Fine properties of the subdifferential for a class of one-homogeneous functionals

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Abstract

We collect here some known results on the subdifferential of one-homogeneous functionals, which are anisotropic and nonhomogeneous variants of the total variation, and establish a new relationship between Lebesgue points of the calibrating field and regular points of the level lines of the corresponding calibrated function.

1 Introduction

In this note we recall some classical results on the structure of the subdifferential of first order one-homogeneous functionals, and we give new regularity results which extend and precise previous work by G. Anzellotti [5, 6, 7].

Given an open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, and a function $u \in C^1(\Omega) \cap BV(\Omega)$, we consider the functional

$$J(u) \ := \ \int_\Omega F(x, Du)$$

where $F : \Omega \times \mathbb{R}^d \to [0, +\infty)$ is continuous in x and $F(x, \cdot)$ is a smooth and uniformly convex norm on $\mathbb{R}^d$, for all $x \in \Omega$.

Since $BV(\Omega) \subset L^{d/(d-1)}(\Omega)$, it is natural to consider J as a convex, l.s.c. function on the whole of $L^{d/(d-1)}(\Omega)$, with value $+\infty$ when $u \notin BV(\Omega)$ (see [2]). In this framework, for any $u \in L^{d/(d-1)}(\Omega)$ we can define the subgradient of a u in the duality $(L^{d/(d-1)}, L^d)$ as

$$\partial J(u) \ = \ \left\{ g \in L^d(\Omega) \, : \, J(v) \geq J(u) \ + \int_\Omega g(x)(v(x) - u(x)) \, dx \ \forall v \in L^{d/(d-1)}(\Omega) \right\}.$$

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The goal of this paper is to investigate the particular structure of the functions u and $g \in \partial J(u)$, when the subgradient is nonempty. Since J can be defined by duality as

$$J(u) \;=\; \sup\left\{-\int_\Omega u(x)\mathrm{div}\, z(x)\,dx \;:\; z\in C_c^\infty(\Omega;\mathbb{R}^d)\,,\; F^*(x,z(x))=0\;\forall x\in\Omega\right\}$$

where $F^*(x,\cdot)$ is the Legendre-Fenchel transform of $F(x,\cdot)$ (it is equivalent to require that $F^\circ(x,z(x))\le 1$, $F^\circ(x,\cdot)$ being the convex polar of F defined in (4)), it is easy to see that such a g has necessarily the form $g=-\mathrm{div}\, z$, for some field $z\in L^\infty(\Omega;\mathbb{R}^d)$ with $F^*(x,z(x))=0$ a.e. in Ω.

Since by a formal integration by parts one gets $z\cdot Du = F(x,Du)$, $|Du|$-a.e., natural questions are: in what sense can this relation be true? can one assign a precise value to z on the support of the measure Du?

The first question has been answered by Anzelotti in the series of papers [5, 6, 7]. However, for the particular vector fields we are interested in, we can be more precise and obtain pointwise properties of z on the level sets of the function u. Indeed, we shall show that z has a pointwise meaning on all level sets of u, up to $\mathscr{H}^{d-1}$-negligible sets (which is much more than $|Du|$-a.e., as illustrated by the function $u=\sum_{n=1}^{+\infty}2^{-n}\chi_{(0,x_n)}$, defined in the interval $(0,1)$, with (x_n) a dense sequence in that interval).

We will therefore focus on the properties of the vector fields $z\in L^\infty(\Omega,\mathbb{R}^d)$ such that $F^*(x,z(x))=0$ a.e. in Ω and $g=-\mathrm{div}\, z\in L^d(\Omega)$, and such that there exists a function u such that for any $\phi\in C_c^\infty(\Omega)$,

$$-\int_\Omega \mathrm{div}\, z(x)u(x)\phi(x)\,dx \;=\; \int_\Omega u(x)\,z(x)\cdot\nabla\phi(x)\,dx + \int_\Omega \phi(x)F(x,Du)\,.$$

In particular, one checks easily that u minimizes the functional

$$\int_\Omega F(x,Du) - \int_\Omega g(x)u(x)\,dx \tag{1}$$

among perturbations with compact support in Ω. Conversely, given $g\in L^d(\Omega)$ with $\|g\|_{L^d}$ sufficiently small, there exist functions u which minimize (1) under various types of boundary conditions, and corresponding fields z.

This kind of functionals appears in many contexts including image processing and plasticity [4, 17]. Notice also that, by the Coarea Formula [2], it holds

$$\int_\Omega F(x,Du) - \int_\Omega gu\,dx = \int_{\mathbb{R}}\left(\int_{\partial^*\{u>s\}} F(x,\nu) - \int_{\{u>s\}} g\,dx\right)ds\,,$$

where ν is the unit normal to $\{u>s\}$, and one can show (see for instance [10]) that the characteristic function of any level set of the form $\{u>s\}$ or $\{u\ge s\}$ is a minimizer of the geometric functional

$$\int_{\partial^* E} F(x,\nu) - \int_E g(x)\,dx\,. \tag{2}$$

The canonical example of such functionals is given by the total variation, corresponding to $F(x, Du) = |Du|$. In this case, (2) boils down to

$$P(E) - \int_E g(x)\, dx. \tag{3}$$

In [8], it is shown that every set with finite perimeter in Ω is a minimizer of (3) for some $g \in L^1(\Omega)$. However, if $g \in L^p(\Omega)$ with $p > d$, and E is a minimizer of (2), then ∂E is locally $C^{1,\alpha}$ for some $\alpha > 0$, out of a closed singular set of zero $\mathscr{H}^{d-3}$-measure [1]. When $g \in L^d(\Omega)$, the boundary ∂E is only of class C^α out of the singular set (see [3]). Since the Euler-Lagrange equation of (2) relates z to the normal to E, understanding the regularity of z is closely related to understanding the regularity of ∂E.

Our main result is that the Lebesgue points of z correspond to regular points of $\partial\{u > s\}$ or $\partial\{u \geq s\}$ (Theorem 3.7), and that the converse is true in dimension $d \leq 3$ (Theorem 3.8).

2 Preliminaries

2.1 BV functions

We briefly recall the definition of function of bounded variation and set of finite perimeter. For a complete presentation we refer to [2].

Definition 2.1. *Let Ω be an open set of $\mathbb{R}^d$, we say that a function $u \in L^1(\Omega)$ is a function of bounded variation if*

$$\int_\Omega |Du| := \sup_{\substack{z \in C_c^1(\Omega) \\ |z|_\infty \leq 1}} \int_\Omega u \operatorname{div} z \; dx < +\infty.$$

We denote by $BV(\Omega)$ the set of functions of bounded variation in Ω (when $\Omega = \mathbb{R}^d$ we simply write BV instead of $BV(\mathbb{R}^d)$).
We say that a set $E \subset \mathbb{R}^d$ is of finite perimeter if its characteristic function χ_E is of bounded variation and denote its perimeter in an open set Ω by $P(E, \Omega) := \int_\Omega |D\chi_E|$, and write simply $P(E)$ when $\Omega = \mathbb{R}^d$.

Definition 2.2. *Let E be a set of finite perimeter and let $t \in [0; 1]$. We define*

$$E^{(t)} := \left\{ x \in \mathbb{R}^d \, : \, \lim_{r \downarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\}.$$

We denote by $\partial E := \left(E^{(0)} \cup E^{(1)}\right)^c$ the measure theoretical boundary of E. We define the reduced boundary of E by:

$$\partial^* E := \left\{ x \in Spt(|D\chi_E|) \, : \, \nu^E(x) := \lim_{r \downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \text{ exists and } |\nu^E(x)| = 1 \right\} \subset E^{(\frac{1}{2})}.$$

The vector $\nu^E(x)$ is the measure theoretical inward normal to the set E.

Proposition 2.3. *If E is a set of finite perimeter then $D\chi_E = \nu^E \ \mathscr{H}^{d-1} \llcorner \partial^* E$, $P(E) = \mathscr{H}^{d-1}(\partial^* E)$ and $\mathscr{H}^{d-1}(\partial E \setminus \partial^* E) = 0$.*

Definition 2.4. *We say that x is an approximate jump point of $u \in BV(\Omega)$ if there exist $\xi \in \mathbb{S}^{d-1}$ and distinct $a, b \in \mathbb{R}$ such that*

$$\lim_{\rho \to 0} \frac{1}{|B_\rho^+(x,\xi)|} \int_{B_\rho^+(x,\xi)} |u(y) - a| \, dy = 0 \qquad \text{and} \qquad \lim_{\rho \to 0} \frac{1}{|B_\rho^-(x,\xi)|} \int_{B_\rho^-(x,\xi)} |u(y) - b| \, dy = 0,$$

where $B_\rho^\pm(x,\xi) := \{y \in B_\rho(x) : \pm(y-x)\cdot\xi > 0\}$. Up to a permutation of a and b and a change of sign of ξ, this characterize the triplet (a, b, ξ) which is then denoted by (u^+, u^-, ν_u). The set of approximate jump points is denoted by J_u.

The following proposition can be found in [2, Proposition 3.92].

Proposition 2.5. *Let $u \in BV(\Omega)$. Then, defining*

$$\Theta_u := \{x \in \Omega \, / \, \liminf_{\rho \to 0} \rho^{1-d} |Du|(B_\rho(x)) > 0\},$$

there holds $J_u \subset \Theta_u$ and $\mathscr{H}^{d-1}(\Theta_u \backslash J_u) = 0$.

2.2 Anisotropies

Let $F(x,p) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a convex one-homogeneous function in the second variable such that there exists c_0 with

$$c_0|p| \le F(x,p) \le \frac{1}{c_0}|p| \qquad \forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d.$$

We say that F is uniformly elliptic if for some $\delta > 0$, the function $p \mapsto F(p) - \delta|p|$ is still a convex function. We define the polar function of F by

$$F^\circ(x,z) := \sup_{\{F(x,p) \le 1\}} z \cdot p \tag{4}$$

so that $(F^\circ)^\circ = F$. It is easy to check that (* denoting the Legendre-Fenchel convex conjugate) $[F(x,\cdot)^2/2]^* = F^\circ(x,\cdot)^2/2$, in particular (if differentiable), $F(x,\cdot)\nabla_p F(x,\cdot)$ and $F^\circ(x,\cdot)\nabla_z F^\circ(x,\cdot)$ are inverse monotone operators. If we denote by F^* the convex conjugate of F with respect to the second variable, then $F^*(x,z) = 0$ if and only if $F^\circ(x,z) \le 1$.
If $F(x,\cdot)$ is differentiable then, for every $p \in \mathbb{R}^d$,

$$F(x,p) = p \cdot \nabla_p F(x,p) \qquad \text{(Euler's identity)}$$

and

$$z \in \{F^\circ(x,\cdot) \le 1\} \text{ with } p \cdot z = F(x,p) \iff z = \nabla_p F(x,p).$$

If F is elliptic and of class $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\})$, then F° is also elliptic and $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\})$. We will then say that F is a smooth elliptic anisotropy. Observe that, in this case, the function $F^2/2$ is also uniformly δ^2-convex (this follows from the inequalities $D^2F(x,p) \geq \delta/|p|(I - p \otimes p/|p|^2)$ and $F(x,p) \geq \delta|p|$). In particular, for every $x, y, z \in \mathbb{R}^d$, there holds

$$F^2(x,y) - F^2(x,z) \geq 2\left(F(x,z)\nabla_p F(x,z)\right) \cdot (y-z) + \delta^2|y-z|^2, \tag{5}$$

and a similar inequality holds for F°. We refer to [16] for general results on convex norms and convex bodies.

2.3 Pairings between measures and bounded functions

Following [5] we define a generalized trace $[z, Du]$ for functions u with bounded variation and bounded vector fields z with divergence in L^d.

Definition 2.6. *Let Ω be an open set with Lipschitz boundary, $u \in BV(\Omega)$ and $z \in L^\infty(\Omega, \mathbb{R}^d)$ with $\operatorname{div} z \in L^d(\Omega)$. We define the distribution $[z, Du]$ by*

$$\langle [z, Du], \psi \rangle = -\int_\Omega u\,\psi \operatorname{div} z - \int_\Omega u\, z \cdot \nabla\psi \qquad \forall \psi \in \mathcal{C}_c^\infty(\Omega).$$

Proposition 2.7. *The distribution $[z, Du]$ is a bounded Radon measure on Ω and if ν is the inward unit normal to Ω, there exists a function $[z, \nu] \in L^\infty(\partial\Omega)$ such that the generalized Green's formula holds,*

$$\int_\Omega [z, Du] = -\int_\Omega u \operatorname{div} z - \int_{\partial\Omega} [z, \nu] u \, d\mathscr{H}^{d-1}.$$

The function $[z, \nu]$ is the generalized (inward) normal trace of z on $\partial\Omega$.

Given $z \in L^\infty(\Omega, \mathbb{R}^d)$, with $\operatorname{div} z \in L^d(\Omega)$, we can also define the generalized trace of z on ∂E, where E is a set of locally finite perimeter. Indeed, for every bounded open set Ω with Lipschitz boundary, we can define as above the measure $[z, D\chi_E]$ on Ω. Since this measure is absolutely continuous with respect to $|D\chi_E| = \mathscr{H}^{d-1} \llcorner \partial^* E$ we have

$$[z, D\chi_E] = \psi_z(x) \mathscr{H}^{d-1} \llcorner \partial^* E$$

with $\psi_z \in L^\infty(\partial^* E)$ independent of Ω. We denote by $[z, \nu^E] := \psi_z$ the generalized (inward) normal trace of z on ∂E. If E is a bounded set of finite perimeter, by taking Ω strictly containing E, we have the generalized Gauss-Green Formula

$$\int_E \operatorname{div} z = -\int_{\partial^* E} [z, \nu^E] d\mathscr{H}^{d-1}.$$

Anzellotti proved the following alternative definition of $[z, \nu^E]$ [6, 7]

Proposition 2.8. *Let* $(x, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$. *For any* $r > 0$, $\rho > 0$ *we let*

$$C_{r,\rho}(x, \alpha) := \{\xi \in \mathbb{R}^d \,:\, |(\xi - x) \cdot \alpha| < r, |(\xi - x) - [(\xi - x) \cdot \alpha]\alpha| < \rho\}.$$

There holds

$$[z, \alpha](x) = \lim_{\rho \to 0} \lim_{r \to 0} \frac{1}{2r\omega_{d-1}\rho^{d-1}} \int_{C_{r,\rho}(x,\alpha)} z \cdot \alpha$$

where ω_{d-1} *is the volume of the unit ball in* $\mathbb{R}^{d-1}$.

3 The subdifferential of anisotropic total variations

3.1 Characterization of the subdifferential

The following characterization of the subdifferential of J is classical and readily follows for example from the representation formula [9, (4.19)].

Proposition 3.1. *Let* F *be a smooth elliptic anisotropy and* $g \in L^d(\Omega)$ *then* u *is a local minimizer of* (1) *if and only if there exists* $z \in L^\infty(\Omega)$ *with* $\operatorname{div} z = g$, $F^*(x, z(x)) = 0$ *a.e. and*

$$[z, Du] = F(x, Du).$$

Moreover, for every $t \in \mathbb{R}$, *for the set* $E = \{u > t\}$ *there holds* $[z, \nu^E] = F(x, \nu^E)$ $\mathscr{H}^{d-1}$*-a.e. on* ∂E. *We will say that such a vector field is a calibration of the set* E *for the minimum problem* (2).

Remark 3.2. In [5], it is proven that if $z_\rho(x) := \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} z(y)\, dy$, then $z_\rho \cdot \nu^E$ weakly* converges to $[z, \nu^E]$ in $L^\infty_{loc}(\mathscr{H}^{d-1} \llcorner \partial^* E)$. Using (5) it is then possible to prove that if z calibrates E then z_ρ converges to $\nabla_p F(x, \nu^E)$ in $L^2(\mathscr{H}^{d-1} \llcorner \partial^* E)$ yielding that up to a subsequence, $z_{\phi(\rho)}$ converges $\mathscr{H}^{d-1}$-a.e. to $\nabla_p F(x, \nu^E)$. Unfortunately this is still a very weak statement since it is a priori impossible to recover from this the convergence of the full sequence z_ρ.

The main question we want to investigate now is whether we can give a classical meaning to $[z, \nu^E]$ (that is understand if $[z, \nu^E] = z \cdot \nu^E$). We observe that a priori the value of z is not well defined on ∂E which has zero Lebesgue measure (since z has Lebesgue points only a.e.). We let $S := \operatorname{supp}(Du) \subset \Omega$ be the smallest closed set in Ω such that $|Du|(\Omega \setminus S) = 0$. The next result is classical.

Lemma 3.3 (Density estimate)**.** *There exists* $\rho_0 > 0$ *(depending on* g*) and a constant* $\gamma > 0$ *(which depends only on* d*), such that for any* $B_\rho(x) \subset \Omega$ *with* $\rho \le \rho_0$, *and any level set* E *of* u *(that is,* $E \in \{\{u > s\}, \{u \ge s\}, \{u < s\}, \{u \le s\}, s \in \mathbb{R}\}$*), if* $|B_\rho(x) \cap E| < \gamma |B_\rho(x)|$ *then* $|B_{\rho/2}(x) \cap E| = 0$. *As a consequence,* E^0 *and* E^1 *are open,* ∂E *is the topological boundary of* E^1, *and (possibly changing slightly* γ*) if* $x \in \partial E$, *then* $\mathscr{H}^{d-1}(\partial E \cap B_\rho(x)) \ge \gamma \rho^{d-1}$.

For a proof we refer to [13, 12]. This is not true anymore if $g \notin L^d(\Omega)$ [12]. If $\partial\Omega$ is Lipschitz, it is true up to the boundary.

Corollary 3.4. *It follows that* $u \in L^\infty_{loc}(\Omega)$ *and* $u \in C(\Omega \backslash \Theta_u)$.

Proof. For any ball $B_\rho(x) \subset \Omega$ and $\inf_{B_{\rho/2}(x)} u < a < b < \sup_{B_{\rho/2}(x)} u$, one has

$$+\infty > |Du|(B_\rho(x)) \geq \int_a^b P(\{u > s\}, B_\rho(x))\, ds \geq (b-a)\gamma \left(\frac{\rho}{2}\right)^{d-1},$$

so that $osc_{B_{\rho/2}(x)}(u)$ must be bounded and thus $u \in L^\infty_{loc}(\Omega)$. Moreover, if $x \in \Omega\backslash\Theta_u$ we find that $\lim_{\rho\to 0} osc_{B_\rho(x)}(u) = 0$ so that u is continuous at the point x. □

It also follows from Lemma 3.3 that all points in the support of Du must be on the boundary of a level set of u:

Proposition 3.5. *For any* $x \in S$*, there exists* $s \in \mathbb{R}$ *such that either* $x \in \partial\{u > s\}$ *or* $x \in \partial\{u \geq s\}$.

Proof. First, if $x \notin S$ then $|Du|(B_\rho(x)) = 0$ for some $\rho > 0$ and clearly x cannot be on the boundary of a level set of u. On the other hand, if $x \in S$, then for any ball $B_{1/n}(x)$ (n large) there is a level s_n (uniformly bounded) with $\partial\{u > s_n\} \cap B_{1/n}(x) \neq \emptyset$ and by Hausdorff convergence, we deduce that either $x \in \partial\{u > s\}$ or $x \in \partial\{u \geq s\}$ where s is the limit of the sequence $(s_n)_n$ (which must actually converge). □

The following stability property is classical (see e.g. [11]).

Proposition 3.6. *Let* E_n *be local minimizers of* (2)*, with a function* $g = g_n \in L^d(\Omega)$*, and converging in the* L^1*-topology to a set* E*. Assume that the sets* E_n *are calibrated by* z_n*, that* $z_n \overset{*}{\rightharpoonup} z$ *weakly-∗ in* L^∞ *and* $g_n \to g = -\operatorname{div} z \in L^d(\Omega)$*, in* $L^1(\Omega)$ *as* $n \to \infty$*. Then* z *calibrates* E*, which is thus also a minimizer of* (2).

In particular, one must notice that when $z_n \overset{*}{\rightharpoonup} z$ and $F^\circ(x, z_n) \leq 1$ a.e., then in the limit one still has $F^\circ(x, z) \leq 1$ a.e. (thanks to the convexity, and continuity w.r. the variable x).

3.2 The Lebesgue points of the calibration.

The next result shows that the regularity of the calibration z implies some regularity of the calibrated set.

Theorem 3.7. *Let* $\bar{x} \in \partial E$ *be a Lebesgue point of* z*, with* $E = \{u > t\}$ *or* $E = \{u \geq t\}$*. Then,* $\bar{x} \in \partial^* E$ *and*

$$z(\bar{x}) = \nabla_p F(\bar{x}, \nu^E(\bar{x})). \tag{6}$$

Proof. We follow [11, Th. 4.5] and let $z_\rho(y) := z(\bar{x}+\rho y)$. Since $\bar{x}$ is a Lebesgue point of z, we have that $z_\rho \to \bar{z}$ in $L^1(B_R)$, hence also weakly-$*$ in $L^\infty(B_R)$ for any $R>0$, where $\bar{z}\in\mathbb{R}^d$ is a constant vector.

We let $E_\rho := (E-\bar{x})/\rho$ and $g_\rho(y) = g(\bar{x}+\rho y)$ (so that $\operatorname{div} z_\rho = \rho g_\rho$). Observe that E_ρ minimizes

$$\int_{\partial^* E_\rho \cap B_R} F(\bar{x}+\rho y, \nu^{E_\rho}(y))\, d\mathscr{H}^{d-1}(y) + \rho \int_{E_\rho\cap B_R} g_\rho(y)\, dy\,,$$

with respect to compactly supported perturbations of the set (in the fixed ball B_R). Also,

$$\|\rho g_\rho\|_{L^d(B_R)} \;=\; \|g\|_{L^d(B_{\rho R})} \;\overset{\rho\to 0}{\longrightarrow}\; 0\,.$$

By Lemma 3.3, the sets E_ρ (and the boundaries ∂E_ρ) satisfy uniform density bounds, and hence are compact with respect to both local L^1 and Hausdorff convergence.

Hence, up to extracting a subsequence, we can assume that $E_\rho \to \bar{E}$, with $0\in\partial\bar{E}$. Proposition 3.6 shows that $\bar{z}$ is a calibration for the energy $\int_{\partial\bar{E}\cap B_R} F(\bar{x},\nu^{\bar{E}}(y))\, d\mathscr{H}^{d-1}(y)$, and that $\bar{E}$ is a minimizer calibrated by $\bar{z}$.

It follows that $[\bar{z},\nu^{\bar{E}}] = F(\bar{x},\nu^{\bar{E}}(y))$ for $\mathscr{H}^{d-1}$-a.e. y in $\partial\bar{E}$, but since $\bar{z}$ is a constant, we deduce that $\bar{E} = \{y\cdot\bar{\nu}\ge 0\}$ with $\bar{\nu}/F(\bar{x},\bar{\nu}) = \nabla_p F^\circ(\bar{x},\bar{z})$[1]. In particular the limit $\bar{E}$ is unique, hence we obtain the global convergence of $E_\rho\to\bar{E}$, without passing to a subsequence.

We want to deduce that $\bar{x}\in\partial^* E$, with $\nu^E(\bar{x}) = F(\bar{x},\nu^E(\bar{x}))\nabla_p F^\circ(\bar{x},\bar{z})$, which is equivalent to (6). The last identity is obvious from the arguments above, so that we only need to show that

$$\lim_{\rho\to 0}\frac{D\chi_{E_\rho}(B_1)}{|D\chi_{E_\rho}|(B_1)} \;=\; \bar{\nu}\,. \tag{7}$$

Assume we can show that

$$\lim_{\rho\to 0}|D\chi_{E_\rho}|(B_R) \;=\; |D\chi_{\bar{E}}|(B_R)\;\big(\,=\;\omega_{d-1}R^{d-1}\big) \tag{8}$$

for any $R>0$, then for any $\psi\in C_c^\infty(B_R;\mathbb{R}^d)$ we would get

$$\begin{aligned}\frac{1}{|D\chi_{E_\rho}|(B_R)}\int_{B_R}\psi\cdot D\chi_{E_\rho} \;&=\; -\frac{1}{|D\chi_{E_\rho}|(B_R)}\int_{B_R\cap E_\rho}\operatorname{div}\psi(x)\,dx\\ &\longrightarrow\; -\frac{1}{|D\chi_{\bar{E}}|(B_R)}\int_{B_R\cap\bar{E}}\operatorname{div}\psi(x)\,dx \;=\; \frac{1}{|D\chi_{\bar{E}}|(B_R)}\int_{B_R}\psi\cdot D\chi_{\bar{E}}\end{aligned}$$

and deduce that the measure $D\chi_{E_\rho}/(|D\chi_{E_\rho}|(B_R))$ weakly-$*$ converges to $D\chi_{\bar{E}}/(|D\chi_{\bar{E}}|(B_R))$. Using again (8)), we then obtain that

$$\lim_{\rho\to 0}\frac{D\chi_{E_\rho}(B_R)}{|D\chi_{E_\rho}|(B_R)} \;=\; \bar{\nu} \tag{9}$$

[1]We use here that $F(\bar{x},\cdot)\nabla F(\bar{x},\cdot) = [F^\circ(\bar{x},\cdot)\nabla F^\circ(\bar{x},\cdot)]^{-1}$, so that $\bar{z} = \nabla F(\bar{x},\nu^{\bar{E}}(y))$ implies both $F^\circ(\bar{x},\bar{z}) = 1$ and $\nu^{\bar{E}}(y)/F(\bar{x},\nu^{\bar{E}})(y) = \nabla F^\circ(\bar{x},\bar{z})$

for almost every $R > 0$. Since $D\chi_{E_\rho}(B_{\mu R})/(|D\chi_{E_\rho}|(B_{\mu R})) = D\chi_{E_{\rho/\mu}}(B_R)/(|D\chi_{E_{\rho/\mu}}|(B_R))$ for any $\mu > 0$, (9) holds in fact for any $R > 0$ and (7) follows, so that $\bar{x} \in \partial^* E$.
It remains to show (8). First, we observe that, by minimality of E_ρ and $\bar{E}$ plus the Hausdorff convergence of ∂E_ρ in balls, we can easily show the convergence of the energies

$$\lim_{\rho\to 0} \int_{\partial E_\rho \cap B_R} F(\bar{x} + \rho y, \nu^{E_\rho}(y))\, d\mathscr{H}^{d-1}(y) + \rho \int_{E_\rho \cap B_R} g_\rho(y)\, dy = \int_{\partial \bar{E} \cap B_R} F(\bar{x}, \nu^{\bar{E}}(y))\, d\mathscr{H}^{d-1}(y)$$

and, by the continuity of F,

$$\lim_{\rho\to 0} \int_{\partial E_\rho \cap B_R} F(\bar{x}, \nu^{E_\rho}(y))\, d\mathscr{H}^{d-1}(y) \;=\; \int_{\partial \bar{E} \cap B_R} F(\bar{x}, \nu^{\bar{E}}(y))\, d\mathscr{H}^{d-1}(y)\,. \tag{10}$$

Then, (7) follows from Reshetnyak's continuity theorem where, instead of using the Euclidean norm as reference norm, we use the uniformly convex norm $F(\bar{x}, \cdot)$ and the convergence of the measures $F(\bar{x}, D\chi_{E_\rho})$ to $F(\bar{x}, D\chi_{\bar{E}})$ (see [15, 11]).

□

In dimension 2 and 3 we can also show the reverse implication, proving that regular points of the boundary corresponds to Lebesgue points of the calibration. The idea is to show that the parameters r, ρ in Proposition 2.8 can be taken of the same order.

Theorem 3.8. *Assume the dimension is $d = 2$ or $d = 3$. Let x, s be as in Proposition 3.5, E be a minimizer of* (2) *and assume $x \in \partial^* E$. Then x is a Lebesgue point of z and*

$$z(x) = \nabla_p F(x, \nu^E)\,.$$

Proof. We divide the proof into two steps.
Step 1. We first consider anisotropies F which are not depending on the x variable. Without loss of generality we assume $x = 0$. By assumption, there exists the limit

$$\overline{\nu} \;=\; \lim_{\rho\to 0} \frac{D\chi_E(B_\rho(0))}{|D\chi_E|(B_\rho(0))|} \tag{11}$$

and, without loss of generality, we assume that it coincides with the vector e_d corresponding to the last coordinate of $y \in \mathbb{R}^d$.
Also, if we let $E_\rho = E/\rho$, the sets E_ρ, E_ρ^c, ∂E_ρ converge in $B_1(0)$, in the Hausdorff sense (thanks to the uniform density estimates), respectively to $\{y_d \geq 0\}$, $\{y_d = 0\}$, $\{y_d \leq 0\}$.
We also let $z_\rho(y) = z(\rho y)$ and $g_\rho(y) = g(\rho y)$, in particular $-\mathrm{div}\, z_\rho = \rho g_\rho$. We let

$$\omega(\rho) \;=\; \sup_{x\in\Omega} \|g\|_{L^d(B_\rho(x)\cap\Omega)} \tag{12}$$

which is continuously increasing and goes to 0 as $\rho \to 0$, since $|g|^d$ is equi-integrable.
We introduce the following notation: a point in $\mathbb{R}^d$ is denoted by $y = (y', y_d)$, with $y' \in \mathbb{R}^{d-1}$. We let $D_s := \{|y'| \le s\}$, $\bar z := \nabla F(\overline{\nu})$ and $D_s^t = \{D_s + \lambda \bar z : |\lambda| \le t\}$ and denote with ∂D_s the relative boundary of D_s in $\{y_d = 0\}$.
We choose $s \le 1$, $0 < t \le s$, (t is chosen small enough so that $D_s^t \subset B_1(0)$, that is $t < (1/|\bar z|)\sqrt{1-s^2}$). We integrate in D_s^t the divergence $\rho g_\rho = -\mathrm{div}\, z_\rho = \mathrm{div}\,(\bar z - z_\rho)$ against the function $(2\chi_E - 1)t - \frac{\overline{\nu}\cdot y}{F(\overline{\nu})}$, which vanishes for $y_d = \pm t F(\overline{\nu})$ if ρ is small enough (given $t > 0$), so that $\partial E_\rho \cap B_1(0) \subset \{|y_d| \le tF(\overline{\nu})\}$. For y on the lateral boundary of the cylinder D_s^t, let $\xi(y)$ be the internal normal to $\partial D_s + (-t,t)\bar z$ at the point y. Using the fact that z_ρ is a calibration for E_ρ, we easily get that for almost all s,

$$\begin{aligned}
\int_{D_s^t} \rho g_\rho \left((2\chi_E - 1)t - \frac{\overline{\nu}\cdot y}{F(\overline{\nu})} \right) dy \\
= \int_{\partial D_s + (-t,t)\bar z} \left((2\chi_E - 1)t - \frac{\overline{\nu}\cdot y}{F(\overline{\nu})} \right) [(\bar z - z_\rho), \xi(y)]\, d\mathscr{H}^{d-1} \\
- 2t \int_{\partial E_\rho \cap D_s^t} \left(\bar z \cdot \nu^{E_\rho} - F(\nu^{E_\rho}) \right) d\mathscr{H}^{d-1} + \int_{D_s^t} \left(1 - \frac{z_\rho \cdot \overline{\nu}}{F(\overline{\nu})} \right) dy\,. \quad (13)
\end{aligned}$$

Now since $F^\circ(\nabla F(\overline{\nu})) = 1$, there holds $\bar z \cdot \nu^{E_\rho} - F(\nu^{E_\rho}) \le 0$ and using that $\bar z \cdot \xi(y) = 0$ on $\partial D_s + (-t,t)\bar z$, we get

$$\begin{aligned}
\int_{D_s^t} \left(1 - \frac{z_\rho \cdot \overline{\nu}}{F(\overline{\nu})} \right) dy \le \int_{D_s^t} \rho g_\rho \left((2\chi_E - 1)t - \frac{\overline{\nu}\cdot y}{F(\overline{\nu})} \right) dy \\
\int_{\partial D_s + (-t,t)\bar z} \left((2\chi_E - 1)t - \frac{\overline{\nu}\cdot y}{F(\overline{\nu})} \right) z_\rho \cdot \xi(y)\, d\mathscr{H}^{d-1}\,. \quad (14)
\end{aligned}$$

We claim that for $|\xi| \le 1$ with $\xi \cdot \bar z = 0$, there holds

$$(\xi \cdot z_\rho)^2 \le C(F(\overline{\nu}) - \overline{\nu} \cdot z_\rho) \quad (15)$$

Since

$$(\xi \cdot z_\rho)^2 \le |z_\rho|^2 - [z_\rho \cdot (\bar z/|\bar z|)]^2$$

it is enough to prove

$$|z_\rho|^2 - [z_\rho \cdot (\bar z/|\bar z|)]^2 \le C(F(\overline{\nu}) - \overline{\nu} \cdot z_\rho).$$

Using that $\overline{\nu}/F(\overline{\nu}) = \nabla F^\circ(\bar z)$, from (5) applied to F° together with $F^\circ(\bar z) = 1 \ge F^\circ(z_\rho)$, we find

$$(F(\overline{\nu}) - \overline{\nu} \cdot z_\rho) = F(\overline{\nu})(1 - z_\rho \cdot \nabla F^\circ(\bar z)) \ge C|z_\rho - \bar z|^2.$$

which readily implies (15). We thus have

$$\int_{\partial D_s+(-t,t)\bar z}\left((2\chi_{E_\rho}-1)t-\frac{\overline{\nu}\cdot y}{F(\overline{\nu})}\right)(z_\rho\cdot\xi)\,d\mathscr{H}^{d-1}$$
$$\le\ 2C\sqrt{F(\overline{\nu})}t\int_{\partial D_s+(-t,t)\bar z}\sqrt{1-\frac{z_\rho\cdot\overline{\nu}}{F(\overline{\nu})}}\,d\mathscr{H}^{d-1}$$
$$\le\ 2CF(\overline{\nu})t\sqrt{t}\left(\int_{\partial D_s+(-t,t)\bar z}\left(1-\frac{z_\rho\cdot\overline{\nu}}{F(\overline{\nu})}\right)d\mathscr{H}^{d-1}\right)^{\frac12}\sqrt{\mathscr{H}^{d-2}(\partial D_s)}\,. \quad (16)$$

Now, we also have

$$\rho\int_{D_s^t}\left((2\chi_{E_\rho}-1)t-\frac{\overline{\nu}\cdot y}{F(\overline{\nu})}\right)g_\rho(y)\,dy\ \le\ 2t\rho^{1-d}\int_{D_{\rho s}^{\rho t}}g(x)\,dx$$
$$\le\ 2t\rho^{1-d}\|g\|_{L^d(B_{\rho s}(0))}|D_{\rho s}^{\rho t}|^{1-1/d}\ \le\ ct^{2-1/d}s^{d-2+1/d}\omega(\rho s) \quad (17)$$

where here, $c=2\mathscr{H}^{d-1}(D_1)^{1-1/d}$, and ω is defined in (12).
We choose $a<1$, close to 1, and choose $t\in(0,(1/|\bar z|)\sqrt{1-a^2})$. If $\rho>0$ is small enough (so that $\partial E_\rho\cap B_1$ is in $\{|y_d|\le tF(\overline{\nu})\}$), letting $f(s):=\int_{D_s^t}\left(1-\frac{z_\rho\cdot\overline{\nu}}{F(\overline{\nu})}\right)dy$, we deduce from (14), (16) and (17) that for a.e. s with $t\le s\le a$, one has (possibly increasing the constant c)

$$f(s)^2\ \le\ c\left(s^{d-2}t^3f'(s)\ +\ t^{4-2/d}s^{2d-4+2/d}\omega(\rho s)^2\right). \quad (18)$$

Unfortunately, this estimate does not give much information for $d>3$. It seems it allows to conclude only whenever $d\in\{2,3\}$. Since the case $d=2$ is simpler, we focus on $d=3$. Estimate (18) becomes

$$f(s)^2\ \le\ c\left(st^3f'(s)\ +\ t^{10/3}s^{8/3}\omega(\rho s)^2\right). \quad (19)$$

Given $M>0$, we fix a value $t>0$ such that $\log(a/t)\ge cM$. If ρ is chosen small enough, then $\partial E_\rho\cap B_1(0)\subset\{|y_d|<tF(\overline{\nu})\}$, and (19) holds. It yields (assuming $f(t)>0$, but if not, then the Proposition is proved)

$$-\frac{f'(s)}{f(s)^2}\ +\ \frac{1}{ct^3}\frac{1}{s}\ \le\ ct^{1/3}s^{5/3}\frac{\omega(\rho s)^2}{f(s)^2}\ \le\ ct^{1/3}s^{5/3}\frac{\omega(a\rho)^2}{f(t)^2} \quad (20)$$

where we have used the fact that $t\le s\le a$ and f,ω are nondecreasing. Integrating (20) from t to a, after multiplication by t^3 we obtain

$$\frac{t^3}{f(a)}-\frac{t^3}{f(t)}+\frac{\log(a/t)}{c}\ \le\ \frac{3c}{8}t^{10/3}(a^{8/3}-t^{8/3})\frac{\omega(a\rho)^2}{f(t)^2}\,.$$

Hence we get

$$\frac{t^3}{f(t)}+ca^{8/3}t^{-8/3}\omega(a\rho)^2\frac{t^6}{f(t)^2}\ \ge\ M. \quad (21)$$

Eventually, we observe that

$$f(t) = \int_{D_t^t} \left(1 - \frac{z(\rho y)\cdot \overline{\nu}}{F(\overline{\nu})}\right) dy = \frac{1}{\rho^d}\int_{D_{\rho t}^{\rho t}} \left(1 - \frac{z(x)\cdot \overline{\nu}}{F(\overline{\nu})}\right) dx\,,$$

so that (21) can be rewritten

$$\left(\frac{\int_{D_{\rho t}^{\rho t}} \left(1 - \frac{z(x)\cdot\overline{\nu}}{F(\overline{\nu})}\right) dx}{(\rho t)^3}\right)^{-1} \;\geq\; \frac{-1+\sqrt{1+4Mca^{8/3}t^{-8/3}\omega(a\rho)^2}}{2ca^{8/3}t^{-8/3}\omega(a\rho)^2} \tag{22}$$

The value of t being fixed, we can choose the value of ρ small enough in order to have $4Mca^{8/3}t^{-8/3}\omega(a\rho)^2 < 1$, and (using $\sqrt{1+X} \geq 1 + X/2 - X^2/8$ if $X \in (0,1)$), (22) yields

$$\left(\frac{\int_{D_{\rho t}^{\rho t}} \left(1 - \frac{z(x)\cdot\overline{\nu}}{F(\overline{\nu})}\right) dx}{(\rho t)^3}\right)^{-1} \;\geq\; M - M^2ca^{8/3}t^{-8/3}\omega(a\rho)^2 \;\geq\; \frac{3}{4}M\,. \tag{23}$$

It follows that

$$\limsup_{\varepsilon\to 0} \frac{\int_{D_\varepsilon^\varepsilon} \left(1 - \frac{z(x)\cdot\overline{\nu}}{F(\overline{\nu})}\right) dx}{\varepsilon^3} \;\leq\; \frac{4}{3}M^{-1} \tag{24}$$

and since M is arbitrary, 0 is indeed a Lebesgue point of z, with value $\bar z = \nabla F(\overline{\nu})$ (recall that $1 - \frac{z(x)\cdot\overline{\nu}}{F(\overline{\nu})} \geq (C/F(\overline{\nu}))|z(x) - \bar z|^2$).

Step 2. When F depends also on the x variable, the proof follows along the same lines as in *Step 1*, taking into account the errors terms in (14) and (16). Keeping the same notations as in *Step 1* and setting $\bar z := \nabla_p F(0,\bar\nu)$ we find that since $F^\circ(0,\bar z) \leq 1$, there holds $\bar z \cdot \nu^{E_\rho} \leq F(0,\nu^{E_\rho})$ and thus

$$\int_{\partial E_\rho\cap D_s^t} \bar z\cdot\nu^{E_\rho} - F(\rho x,\nu^{E_\rho})d\mathscr{H}^{d-1} \leq \int_{\partial E_\rho\cap D_s^t} |F(0,\nu^{E_\rho}) - F(\rho x,\nu^{E_\rho})|d\mathscr{H}^{d-1} \leq C\rho s^{d-1}$$

where the last inequality follows from $t \leq s$ and the minimality of E_ρ inside D_s^t. Now since

$$\left(F^\circ\right)^2(0,z_\rho) - \left(F^\circ\right)^2(\rho x,z_\rho) \geq \left(F^\circ\right)^2(0,z_\rho) - 1 \geq 2\frac{\bar\nu}{F(0,\bar\nu)}\cdot(z_\rho - z) + \delta^2|z_\rho - z|^2$$

we find that (15) transforms into,

$$(\xi\cdot z_\rho)^2 \leq C\left[(F(0,\bar\nu) - \bar\nu\cdot z_\rho) + (\left(F^\circ\right)^2(0,z_\rho) - \left(F^\circ\right)^2(\rho x,z_\rho))\right]$$

for every $|\xi| \le 1$ and $\xi \cdot \bar{z} = 0$, from which we get

$$\int_{\partial D_s + (-t,t)\bar{z}} \left((2\chi_{E_\rho} - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})} \right) (z_\rho \cdot \xi) \, d\mathscr{H}^{d-1}$$
$$\le 2CF(0,\overline{\nu})t\sqrt{t} \left(\int_{\partial D_s + (-t,t)\bar{z}} \left(1 - \frac{z_\rho \cdot \overline{\nu}}{F(0,\overline{\nu})} \right) d\mathscr{H}^{d-1} \right)^{\frac{1}{2}} \sqrt{\mathscr{H}^{d-2}(\partial D_s)}$$
$$+ 2Ct \int_{\partial D_s + (-t,t)\bar{z}} \left| (F^\circ)^2 (0, z_\rho) - (F^\circ)^2 (\rho x, z_\rho) \right|^{1/2} d\mathscr{H}^{d-1}$$
$$\le CF(0,\overline{\nu})t\sqrt{t} \left(\int_{\partial D_s + (-t,t)\bar{z}} \left(1 - \frac{z_\rho \cdot \overline{\nu}}{F(0,\overline{\nu})} \right) d\mathscr{H}^{d-1} \right)^{\frac{1}{2}} \sqrt{\mathscr{H}^{d-2}(\partial D_s)} + Ct\rho^{1/2} s^{d-1} t \,.$$

Using these estimates, we finally get that, setting as before $f(s) := \int_{D_s^t} \left(1 - \frac{z_\rho \cdot \overline{\nu}}{F(0,\overline{\nu})}\right) dy$, there holds

$$f(s)^2 \;\le\; c \left(s^{d-2} t^3 f'(s) \;+\; t^{4-2/d} s^{2d-4+2/d} \omega(\rho s)^2 + \rho t s^{d-1} + \rho^{1/2} t^2 s^{d-1} \right) .$$

From this inequality, the proof can be concluded exactly as in *Step 1*. □

Eventually, we can also give a locally uniform convergence result.

Proposition 3.9. *For all $x \in \Omega$ we let*

$$z_\rho(x) := \frac{1}{|B_\rho(0)|} \int_{B_\rho(x) \cap \Omega} z(y) \, dy \,.$$

Then, $F^\circ(x, z_\rho(x)) \to 1$ locally uniformly on S.

Proof. Given $K \subset \Omega$ a compact set, we can check that for any $t > 0$, there exists $\rho_0 > 0$ such that for any $x \in K \cap S$, if E^x is the level set of u through x, then for any $\rho \le \rho_0$, the boundary of $(E^x - x)/\rho \cap B_1(0)$ lies in a strip of width $2t$, that is, there is $\overline{\nu}^x \in \mathbb{S}^{d-1}$ with $\partial((E^x - x)/\rho) \cap B_1(0) \subset \{|y \cdot \overline{\nu}^x| \le t\})$.
Indeed, if this is not the case, one can find $t > 0$, $\rho_k \to 0$, $x_k \in K \cap S$, such that $\partial((E^{x_k} - x_k)/\rho_k) \cap B_1(0)$ is not contained in any strip of width $2t$. Up to a subsequence we may assume that $x_k \to x \in K \cap S$, and from the bound on the perimeter, that $(E^{x_k} - x_k)/\rho_k \cap B_1(0)$ converges to a local minimizer of $\int_{\partial E} F(0, \nu^E) d\mathscr{H}^{d-1}$ and is thus a halfspace.[2] Moreover, $\partial((E^{x_k} - x_k)/\rho_k) \cap B_1(0)$ converges in the Hausdorff sense (thanks to the density estimates) to a hyperplane. We easily obtain a contradiction.
The thesis follows when we observe that the proof of Proposition 3.8 can be reproduced by replacingthe direction $\nu^{E^x}(x)$ (which exists only if x lies in the reduced boundary of E^x) with the direction $\overline{\nu}^x$ given above. □

[2] If $d = 2$, this Bernstein result readily follows from the strict convexity of F, see [11, Prop 3.6] whereas for $d = 3$, see [18]. In the case of the area i.e when $F(x, Du) = |Du|$ and $d \le 7$, see also [12, Rem 3.2].

3.3 A counterexample.

We provide an example where $g \in L^{d-\varepsilon}(\Omega)$, with $\varepsilon > 0$ arbitrarily small, and Theorem 3.8 does not hold.

Let $\Omega = B_1(0)$ be the unit ball of $\mathbb{R}^d$ and let $E = \Omega \cap \{x_d \leq 0\}$. We shall construct a vector field $z : \Omega \to \mathbb{R}^d$ such that $z = \nu^E$ on $\partial E \cap \Omega$, $|z| \leq 1$ everywhere in Ω, $\mathrm{div} z \in L^{d-\varepsilon}(\Omega)$, but 0 is not a Lebesgue point of z. Notice that E minimizes the functional (3) with $g = \mathrm{div} z$.

Letting $r_n \to 0$ be a decreasing sequence to be determined later, and let $B_n = B_{r_n}(x_n)$ with $x_n = 2r_n e_d$. Without loss of generality, we may assume $r_{n+1} < r_n/4$ so that the balls B_n are all disjoint. We define the vector field z as follows: $z(x) = e_d$ if $x \in \Omega \setminus \cup_n B_n$, and $z(x) = |x - x_n| e_d$ if $x \in B_n$. It follows that $\mathrm{div} z = 0$ in $\Omega \setminus \cup_n B_n$ and $|\mathrm{div} z| \leq 1/r_n$ in B_n, so that

$$\int_\Omega |\mathrm{div} z|^{d-\varepsilon}\, dx = \sum_n \int_{B_n} |\mathrm{div} z|^{d-\varepsilon}\, dx \leq \omega_d \sum_n r_n^\varepsilon < +\infty$$

if we choose r_n converging to zero sufficiently fast, so that $g = -\mathrm{div}\, z \in L^{d-\varepsilon}(\Omega)$.

However, since $z \cdot e_d \leq 1/2$ in $B_{r_n/2}(x_n)$, we also have

$$\int_{B_{3r_n}(0)} z \cdot e_d\, dx \leq |B_{3r_n}(0)| - \frac{1}{2} \left|B_{r_n/2}(x_n)\right|$$

so that

$$\frac{1}{|B_{3r_n}(0)|} \int_{B_{3r_n}(0)} z \cdot e_d\, dx \leq 1 - \frac{1}{6^d} < 1\,.$$

On the other hand, for $\delta \in (0, 1/6^d)$ we have

$$\frac{1}{|B_{r_n}(0)|} \int_{B_{r_n}(0)} z \cdot e_d\, dx \geq \frac{1}{|B_{r_n}(0)|} \left(|B_{r_n}(0)| - \sum_{i=n+1}^{\infty} |B_{r_i}(x_i)| \right) \geq 1 - \delta\,,$$

if we take the sequence r_n converging to 0 sufficiently fast. It follows that 0 is not a Lebesgue point of z.

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