langle \nu_x, \nu_y\rangle}{\|\bar{u}' + \hat{q}\nu_y\|^2},$$

then we get

$$\int_{\mathbb{R}} |\hat{u}_x|^2 dy = (\hat{q}')^2 + \hat{q}^2\|\nu_x\|^2 - \hat{q}^4 \frac{\langle \nu_x, \nu_y\rangle^2}{\|\bar{u}' + \hat{q}\nu_y\|^2}.$$

Therefore the energy density of the test map $\hat{u}$ is given by

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{2}|\hat{u}_x|^2 dy + \int_{\mathbb{R}} (W(\hat{u}) + \frac{1}{2}|\hat{u}_y|^2) dy \\ &= \frac{1}{2}\Big((\hat{q}')^2 + \hat{q}^2\|\nu_x\|^2 - \hat{q}^4 \frac{\langle \nu_x, \nu_y\rangle^2}{\|\bar{u}' + \hat{q}\nu_y\|^2}\Big) + \mathcal{W}(\hat{q}\nu) + c_0. \end{aligned} \tag{3.47}$$

Note that, since we do not change the direction vector $\nu(x, \cdot)$, this expression is completely determined once we fix the function $\hat{q}$.

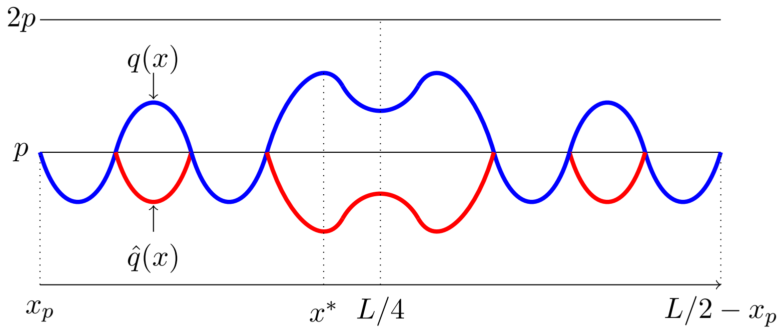


Figure 3: The maps $x \to q(x)$ and $x \to \hat{q}(x)$ in Lemma 3.10, $q(x^*) \leq 2p$

**Lemma 3.10.** *If $u = u^L$ satisfies (3.44), then*

$$q(x) \leq p, \quad x \in \Big[x_p, \frac{L}{2} - x_p\Big].$$

*Proof.* Assume instead that $q(x^*) > p$ for some $x^* \in (x_p, \frac{L}{2} - x_p)$. We can assume that $q(x^*) = \max_{x \in [x_p, \frac{L}{2} - x_p]} q(x)$. We show that this implies the existence of a competing map $\tilde{u}$ with less energy than $u$. Consider first the case where $q(x^*) \in (p, 2p]$. In this case we set $\tilde{u} = \hat{u}$ with $\hat{u}$ defined in (3.45) and, see Figure 3,

$$\begin{aligned} &\hat{q}(x) = q(x), \quad \text{if } \ q(x) \leq p, \\ &\hat{q}(x) = 2p - q(x), \quad \text{if } \ q(x) \in (p, 2p]. \end{aligned} \tag{3.48}$$

With this definition we have

$$\tilde{u}(x_p) = u(x_p) = u\Big(\frac{L}{2} - x_p\Big) = \tilde{u}\Big(\frac{L}{2} - x_p\Big). \tag{3.49}$$

To see this we note that $\max_{x \in [x_p, \frac{L}{2} - x_p]} q(x) = q(x^*) \leq q^*$ implies that $\mathrm{sg}(x)$ is constant in $[x_p, \frac{L}{2} - x_p]$ therefore from (3.39) and $u(x, y) = u(\frac{L}{2} - x, y)$ it follows

$$v_x(x, y) = -v_x\Big(\frac{L}{2} - x, y\Big) \quad v_y(x, y) = v_y\Big(\frac{L}{2} - x, y\Big),$$

and by consequence

$$\hat{h}'(x) = -\hat{h}'\Big(\frac{L}{2} - x\Big),$$

which yields

$$\hat{h}\Big(\frac{L}{2} - x_p\Big) = h(x_p) + \int_{x_p}^{\frac{L}{2} - x_p} \hat{h}'(x) dx = h(x_p) = h\Big(\frac{L}{2} - x_p\Big).$$

This and $q(x_p) = q(\frac{L}{2} - x_p)$ imply (3.49). It remains to show that the energy of $\tilde{u}$ is strictly less than the energy of $u$. By comparing (3.47) with the analogous expression of the energy of $u$ and observing that $(\hat{q}')^2 = (q')^2$ and $\hat{q}(x) \leq q(x)$ with strict inequality near $x^*$ we see that this is indeed the case.

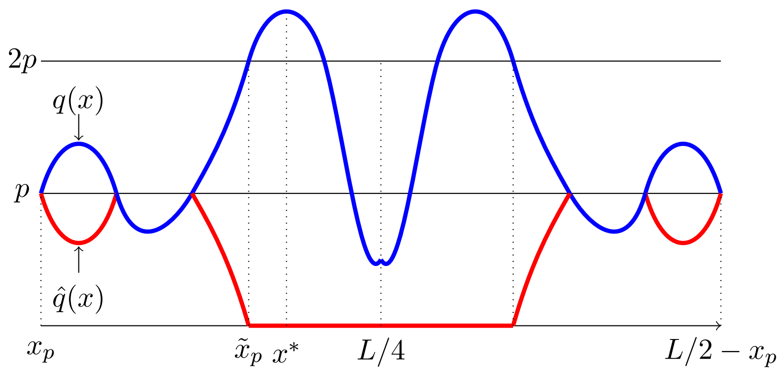


Figure 4: The maps $x \to q(x)$ and $x \to \hat{q}(x)$ in Lemma 3.10, $q(x^*) > 2p$

Assume now that $q(x^*) > 2p$, see Figure 4. Let $\tilde{x}_p \in (x_p, \frac{L}{4})$ be the number

$$\tilde{x}_p = \max\{x > x_p : q(s) \leq 2p, \ s \in (x_p, x]\}.$$

Note that from $\bar{u}_{\mathrm{sg}(x_p)} = \bar{u}_{\mathrm{sg}(\frac{L}{2} - x_p)}$ and the symmetry of $u$ it follows that $\mathrm{sg}(x)$ is equal to a constant, say $+$, in $[x_p, \tilde{x}_p] \cup [\frac{L}{2} - \tilde{x}_p, \frac{L}{2} - x_p]$.

We define the competing map $\tilde{u}$ as follows. In the interval $[x_p, \tilde{x}_p]$ we set $\tilde{u} = \hat{u}$ with $\hat{q}$ exactly as in (3.48) and

$$\hat{h}(x) = h(x_p) + \int_{x_p}^{x} \hat{h}'(s)dx, \ \ x \in [x_p, \tilde{x}_p].$$

In the interval $(\tilde{x}_p, \frac{L}{2} - \tilde{x}_p)$ we take

$$\tilde{u}(x, y) = \bar{u}_+(y - \hat{h}(\tilde{x}_p)).$$

Finally in the interval $[\frac{L}{2} - \tilde{x}_p, \frac{L}{2} - x_p]$ we set again $\tilde{u} = \hat{u}$ with $\hat{q}$ as in (3.48) but with

$$\hat{h}(x) = \hat{h}(\tilde{x}_p) + \int_{\frac{L}{2} - \tilde{x}_p}^{x} \hat{h}'(s)dx, \ \ x \in \Big[\frac{L}{2} - \tilde{x}_p, \frac{L}{2} - x_p\Big].$$

With these definitions $\tilde{u}$ is a continuous piece-wise smooth map that satisfies (3.49) and, as in the case $q(x^*) \leq 2p$, one checks that $\tilde{u}$ has energy strictly less than $u$. The proof is complete. □

Next we show that the statement of Lemma 3.10 can be upgraded to exponential decay. We have indeed

**Lemma 3.11.** *There exists a positive constants $c^*, C^*$ independent of $L \geq L_0$ such that*

$$\|v(x, \cdot)\| \leq C^* e^{-c^* x}, \ \ x \in \Big[0, \frac{L}{4}\Big].$$

*Proof.* We show that, under the standing assumption that $2p = q^* > 0$ is sufficiently small, for $L \geq 4x_p$ it results

$$q(x) \leq \sqrt{2} p e^{-\frac{1}{2}\sqrt{\mu}(x - x_p)}, \ \ x \in \Big[x_p, \frac{L}{4}\Big], \tag{3.50}$$

where $\mu > 0$ is the constant in (3.26). Then the lemma follows from (3.50) and (3.40) that implies $q(x) = \|v(x,\cdot)\| \leq \frac{K}{\sqrt{k}}$. To prove (3.50) we proceed as in the proof of Lemma 3.5 in [11]. We first establish the inequality

$$\frac{d^2}{dx^2}\|v(x,\cdot)\|^2 \geq \mu\|v(x,\cdot)\|^2, \quad x \in \Big[x_p, \frac{L}{2} - x_p\Big]. \tag{3.51}$$

We begin by the elementary inequality

$$\begin{aligned} &\frac{d^2}{dx^2}\|v(x,\cdot)\|^2 = \frac{d^2}{dx^2}\|u(x,\cdot) - \bar{u}_+(\cdot - h(x))\|^2 \\ &\geq 2\Big\langle \frac{d^2}{dx^2}\Big(u(x,\cdot) - \bar{u}_+(\cdot - h(x))\Big), u(x,\cdot) - \bar{u}_+(\cdot - h(x))\Big\rangle. \end{aligned} \tag{3.52}$$

From

$$\begin{aligned} &\frac{d^2}{dx^2}\Big(u(x,\cdot) - \bar{u}_+(\cdot - h(x))\Big) \\ &= u_{xx}(x,\cdot) - \bar{u}_+''(\cdot - h(x))(h'(x))^2 + \bar{u}_+'(\cdot - h(x))h''(x), \end{aligned}$$

and (3.52), using also (3.22) (and $\langle \phi, \psi\rangle = \langle \phi(\cdot - r), \psi(\cdot - r)\rangle$), it follows

$$\begin{aligned} &\frac{d^2}{dx^2}\|v(x,\cdot)\|^2 \geq 2\langle u_{xx}(x,\cdot), v(x, \cdot - h(x))\rangle \\ &- 2\langle \bar{u}_+'', v(x,\cdot)\rangle \frac{\langle v_x(x,\cdot), v_y(x,\cdot)\rangle^2}{\|\bar{u}_+' + v_y(x,\cdot)\|^4} = 2I_1 + 2I_2. \end{aligned}$$

Since $u$ is a solution of (1.10) and $\bar{u}_+$ solves (1.7) we have

$$u_{xx}(x,\cdot) = W_u(u(x,\cdot)) - W_u(\bar{u}_+(\cdot - h(x))) - \Big(u(x,\cdot) - \bar{u}_+(\cdot - h(x))\Big)_{yy}.$$

Then, recalling the definition of the operator $T$ and that $v(x,\cdot) = u(x, \cdot + h(x)) - \bar{u}_+$, we obtain

$$\begin{aligned} &I_1 = \langle W_u(\bar{u}_+ + v(x,\cdot)) - W_u(\bar{u}_+) - v_{yy}(x,\cdot), v(x,\cdot)\rangle \\ &= \langle W_u(\bar{u}_+ + v(x,\cdot)) - W_u(\bar{u}_+) - W_{uu}(\bar{u}_+)v(x,\cdot), v(x,\cdot)\rangle + \langle Tv(x,\cdot), v(x,\cdot)\rangle. \end{aligned} \tag{3.53}$$

Now we observe that a standard computation yields

$$J_{\mathbb{R}}(u(x,\cdot)) - c_0 = \frac{1}{2}\langle Tv(x,\cdot), v(x,\cdot)\rangle + \int_{\mathbb{R}} f_W(x,y)dy,$$

where

$$f_W = W(\bar{u}_+ + v) - W(\bar{u}_+) - W_u(\bar{u}_+)v - \frac{1}{2}W_{uu}(\bar{u}_+)v\cdot v.$$

From (3.41), $q(x) = \|v(x,\cdot)\| \leq p$ and (3.25) it follows, with $C_W > 0$ a suitable constant,

$$|f_W(x,y)| \leq C_W|v(x,y)|^3 \leq C_1 C_W\|v(x,\cdot)\|^{\frac{2}{3}}|v(x,y)|^2,$$

and therefore

$$\langle Tv(x,\cdot), v(x,\cdot)\rangle \geq 2(J_{\mathbb{R}}(u(x,\cdot)) - c_0) - C\|v(x,\cdot)\|^{\frac{8}{3}}. \tag{3.54}$$

Introducing this estimate into (3.53) and observing that the other term in the right hand side of (3.53) can also be estimated by a constant times $\|v(x,\cdot)\|^{\frac{8}{3}}$ we finally obtain

$$I_1 \geq 2(J_{\mathbb{R}}(u(x,\cdot)) - c_0) - C\|v(x,\cdot)\|^{\frac{8}{3}}.$$

To estimate $I_2$ we note that from (3.41) and (3.23), provided $q^* = 2p$ is sufficiently small, we get

$$\|v_x(x,\cdot)\|^2 \leq 2\|u_x(x,\cdot)\|^2 \leq 4(J_{\mathbb{R}}(u(x,\cdot)) - c_0), \;\; x \in \Big[x_p, \frac{L}{2} - x_p\Big],$$

where we have also used (3.31). This and (3.41) imply

$$|I_2| \leq Cp(J_{\mathbb{R}}(u(x,\cdot)) - c_0),$$

for some constant $C > 0$ and we obtain

$$(I_1 + I_2) \geq (2 - Cp)(J_{\mathbb{R}}(u(x,\cdot)) - c_0) \geq \frac{1}{2}\mu\|v(x,\cdot)\|^2, \;\; x \in \Big[x_p, \frac{L}{2} - x_p\Big]$$

and (3.51) is established.

From (3.51) and the comparison principle we have

$$\|v(x,\cdot)\|^2 \leq \varphi(x), \;\; x \in \Big[x_p, \frac{L}{2} - x_p\Big] \tag{3.55}$$

where

$$\varphi(x) = p^2 \frac{\cosh\sqrt{\mu}(x - \frac{L}{4})}{\cosh\sqrt{\mu}(x_p - \frac{L}{4})}$$

is the solution of the problem

$$\begin{cases} \varphi'' = \mu\varphi, \;\; x \in [x_p, \frac{L}{2} - x_p], \\ \\ \varphi(x_p) = \varphi(\frac{L}{2} - x_p) = p^2. \end{cases}$$

Then (3.50) follows from (3.55) and

$$\varphi(x) \leq 2p^2 e^{-\sqrt{\mu}(x - x_p)}, \;\; x \in \Big[x_p, \frac{L}{4}\Big].$$

This concludes the proof. □

To finish the proof of Theorem 1.2 it remains to show that there is a sequence $u^{L_j}$, $L_j \to +\infty$, that converges to a heteroclinic connection between suitable translates of $\bar{u}_\pm$. Indeed, once this is established, a suitable translation $\eta$ in the $y$ direction yields the sequence $u^{L_j}(x, y-\eta)$ and the heteroclinic $u^H$ in Theorem 1.2. From (3.11) it follows that there exists a subsequence, still denoted by $L_j$, and a classical solution $u^\infty : \mathbb{R}^2 \to \mathbb{R}^m$ of (1.10) such that we have

$$\lim_{j \to +\infty} u^{L_j}(x,y) = u^\infty(x,y), \tag{3.56}$$

in the $C^2$ sense in compacts. Moreover $u^\infty$ satisfies the exponential estimates (3.37) and (3.38). This implies that the convergence in (3.56) is in the $C^2$ sense in any set of the form $[-\lambda, \lambda] \times \mathbb{R}$. Set $u_j = u^{L_j}$ and denote by $h_j$ and $v_j$ the functions determined by the decomposition (3.17) of $u_j$:

$$\begin{aligned} & u_j(x, y) = \bar{u}_+(y - h_j(x)) + v_j(x, y - h_j(x)), \\ & \langle v_j, \bar{u}'_+ \rangle = 0. \end{aligned} \tag{3.57}$$

On the basis of Remark 5, $v_j$ and its first and second derivatives satisfy (3.40). Therefore (3.41) shows that, under the standing assumption of $q^* > 0$ small, we can control the size of $\|(v_j)_y(x, \cdot)\|$ and, proceeding as in the derivation of (3.30), we obtain from (3.22)

$$|h'_j(x)| \leq C\|(v_j)_x(x, \cdot)\| \leq C(J_{\mathbb{R}}(u_j(x, \cdot)) - c_0)^{\frac{1}{2}}, \;\; x \in \Big[l_p, \frac{L_j}{4}\Big].$$

On the other hand from (3.54) and (3.41) we get

$$J_{\mathbb{R}}(u(x, \cdot)) - c_0 \leq C(\|v_y(x, \cdot)\|^2 + \|v(x, \cdot)\|^2 + \|v(x, \cdot)\|^{\frac{8}{3}}) \leq C\|v(x, \cdot)\|,$$

and we conclude

$$|h'_j(x)| \leq C\|v_j(x, \cdot)\|^{\frac{1}{2}} \leq Ce^{-\frac{1}{4}\sqrt{\mu}(x - l_p)}, \;\; x \in \Big[l_p, \frac{L_j}{4}\Big] \tag{3.58}$$

where we have also used (3.50).

This and the fact that, as we have seen in Remark 5, $h_j(x)$ is bounded independently of $j$, imply that by passing to a subsequence if necessary, we can assume that there is a Lipschitz continuous and bounded map $h^\infty : [l_p, +\infty) \to \mathbb{R}$ such that

$$\lim_{j \to +\infty} h_j(x) = h^\infty(x), \;\; x \in [l_p, +\infty),$$

uniformly in compacts. It follows that we can pass to the limit in (3.57) and obtain in particular that there exists the limit $v^\infty : [l_p, +\infty) \times \mathbb{R} \to \mathbb{R}^m$ of

$$\lim_{j \to +\infty} v_j(x, y) = v^\infty(x, y),$$

and the convergence is in $L^2$ and in $L^\infty$ in sets of the form $[l_p, l] \times \mathbb{R}$. The functions $h^\infty$ and $v^\infty$ coincide with the functions determined by the decomposition (3.17) of $u^\infty$. Moreover from (3.50) and (3.58) we have that $q^\infty(x) = \|v^\infty(x, \cdot)\|$ and $h^\infty$ satisfy

$$\begin{aligned} & q^\infty(x) \leq C^* e^{-c^* x}, \;\; x \geq 0, \\ & h^\infty(x) \leq Ce^{-\frac{1}{4}\sqrt{\mu}(x - l_p)}, \;\; x \geq l_p. \end{aligned}$$

The first of these estimates shows that , for $x \to +\infty$, $u^\infty(x, \cdot)$ converges in the $L^2$ sense to the manifold of the translates of $\bar{u}_+$. The estimate for $h^\infty$ shows that there exists $\eta = \lim_{x \to +\infty} h^\infty(x)$ and therefore that actually, for $x \to \infty$, $u^\infty(x, \cdot)$ converges, to a specific element of that manifold. This, taking also into account the symmetry properties of $u^\infty$ implies that indeed $u^\infty$ is a heteroclinic solution of (1.10) that connects translates of $\bar{u}_\pm$.

This concludes the proof of Theorem 1.2.

# A Appendix

We present an elementary proof of Lemma 3.6, that we restate as PropositionA.1.

**Proposition A.1.** *Assume that* $W : \mathbb{R}^m \to \mathbb{R}$ *is of class* $C^3$, $a_\pm$ *are non degenerate, and* $u \in \mathcal{H}^1 = \bar{\mathrm{u}} + H^1(\mathbb{R}; \mathbb{R}^m)$.

*Then, for each* $p > 0$ *there is* $e_p > 0$ *such that*

$$\|u - \bar{u}_\pm(\cdot - r)\|_1 \geq q_1^u \geq p, \quad r \in \mathbb{R}. \tag{A.1}$$

*implies*

$$J_{\mathbb{R}}(u) - c_0 \geq e_p.$$

*Moreover* $e_p$ *is continuous in* $p$ *and for* $p \leq \|v\|_1$ *small it results*

$$e_p \leq J_{\mathbb{R}}(\bar{u} + v) - c_0 \leq C^1 \|v\|_1^2, \;\; v \in H^1(\mathbb{R}; \mathbb{R}^m), \;\bar{u} \in \{\bar{u}_-, \bar{u}_+\},$$

*with* $C^1 > 0$ *a constant.*

*Proof.* If $u$ satisfies (A.1) and has $J_{\mathbb{R}}(u) \geq 2c_0$ we can take $e_p = c_0$. It follows that in the proof we can assume

$$J_{\mathbb{R}}(u) < 2c_0. \tag{A.2}$$

Note also that $u \in \mathcal{H}^1$ implies

$$\lim_{s \to \pm\infty} u(s) = a_\pm. \tag{A.3}$$

and set

$$q_0 = \min\Big\{ r_0, \frac{\gamma^2}{8C_W} \Big\}, \tag{A.4}$$

where $r_0$ and $\gamma$ are the constants in (3.5) and

$$C_W = \max\{|W_{uuu}(a_\pm + z)| : |z| \leq 3r_0\}.$$

Given $q \in (0, q_0)$ define

$$\begin{aligned}
&J_z^+(q) = \min_{v \in \mathcal{V}_z^+(q)} J(v),\\
&\mathcal{V}_z^+(q) = \{v \in H^1_{loc}((0, \tau^v); \mathbb{R}^m) : v(0) = z, |z - a_+| = q, \;\lim_{s \to \tau^v} v(s) = a_+\},\\
&J^-(q) = \min_{v \in \mathcal{V}^-(q)} J(v),\\
&\mathcal{V}^-(q) = \{v \in H^1_{loc}((0, \tau^v); \mathbb{R}^m) : |v(0) - a_+| = q, \;\lim_{s \to \tau^v} v(s) = a_-\},\\
&J_0(q) = \min_{v \in \mathcal{V}_0(q)} J(v),\\
&\mathcal{V}_0(q) = \{v \in H^1((0, \tau^v); \mathbb{R}^m) : |v(0) - a_+| = q_0, \;|v(\tau^v) - a_+| = q\}.
\end{aligned}$$

Observe that there exists a positive functions $\psi : (0, q_0) \to \mathbb{R}$ that converges to zero with $q$ and satisfies

$$J_z^+(q) \leq \psi(q).$$

Note also that $J_{\mathbb{R}}(\bar{u}_\pm) = c_0$ and the minimality of $\bar{u}_\pm$ imply $J^-(q) + \psi(q) \geq c_0$ and therefore we have

$$c_0 - \psi(q) \leq J^-(q). \tag{A.5}$$

For $u \in \mathcal{H}^1$ define

$$\begin{aligned} s^{u,-}(\rho) &= \max\{s : |u(t) - a_-| \leq \rho, \text{ for } t \leq s\}, \\ s^{u,+}(\rho) &= \min\{s : |u(t) - a_+| \leq \rho, \text{ for } t \geq s\}. \end{aligned}$$

Since $\psi(q) \to 0$ as $q \to 0$ while $\lim_{q\to 0} J_0(q) = \mathsf{J}_0$, $\mathsf{J}_0$ a positive constant, we can fix $q = q(q_0)$ in such a way that

$$2J_0(q(q_0)) - \psi(q(q_0)) \geq \mathsf{J}_0. \tag{A.6}$$

We claim that in this proposition it suffices to consider only maps that satisfy the condition

$$s^{u,+}(q_0) - s^{u,-}(q_0) \leq \frac{2c_0}{W_m(q(q_0))}, \tag{A.7}$$

where $W_m(t) = \min_{a\in\{a_-,a_+\},|z|\geq t} W(a + z)$. To see this set

$$\begin{aligned} \bar{s}^{u,-} &= \max\{s : |u(s) - a_-| = q(q_0)\}, \\ \bar{s}^{u,+} &= \min\{s : |u(s) - a_+| = q(q_0)\}, \end{aligned}$$

and observe that the definition of $\bar{s}^{u,\pm}$ implies $|u(s) - a_\pm| > q(q_0)$, for $s \in (\bar{s}^{u,-}, \bar{s}^{u,+})$. It follows

$$(\bar{s}^{u,+} - \bar{s}^{u,-})W_m(q(q_0)) \leq 2c_0. \tag{A.8}$$

Assume first that

$$\begin{aligned} &|u(s) - a_-| < q_0, \text{ for } s \in (-\infty, \bar{s}^{u,-}), \\ &|u(s) - a_+| < q_0, \text{ for } s \in (\bar{s}^{u,+}, +\infty). \end{aligned} \tag{A.9}$$

In this case we have

$$\bar{s}^{u,-} < s^{u,-}(q_0) < s^{u,+}(q_0) < \bar{s}^{u,+},$$

that together with (A.8) implies (A.7). Now assume that (A.9) does not hold and there exists $s^* \in (\bar{s}^{u,+}, +\infty)$ such that $|u(s^*) - a_+| = q_0$ (or $s^* \in (-\infty, \bar{s}^{u,-})$ such that $|u(s^*) - a_-| = q_0$). For definiteness we consider the first eventuality, the other possibility is discussed in a similar way. To estimate the energy of $u$ we focus on the intervals $(-\infty, \bar{s}^{u,+})$, $(\bar{s}^{u,+}, s^{u,+}(q(q_0)))$, and $(s^{u,+}(q(q_0)), +\infty)$. We have $J_{(-\infty,\bar{s}^{u,+})}(u) \geq J^-(q(q_0))$ and since $s^* \in (\bar{s}^{u,+}, s^{u,+}(q(q_0)))$ we also have $J_{(\bar{s}^{u,+},s^{u,+}(q(q_0)))}(u) \geq 2J_0(q(q_0))$. This, (A.5) and (A.6) imply

$$\begin{aligned} J_{\mathbb{R}}(u) &\geq J_{(-\infty,\bar{s}^{u,+})}(u) + J_{(\bar{s}^{u,+},s^{u,+}(q(q_0)))}(u) \geq J^-(q(q_0)) + 2J_0(q(q_0)) \\ &\geq c_0 - \psi(q(q_0)) + 2J_0(q(q_0)) \geq c_0 + \mathsf{J}_0. \end{aligned}$$

This completes the proof of the claim. Indeed this computation shows that, if $s^*$ with the above properties exists, then we can take $e_p = \mathsf{J}_0$.

Since $J_{\mathbb{R}}$ is translation invariant we can also restrict ourselves to the set of the maps that satisfy

$$-s^{u,-}(q(q_0)) = s^{u,+}(q(q_0)) \leq \frac{c_0}{W_m(q(q_0))}. \tag{A.10}$$

and assume that also $\bar{u}_\pm$ satisfy (A.10). We remark that the set of maps that satisfy (A.2) and (A.7) is equibounded and equicontinuous. Indeed (A.2) implies

$$|u(s_1) - u(s_2)| \leq \sqrt{2c_0}|s_1 - s_2|^{\frac{1}{2}},$$

which together with (A.7) yield

$$|u(s)| \leq M_0 := |a_-| + 3q_0 + \sqrt{2c_0}\Big(\frac{2c_0}{W_m(q(q_0))}\Big)^{\frac{1}{2}}.$$

We first prove the proposition with (A.1) replaced by

$$\|u - \bar{u}_\pm(\cdot - r)\| \geq q^u \geq p, \quad r \in \mathbb{R}. \tag{A.11}$$

Assume the proposition is false. Then there is a sequence $\{u_j\} \subset \mathcal{H}^1$ that satisfies (A.3) and

$$\begin{aligned}&\lim_{j\to+\infty} J_{\mathbb{R}}(u_j) = c_0,\\ &\|u_j - \bar{u}_\pm(\cdot - r)\| \geq p, \quad r \in \mathbb{R}.\end{aligned}$$

Since the sequence $\{u_j\}$ is equibounded and equicontinuous there is a subsequence, still labeled $\{u_j\}$ and a continuous map $\bar{u} : \mathbb{R} \to \mathbb{R}^m$ such that

$$\lim_{j\to+\infty} u_j(s) = \bar{u}(s),$$

uniformly in compact sets. From $\int_{\mathbb{R}} |u_j'|^2 < 4c_0$ and the fact that $u_j$ is uniformly bounded, by passing to a further subsequence if necessary, we have that $u_j$ converges to $\bar{u}$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m)$. A standard argument then shows that

$$J_{\mathbb{R}}(\bar{u}) = c_0,$$

and therefore, by the assumption that $\bar{u}_\pm$ and their translates are the only minimizers of $J_{\mathbb{R}}$, we conclude that $\bar{u}$ coincides either with $\bar{u}_-(\cdot - r)$ or with $\bar{u}_+(\cdot - r)$ with $|r| \leq \lambda_0$ where $\lambda_0$ is determined by the condition that $\bar{u}$ satisfies (A.10).

Since $\lambda_0$ is fixed, from (1.19) it follows that we can assume $|\bar{u(s)} - a_+| \leq Ke^{-ks}$ for $s > 0$. Fix a number $l > \lambda_0$ such that

$$Ke^{-kl} \leq q_0, \quad \text{and} \quad \frac{K}{C_W}e^{-kl} \leq \frac{p^2}{8}, \tag{A.12}$$

and observe that $\bar{u}$ restricted to the interval $[-l, l]$ is a minimizer of $J_{(-l,l)}(u)$ in the class of $u$ that satisfy $u(\pm l) = \bar{u}(\pm l)$. From this observation it follows

$$J_{(-l,l)}(u_j) \geq J_{(-l,l)}(\bar{u}) - Cl\delta_j, \tag{A.13}$$

where $C > 0$ is a constant and $\delta_j = \max_{\pm} |u_j(\pm l) - \bar{u}(\pm l)|$.

From the properties of $u$ and (1.19) we have

$$|u_j(s) - \bar{u}(s)| \leq |u_j(s) - a_+| + |\bar{u}(s) - a_+| \leq q_0 + Ke^{-kl} \leq 2q_0, \quad \text{for } s \geq l. \tag{A.14}$$

We estimate the differences $J_{(-\infty,-l)}(u_j) - J_{(-\infty,-l)}(\bar{u})$ and $J_{(l,+\infty)}(u_j) - J_{(l,+\infty)}(\bar{u})$. We have with $u_j = \bar{u} + v_j$

$$\begin{aligned}
J_{(l,+\infty)}(u_j) - J_{(l,+\infty)}(\bar{u}) &= \int_l^{+\infty} \Big(\bar{u}' \cdot v_j' + \frac{1}{2}|v_j'|^2 + W(\bar{u} + v_j) - W(\bar{u})\Big) ds \\
&= -\bar{u}'(l) \cdot v_j(l) + \int_l^{+\infty} \Big( - \bar{u}'' \cdot v_j + \frac{1}{2}|v_j'|^2 + W(\bar{u} + v_j) - W(\bar{u})\Big) ds \\
&= -\bar{u}'(l) \cdot v_j(l) + \int_l^{+\infty} \Big(\frac{1}{2}|v_j'|^2 + W(\bar{u} + v_j) - W(\bar{u}) - W_u(\bar{u}) \cdot v_j\Big) ds \\
&\geq -2q_0 Ke^{-kl} + \int_l^{+\infty} \Big(\frac{1}{2}(|v_j'|^2 + W_{uu}(\bar{u})v_j \cdot v_j) \\
&\quad + W(\bar{u} + v_j) - W(\bar{u}) - W_u(\bar{u}) \cdot v_j - \frac{1}{2}W_{uu}(\bar{u})v_j \cdot v_j\Big) ds
\end{aligned} \tag{A.15}$$

Set $I(v_j) = W(\bar{u} + v_j) - W(\bar{u}) - W_u(\bar{u}) \cdot v_j - \frac{1}{2}W_{uu}(\bar{u})v_j \cdot v_j$. Then we have

$$\begin{aligned}
I(v_j) &= \int_0^1 \int_0^1 \int_0^1 \rho^2 \sigma W_{uuu}(\bar{u} + \rho\sigma\tau v_j)(v_j, v_j, v_j) d\tau d\sigma d\rho \\
&= \int_0^1 \int_0^1 \int_0^1 \rho^2 \sigma W_{uuu}(a_+ + (\bar{u} - a_+) + \rho\sigma\tau v_j)(v_j, v_j, v_j) d\tau d\sigma d\rho.
\end{aligned}$$

It follows $|I(v_j)| \leq 2q_0 C_W |v_j|^2$. This and (A.15) imply

$$\begin{aligned}
&J_{(l,+\infty)}(u_j) - J_{(l,+\infty)}(\bar{u}) \\
&\geq -2q_0 Ke^{-kl} + \int_l^\infty \frac{1}{2}(|v_j'|^2 + \gamma^2 |v_j|^2) ds - 2q_0 C_W \int_l^\infty |v_j|^2 ds \\
&\geq -\frac{\gamma^2}{4C_W} Ke^{-kl} + \frac{1}{4}\gamma^2 \int_l^\infty |v_j|^2 ds \\
&\geq -\gamma^2 \frac{p^2}{32} + \frac{1}{4}\gamma^2 \int_l^\infty |v_j|^2 ds,
\end{aligned}$$

where we have used (A.4) and (A.12). From this, the analogous estimate valid in the interval $(-\infty, -l)$, and (A.13) we obtain

$$\begin{aligned}
0 &= \lim_{j \to +\infty} (J_{\mathbb{R}}(\bar{u} + v_j) - c_0) \\
&\geq \lim_{j \to +\infty} \Big( - Cl\delta_j - \gamma^2 \frac{p^2}{16} + \frac{1}{4}\gamma^2 \Big( \int_{-\infty}^{-l} |v_j|^2 ds + \int_l^\infty |v_j|^2 ds \Big)\Big).
\end{aligned} \tag{A.16}$$

Since $v_j$ converges to 0 uniformly in $[-l, l]$, for $j$ large we have

$$\int_{-l}^l |v_j|^2 \leq \frac{p^2}{2}$$

and therefore from (A.11)

$$\int_{-\infty}^{-l} |v_j|^2 ds + \int_l^\infty |v_j|^2 ds \geq \frac{p^2}{2}.$$

This and (A.16) imply

$$
\begin{aligned}
0 &= \lim_{j\to+\infty}(J_{\mathbb{R}}(\bar{u}+v_j)-c_0)\\
&\geq \lim_{j\to+\infty}\Big(-Cl\delta_j-\gamma^2\frac{p^2}{16}+\gamma^2\frac{p^2}{8}\Big)=\gamma^2\frac{p^2}{16}.
\end{aligned}
$$

This contradiction concludes the proof of the proposition when (A.1) is replaced by (A.11). To complete the proof we note that it suffices to consider the case $p \leq 2(2+\sqrt{2})\sqrt{c_0} =: 2p_0$. Indeed (A.2) implies $\|u'\| \leq 2\sqrt{c_0}$ that together with $\|\bar{u}_\pm'\| \leq \sqrt{2c_0}$ yields

$$
\|u' - \bar{u}_\pm'(\cdot - r)\| \leq p_0, \quad r \in \mathbb{R}. \tag{A.17}
$$

It follows that $p > 2p_0$ implies $\|u - \bar{u}_\pm'(\cdot - r)\| > p_0$ and the existence of $e_p$ follows from the first part of the proof.

Set

$$
C_W^0 = \max\{|W_{uu}(\bar{u}_\pm(s)+z)| : s \in \mathbb{R},\ |z| \leq 2p_0\},
$$

and define $\tilde{p} = \tilde{p}(p)$ by

$$
\tilde{p}(p) = \frac{p}{\sqrt{2(1+C_W^0)}}.
$$

We distinguish the following alternatives:

a) $\|u - \bar{u}_\pm(\cdot - r)\| \geq \tilde{p}$, for $r \in \mathbb{R}$,

b) there exists $\bar{r} \in \mathbb{R}$ and $\bar{u} \in \{\bar{u}_-, \bar{u}_+\}$ such that

$$
\|u - \bar{u}(\cdot - \bar{r})\| < \tilde{p}. \tag{A.18}
$$

In case a) the proposition is true from the first part of the proof with $e_p = e_{\tilde{p}}$.

Case b). From (A.1) and (A.18) it follows

$$
\|u' - \bar{u}'(\cdot - \bar{r})\|^2 > p^2 - \tilde{p}^2. \tag{A.19}
$$

For simplicity we write $\bar{u}$ instead of $\bar{u}(\cdot - \bar{r})$ and set $v = u - \bar{u}$. Note that from (A.17), (A.18) and $\tilde{p} \leq p_0$ it follows

$$
|v(s)|^2 \leq 2\int_{-\infty}^{s} |v(s)||v'(s)|ds \leq 2\|v\|\|v'\| \leq 4p_0^2.
$$

We compute

$$
J(u) - c_0 = \frac{1}{2}\|v'\|^2 + \int_{\mathbb{R}}\int_0^1 \Big(W_u(\bar{u}+\tau v) - W_u(\bar{u})\Big)v\,d\tau ds. \tag{A.20}
$$

Since

$$
\Big|\int_0^1 \Big(W_u(\bar{u}+\tau v) - W_u(\bar{u})\Big)v\,d\tau\Big| \leq \frac{1}{2}C_W^0|v|^2, \tag{A.21}
$$

we have from (A.19) and (A.20)

$$
J(u) - c_0 \geq \frac{1}{2}(p^2 - \bar{p}^2) - \frac{1}{2}C_W^0\bar{p}^2 = \frac{1}{4}p^2.
$$

This concludes the first part of the lemma. The last statement is a consequence of the fact that $J_{\mathbb{R}}(u)$ is continuous in $\mathcal{H}^1$ and of (A.20) and (A.21). $\square$

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