

# Homogenization of fronts in highly heterogeneous media

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## Abstract

We consider the evolution by mean curvature in a highly heterogeneous medium, modeled by a periodic forcing term, with large  $L^\infty$ -norm but with zero average. We prove existence of a homogenization limit, when the dimension of the periodicity cell tends to zero, and show some properties of the effective velocity.

## 1 Introduction

This paper deals with mean curvature flow in a heterogeneous medium, represented by a  $\mathbb{Z}^n$ -periodic function  $g \in \text{Lip}(\mathbb{R}^n)$  acting on the flow as an additive forcing term. This problem has been already considered in the literature, starting from the papers [12, 20]. In particular, in [14, 10] the authors consider the following scaling

$$V^\varepsilon(x) = \varepsilon H(x) + g\left(\frac{x}{\varepsilon}\right) \quad \text{for } x \in \Gamma^\varepsilon(t), \quad (1)$$

where  $\Gamma^\varepsilon(t)$  is the evolving hypersurface,  $V^\varepsilon(x)$  is its normal velocity at  $x \in \Gamma^\varepsilon(t)$ , and  $H(x)$  its mean curvature. Under appropriate assumptions on the function  $g$ , one can prove that the evolution law (1) converges, as  $\varepsilon \rightarrow 0$ , to the first-order anisotropic law

$$V(x) = c(\nu(x)) \quad \text{for } x \in \Gamma(t),$$

where  $\Gamma(t)$  denotes the limit of  $\Gamma^\varepsilon(t)$ ,  $V(x)$  the limiting normal velocity at  $x \in \Gamma(t)$  and  $c$  is a continuous function of the normal vector  $\nu(x)$  to  $\Gamma(t)$  at  $x$ , and can be determined by solving a suitable cell problem.

Since  $c = 0$  when  $g$  has zero average, a natural question is what happens if we consider a different time-scaling of (1), namely

$$V^\varepsilon(x) = H(x) + \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}\right) \quad \text{for } x \in \Gamma^\varepsilon(t), \quad (2)$$

under the additional assumption that  $g$  has zero average.

The object of the present paper is the study of the limit of the evolution law (2) as  $\varepsilon \rightarrow 0$ . We assume that the evolving hypersurface  $\Gamma^\varepsilon(t)$  is a graph of a function  $u^\varepsilon(\cdot, t)$  with respect to a fixed hyperplane, independent of  $\varepsilon$ , and that the forcing term  $g$  does

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not depend on the variable orthogonal to such hyperplane. In this case, the function  $u^\varepsilon$  satisfies the equation

$$u_t^\varepsilon(t, x) = \operatorname{tr} \left[ \left( \mathbf{I} - \frac{Du^\varepsilon \otimes Du^\varepsilon}{1 + |Du^\varepsilon|^2} \right) D^2 u^\varepsilon \right] + \frac{1}{\varepsilon} g \left( \frac{x}{\varepsilon} \right) \sqrt{1 + |Du^\varepsilon|^2} \quad \text{in } (0, +\infty) \times \mathbb{R}^n, \quad (3)$$

where we require that the initial datum  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$  is a uniformly continuous function such that  $u_0(x) - qx$  is bounded for some  $q \in \mathbb{R}^n$ .

Another motivation for considering (3) comes from the following homogenization problem:

$$u_t^\varepsilon = \operatorname{tr} \left[ \left( \mathbf{I} - \frac{Du^\varepsilon \otimes Du^\varepsilon}{1 + |Du^\varepsilon|^2} \right) D^2 u^\varepsilon \right] + g \left( \frac{x}{\varepsilon} \right) \sqrt{1 + |Du^\varepsilon|^2} \quad \text{in } (0, +\infty) \times \mathbb{R}^n. \quad (4)$$

Since the function  $g$  is bounded, one can show that, as  $\varepsilon \rightarrow 0$ ,  $u^\varepsilon \rightarrow u$  locally uniformly in  $\mathbb{R}^n \times [0, +\infty)$ . Moreover,  $u$  solves the limit equation

$$u_t = \operatorname{tr} \left[ \left( \mathbf{I} - \frac{Du \otimes Du}{1 + |Du|^2} \right) D^2 u \right] + \left( \int_{[0,1]^n} g \, dx \right) \sqrt{1 + |Du|^2} \quad \text{in } (0, +\infty) \times \mathbb{R}^n. \quad (5)$$

This result, derived in [11] when  $n = 1$ , can be obtained in general dimensions by the so-called *perturbed test function method* [15], which is by now a standard tool in viscosity solutions theory applied to homogenization problems. More precisely, one considers the formal asymptotic expansion

$$u^\varepsilon(x, t) = u(x, t) + \varepsilon^2 \chi \left( \frac{x}{\varepsilon}, x, t \right) \quad (6)$$

where the corrector  $\chi(\xi, x, t) = \psi(\xi, Du(x, t))$  is periodic in  $\xi$  and solves, for every fixed  $p = Du(x, t)$ , the cell problem

$$\operatorname{tr} \left[ \left( \mathbf{I} - \frac{p \otimes p}{1 + |p|^2} \right) D_{\xi\xi}^2 \chi \right] = \left( \int_{[0,1]^n} g(x) \, dx - g(\xi) \right) \sqrt{1 + |p|^2} \quad \text{in } \mathbb{R}^n.$$

Plugging the expansion (6) in (4) and using the comparison principle for viscosity solutions, one obtains that  $u$  solves (5).

In this paper, we apply the same method to Equation (3), and we prove that, under a suitable assumption on the function  $g$  (see Section 2), the limit function  $u$  solves the anisotropic parabolic equation

$$u_t = \operatorname{tr} (A(Du) D^2 u) \quad \text{in } (0, +\infty) \times \mathbb{R}^n,$$

where  $A(p)$  is a smooth function depending on  $g$ , with values in the set of positive definite symmetric matrices. Obviously, when  $g \equiv 0$  we have  $A(p) = \mathbf{I} - \frac{p \otimes p}{1 + |p|^2}$ .

When  $n = 1$ , thanks to an explicit representation formula for  $A(p)$ , we can further show that

$$0 < A(p) \leq \frac{1}{1 + p^2} \quad \text{for all } p \in \mathbb{R}$$

and that  $\lim_{|p| \rightarrow \infty} A(p)(1 + p^2) = 0$ , when  $g \not\equiv 0$ . In particular, this implies that the presence of  $g$  has the effect of decreasing the speed of the front in the limit, without stopping the motion.

A further consequence of our result is the nonexistence of compact embedded solutions of the prescribed curvature problem

$$H + g = 0, \tag{7}$$

for all  $g$  such that  $A(p) \neq 0$  for some  $p \in \mathbb{R}^n$ . For similar results in the nonperiodic setting, we refer to [18] and references therein.

The paper is organized as follows. In Section 2 we describe the standing assumptions on the forcing term  $g$ . Section 3 and 4 are devoted to the analysis of two ergodic problems in  $\mathbb{R}^n$ , which permit to define the limit parabolic operator  $F(p, X) = \text{tr}(A(p)X)$ . In Section 4 we consider the case of planar curves (i.e.  $n = 1$ ), and obtain a more explicit description of the function  $A(p)$  (see Proposition 4.1). Finally, Section 5 contains the main homogenization result (see Theorem 5.3) and a discussion of its consequences for the related prescribed curvature problem (7).

## 2 Standing assumptions

In this section we state some conditions on the forcing term  $g$ , which will hold throughout the paper. Setting  $Q := (0, 1)^n$ , our first condition is:

$$(G1) \quad g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is Lipschitz continuous, } \mathbb{Z}^n\text{-periodic and } \int_Q g(y)dy = 0.$$

The requirement that  $g$  has zero average on  $Q$  is necessary due to the homogenization result for Equations (1) and (4) discussed in the Introduction.

We also need a condition ensuring that the oscillation of  $g$  on  $Q$  is not too large. Let us first define the space  $BV_{\text{per}}(Q)$  of functions which have periodic bounded variation in  $Q$ . We refer to [3] for a general introduction to functions of bounded variation and sets of finite perimeter.

It is a classical result that any  $u \in BV(Q)$  admits a trace  $u^Q$  on  $\partial Q$  (see e.g. [3, Thm 3.87]). Let  $\partial_0 Q := \partial Q \cap \{y : \prod_{i=1}^n y_i = 0\}$  and let  $\sigma : \partial_0 Q \rightarrow \partial Q$  be the function  $\sigma(y) := y + \sum_{i=1}^n \lambda_i(y)e_i$ , where  $\lambda_i(y) = 1$  if  $y_i = 0$  and  $\lambda_i(y) = 0$  otherwise. The periodic total variation of  $u \in BV(Q)$  is defined as

$$|Du|_{\text{per}}(Q) := |Du|(Q) + \int_{\partial_0 Q} |u^Q(y) - u^Q(\sigma(y))| d\mathcal{H}^{n-1}(y). \tag{8}$$

The space  $BV_{\text{per}}(Q)$  is the space  $BV(Q)$  endowed with the norm

$$\|u\|_{BV_{\text{per}}(Q)} := \|u\|_{L^1(Q)} + |Du|_{\text{per}}(Q).$$

For every  $E \subseteq Q$  we define the periodic perimeter of  $E$  as

$$\text{Per}_{\text{per}}(E, Q) := |D\chi_E|_{\text{per}}(Q) \tag{9}$$

where  $\chi_E$  is the characteristic function of  $E$ . We observe that  $BV_{\text{per}}(Q)$  coincides with  $BV(\mathbb{T}^n)$ , where  $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$  is the  $n$ -dimensional torus. In particular, the following Coarea Formula holds [3, Thm 3.40]

$$|Du|_{\text{per}}(Q) = \int_{\mathbb{R}} \text{Per}_{\text{per}}(\{u > t\}, Q) dt \quad \text{for all } u \in BV_{\text{per}}(Q). \tag{10}$$

The second condition we assume on  $g$  is:

(G2) there exists  $\delta < 1$  such that for every  $E \subseteq Q$  of finite perimeter

$$\int_E g(y) dy \leq \delta \text{Per}_{\text{per}}(E, Q). \quad (11)$$

Note that, when  $n > 1$ , condition (G2) is satisfied whenever  $\|g\|_{L^n(Q)} < C(Q)$ , where  $C(Q)$  is the isoperimetric constant of  $\mathbb{T}^n$ . Indeed, since  $g$  has zero average, possibly exchanging  $E$  with  $Q \setminus E$  in (11) we can assume that  $|E| \leq 1/2$ . By the isoperimetric inequality on  $\mathbb{T}^n$  [23], we then obtain

$$\int_E g(y) dy \leq \|g\|_{L^n(Q)} |E|^{\frac{n-1}{n}} \leq \frac{\|g\|_{L^n(Q)}}{C(Q)} \text{Per}_{\text{per}}(E, Q) = \delta \text{Per}_{\text{per}}(E, Q)$$

where  $\delta = \|g\|_{L^n(Q)}/C(Q) < 1$ .

When  $n = 1$ , condition (G2) is equivalent to

$$\max_{y \in [0,1]} \int_0^y g(s) ds - \min_{y \in [0,1]} \int_0^y g(s) ds < 2 \quad (12)$$

which is the same condition assumed in [10].

### 3 The cell problems and the effective operator

In this section, we consider two ergodic problems in  $Q$ , see (13), (15), whose solutions are useful to define the limit problem as  $\varepsilon \rightarrow 0$  of the singularly perturbed Equations (3).

**Lemma 3.1** *Under the standing assumptions, for every  $p \in \mathbb{R}^n$  the equation*

$$-\text{div} \left( \frac{D\chi + p}{\sqrt{1 + (D\chi + p)^2}} \right) = g(y) \quad \text{in } \mathbb{R}^n \quad (13)$$

*admits a  $\mathbb{Z}^n$ -periodic solution  $\chi(y; p) \in C^{2+\alpha}(\mathbb{R}^n)$ , for all  $\alpha < 1$ , which is unique up to an additive constant. Moreover  $\chi$  depends smoothly on  $p$ .*

**PROOF** We observe that (13) is the Euler-Lagrange equation of the functional

$$J_p(u) := \int_Q \left( \sqrt{1 + (Du + p)^2} - gu \right) dy + \int_{\partial_0 Q} |u^Q(y) - u^Q(\sigma(y))| d\mathcal{H}^{n-1}(y)$$

which is a convex lower semicontinuous functional on  $BV_{\text{per}}(Q)$  (see [3]).

We claim that  $J_p$  is coercive on the subspace

$$BV_{\text{per}}^0(Q) := \left\{ u \in BV_{\text{per}}(Q) : \int_Q u = 0 \right\}.$$

Notice that  $|Du|_{\text{per}}(Q)$  is an equivalent norm on  $BV_{\text{per}}^0(Q)$  [3, Thm 3.44]. By condition (G2) and the Coarea Formula (10), recalling that  $\int_Q g dy = 0$ , we have

$$\int_Q gu dy = \int_{\mathbb{R}} \int_{\{u>t\}} g dy dt \leq \delta \int_{\mathbb{R}} \text{Per}_{\text{per}}(\{u > t\}, Q) dt = \delta |Du|_{\text{per}}(Q), \quad (14)$$

for all  $u \in BV_{\text{per}}(Q)$ . Therefore, we obtain

$$\begin{aligned} J_p(u) &\geq \int_Q |Du + p| - \int_Q gu \, dy + \int_{\partial_0 Q} |u^Q(y) - u^Q(\sigma(y))| \, d\mathcal{H}^{n-1}(y) \\ &\geq -|p| - \int_Q gu \, dy + |Du|_{\text{per}}(Q) \\ &\geq -|p| + (1 - \delta)|Du|_{\text{per}}(Q), \end{aligned}$$

which proves that  $J_p$  is coercive on  $BV_{\text{per}}^0(Q)$ .

Since  $J_p$  is coercive and lower semicontinuous on  $BV_{\text{per}}^0(Q)$ , it admits a minimizer  $\chi(\cdot; p)$ . Moreover, by convexity,  $\chi(\cdot; p)$  is the unique minimizer of  $J_p$  on  $BV_{\text{per}}(Q)$ , up to an additive constant.

Finally, by [16], any solution  $\chi(y; p)$  of (13) is Lipschitz continuous, and hence of class  $\mathcal{C}^{2+\alpha}(\mathbb{R}^n)$  by standard elliptic regularity [17]. Also, by differentiation of (13) in the  $p$ -variables, the same regularity holds for any derivative of  $\chi(y; p)$  with respect to  $p$ .  $\square$

**REMARK** Notice that the periodic function  $\psi_p(y) = \chi(y; p) + p \cdot y$  solves the prescribed curvature problem

$$-\text{div} \left( \frac{D\psi_p}{\sqrt{1 + (D\psi_p)^2}} \right) = g(y), \quad y \in \mathbb{R}^n.$$

In particular, the graph of  $\psi_p$  is a plane-like solution of the geometric equation  $H = g$ , lying at a bounded distance from the hyperplane  $\{(y, p \cdot y) : y \in \mathbb{R}^n\}$ . We refer to [9] for a general analysis of such solutions.

**Lemma 3.2** *For any  $p \in \mathbb{R}^n$  and  $M \in \mathbf{S}_n$  there exists a unique constant  $F(p, M)$  such that there exists a  $\mathbb{Z}^n$ -periodic solution  $\psi(y; p, M) \in \mathcal{C}^{2+\alpha}(\mathbb{R}^n)$ , for all  $\alpha < 1$ , to the cell problem*

$$\begin{aligned} F(p, M) &= \text{tr} \left[ \left( \mathbf{I} - \frac{(p + D_y \chi) \otimes (p + D_y \chi)}{1 + |p + D_y \chi|^2} \right) (D^2 \psi + M + 2D_{py}^2 \chi M) \right] \\ &\quad - 2(p + D_y \chi)^T D_{yy}^2 \chi \left( \frac{D\psi + (D_p \chi)^T M}{1 + |p + D_y \chi|^2} - \frac{(p + D_y \chi) \cdot (D\psi + (D_p \chi)^T M)}{(1 + |p + D_y \chi|^2)^2} (p + D_y \chi) \right) \\ &\quad + g(y) \frac{(p + D_y \chi) \cdot (D\psi + (D_p \chi)^T M)}{\sqrt{1 + |p + D_y \chi|^2}}, \end{aligned} \tag{15}$$

where  $\chi(y; p)$  is the solution to (13) with  $\chi(0; p) = 0$ . Moreover  $\psi(y; p, M)$  is in  $\mathcal{C}^2(\mathbb{R}^n)$  and is unique up to an additive constant.

Finally, there exists a  $n \times n$  symmetric matrix  $A(p)$ , depending smoothly on  $p$ , such that

$$F(p, M) = \text{tr}(A(p)M). \tag{16}$$

**PROOF** Observe that, using the fact that  $\chi$  solves (13), Equation (15) can be rewritten as

$$\begin{aligned} \text{tr} (B(y, p)D^2 \psi) + b(y, p) \cdot D\psi &= F(p, M) - \text{tr} [B(y, p)(\mathbf{I} + 2D_{py}^2 \chi)M] \\ &\quad - (b(y, p))^T M D_p \chi, \end{aligned} \tag{17}$$

where  $B(y, p) := \mathbf{I} - \frac{(p+D_y\chi)\otimes(p+D_y\chi)}{1+|p+D_y\chi|^2}$  is a symmetric, positive definite matrix and

$$b(y, p) := -\frac{2(p+D_y\chi)^T D_{yy}^2 \chi}{1+|p+D_y\chi|^2} + \left[ \frac{3g(y)}{\sqrt{1+|p+D_y\chi|^2}} + \frac{2\Delta_{yy}\chi}{1+|p+D_y\chi|^2} \right] (p+D_y\chi).$$

Moreover, recalling that  $\chi$  is  $\mathbb{Z}^n$ -periodic and smooth, we get that (17) is a uniformly elliptic equation and that both  $B(y, p)$  and  $b(y, p)$  are  $\mathbb{Z}^n$ -periodic in  $y$ . The existence of a unique constant  $F(p, M)$  such that (17) admits a continuous periodic solution  $\psi$  is then a well-known fact (see [4, Thm II.2], [1, Thm 7.1]). Finally,  $\psi$  is unique up to an additive constant and is of class  $\mathcal{C}^{2+\alpha}$  with respect to  $y$ , for all  $\alpha < 1$ , by elliptic regularity [17].

We also have an explicit characterization of  $F(p, M)$ . Indeed, consider the differential operator

$$\mathcal{L}_p(\phi) := \operatorname{tr}(B(y, p)D^2\phi) + b(y, p) \cdot D\phi$$

and let  $\mathcal{L}_p^*$  be its formal adjoint. Then, the equation  $\mathcal{L}_p^*(m) = 0$  admits a  $\mathbb{Z}^n$ -periodic solution  $m(y; p) > 0$ , which is unique up to a multiplicative constant (see [7, Thm II.4.2]), so that we may fix  $\int_Q m(y; p) dy = 1$ . Notice that  $m(y; p)$ , as well as  $\chi(y; p)$ , depends smoothly on  $p$  by elliptic regularity.

In [7, Thm II.6.1] (see also [15, Thm 2.1], [1, Corollary 6.2]) it is proved that

$$F(p, M) = \int_Q [\operatorname{tr}(B(y, p)(\mathbf{I} + 2D_{py}^2 \chi(y; p))M) + (b(y, p))^T M D_p \chi(y; p)] m(y; p) dy.$$

This formula implies, in particular, the regularity of  $F$  with respect to  $p$ , since the functions  $b(y, \cdot)$  and  $B(y, \cdot)$  are smooth, due to the regularity properties of  $\chi$  and  $m$ .

Finally, the above representation formula for  $F$  also implies (16), since  $F(p, \cdot)$  is a linear function of  $M$  for any fixed  $p$ , and hence which can be written as  $\operatorname{tr}[A(p)M]$  for some symmetric matrix  $A(p)$ . □

**REMARK** Notice that, always by elliptic regularity, from (15) it follows that the map  $g \mapsto A(p)$  (when defined) is continuous with respect to the Lipschitz norm of  $g$ . In particular, since  $A(p) = \mathbf{I} - \frac{p\otimes p}{1+|p|^2}$  when  $g = 0$ , we get that there exists  $\delta > 0$  such that, if  $\|g\|_{\text{Lip}} < \delta$ , then  $A(0) \neq 0$ . Notice that, possibly reducing  $\delta$ , this condition implies (11).

## 4 The effective operator in dimension 1

When  $n = 1$ , we obtain a much more explicit description of the effective operator  $F(p, M)$  (see Proposition 4.1). This is done by solving explicitly the two cell problems (13) and (15).

**Lemma 4.1** *Under the standing assumptions, for every  $p \in \mathbb{R}$  the periodic solution to (13) such that  $\chi(0; p) = 0$  is given by*

$$\chi(y; p) = -py + \int_0^y \frac{c(p) - G(s)}{\sqrt{1 - (c(p) - G(s))^2}} ds, \quad (18)$$

where  $G'(y) = g(y)$ ,  $\int_0^1 G(y) dy = 0$ , and  $c = f^{-1}$  with  $f(c) = \int_0^1 \frac{c - G(y)}{\sqrt{1 - (c - G(y))^2}} dy$ .

PROOF We denote by  $\overline{m} := \max_{[0,1]} G$  and  $\underline{m} = \min_{[0,1]} G$ .

Integrating once the equation, easy computations show that every solution to (13) satisfies  $\frac{1}{1+(p+\chi_y)^2} = 1 - (c - G(y))^2$  for some constant  $c$ . Therefore solving the problem necessarily requires  $\max_{[0,1]}(G - c) < 1$  and  $\min_{[0,1]}(G - c) > -1$ . It is possible to find  $c$  satisfying this condition if and only if  $\overline{m} - \underline{m} < 2$ . This is ensured by (11). In this case it is sufficient to choose  $c \in (\overline{m} - 1, \underline{m} + 1)$ . For such constants  $c$ , we get that  $\chi_y(y) = -p + \frac{c - G(y)}{\sqrt{1 - (c - G(y))^2}}$ .

We define the function  $f : (\overline{m} - 1, \underline{m} + 1) \rightarrow \mathbb{R}$  by

$$f(c) := \int_0^1 \frac{c - G(y)}{\sqrt{1 - (c - G(y))^2}} dy.$$

Straightforward computations show that  $f$  is strictly increasing and we claim that  $f(\overline{m} - 1, \underline{m} + 1) = (-\infty, +\infty)$ . Indeed we remark that, by assumption (G1),  $G \in \mathcal{C}^{1,1}(\mathbb{R})$  and, if  $y_0$  is a maximum point of  $G$ , then for  $y$  close to  $y_0$  we have  $|G(y) - G(y_0)| \leq k|y - y_0|^2$ . As a consequence, if  $c = \overline{m} - 1$ , in a small neighborhood of  $y_0$  we have

$$1 - (\overline{m} - 1 - G(y))^2 = 1 - (G(y_0) - 1 - G(y))^2 = 2(G(y) - G(y_0)) + (G(y) - G(y_0))^2 \leq \tilde{k}|y - y_0|^2,$$

for some constant  $\tilde{k} > 0$ . Possibly changing the constant  $\tilde{k}$ , we then get

$$\frac{c - G(y)}{\sqrt{1 - (c - G(y))^2}} \leq -\frac{\tilde{k}}{|y - y_0|}.$$

This inequality shows that the function is not integrable for  $c = \overline{m} - 1$  and therefore  $f(c) \rightarrow -\infty$  as  $c \rightarrow \overline{m} - 1$ . An analogous argument holds when  $c = \underline{m} + 1$ .

Since we are looking for periodic solutions to Equation (13), we impose the condition  $\int_0^1 \chi_y(y) dy = 0$ . This gives  $f(c) = p$ , that is,  $c = f^{-1}$ .

Notice that the function  $c$  is smooth and

$$c'(p) = \frac{1}{\int_0^1 [1 - (c(p) - G(y))^2]^{-\frac{3}{2}} dy}.$$

□

**Proposition 4.1** *Under the standing assumptions and for  $n = 1$ , the function  $A(p)$  in Lemma 3.2 is given by*

$$0 < A(p) = \frac{c'(p)}{\int_0^1 \sqrt{1 - (c(p) - G(y))^2} dy} \leq \frac{1}{1 + p^2} \quad \forall p \in \mathbb{R}, \quad (19)$$

where  $G(y)$  and  $c(p)$  are as in Lemma 4.1. Moreover if  $g \neq 0$  there exists a constant  $K_g$  such that

$$0 < A(p)(1 + p^2) \leq \frac{K_g}{\sqrt{1 + p^2}}. \quad (20)$$

In particular  $A(p)(1 + p^2) \rightarrow 0$  as  $|p| \rightarrow +\infty$ .

PROOF To obtain the characterization of  $A$ , we explicitly solve Equation (15). Due to the homogeneity properties of (15) with respect to  $M$ , we immediately get  $A(p) = F(p, 1)$ .

We rewrite (15) with  $M = 1$ . We consider the coefficient of  $\psi_y$  and obtain, recalling the characterization (18) of  $\chi$ ,

$$\begin{aligned} & \frac{g(y)(p + \chi_y)}{\sqrt{1 + (p + \chi_y)^2}} - \frac{2\chi_{yy}(p + \chi_y)}{(1 + (p + \chi_y)^2)^2} \\ &= \frac{p + \chi_y}{1 + (p + \chi_y)^2} \left[ g(y)\sqrt{1 + (p + \chi_y)^2} - \frac{2\chi_{yy}}{1 + (p + \chi_y)^2} \right] = 3g(y)(c(p) - G(y)). \end{aligned}$$

The last two terms on the right-hand side of (15) coincide with

$$\frac{2\chi_{yy}\chi_p(p + \chi_y)}{(1 + (p + \chi_y)^2)^2} - \frac{g(y)\chi_p(p + \chi_y)}{\sqrt{1 + (p + \chi_y)^2}} = -3g(y)(c(p) - G(y))\chi_p.$$

Then, using the explicit formula for  $\chi_{yp}$  deduced from (18), (15) can be rewritten as

$$\psi_{yy} + \chi_{yp} + \frac{3g(y)(c(p) - G(y))}{1 - (c(p) - G(y))^2}(\psi_y + \chi_p) = \frac{F(p, 1)}{1 - (c(p) - G(y))^2} - \frac{c'(p)}{(1 - (c(p) - G(y))^2)^{\frac{3}{2}}}$$

Note that  $\frac{3g(y)(c(p) - G(y))}{1 - (c(p) - G(y))^2} = [\log(1 - (c(p) - G(y))^2)^{\frac{3}{2}}]_y$ . Therefore we obtain

$$\left[ (1 - (c(p) - G(y))^2)^{\frac{3}{2}} (\psi_y + \chi_p) \right]' = F(p, 1)\sqrt{1 - (c(p) - G(y))^2} - c'(p)$$

and integrating we get, for some constant  $d(p)$ ,

$$(1 - (c(p) - G(y))^2)^{\frac{3}{2}} (\psi_y + \chi_p) = d(p) + F(p, 1) \int_0^y \sqrt{1 - (c(p) - G(s))^2} ds - c'(p)y.$$

Then

$$\psi_y = -\chi_p + \frac{d(p) + F(p, 1) \int_0^y \sqrt{1 - (c(p) - G(s))^2} ds - c'(p)y}{(1 - (c(p) - G(y))^2)^{\frac{3}{2}}}.$$

We look for a periodic solution  $\psi$ , then we impose that  $\psi_y(0) = \psi_y(1)$ . Recalling the formula for  $c'(p)$  and  $\chi_p$ , we get  $\chi_p(0) = \chi_p(1) = 0$ , so  $\psi_y(0) = \frac{d(p)}{(1 - (c(p) - G(0))^2)^{\frac{3}{2}}}$  and

$$\psi_y(1) = \frac{d(p) + F(p, 1) \int_0^1 \sqrt{1 - (c(p) - G(s))^2} ds - c'(p)}{(1 - (c(p) - G(1))^2)^{\frac{3}{2}}}.$$

Then, since  $G$  is periodic, we obtain the condition  $F(p, 1) \int_0^1 \sqrt{1 - (c(p) - G(y))^2} dy = c'(p)$ , which gives the desired characterization (19) of  $A(p)$ . In particular, recalling the definition of  $c'(p)$  we get that  $A(p) > 0$  for every  $p$ .

We prove now that  $A(p)(1 + p^2) \leq 1$ . By definition of  $c(p)$  and Hölder inequality we get

$$|p| \leq \int_0^1 \left| \frac{c(p) - G(y)}{\sqrt{1 - (c(p) - G(y))^2}} \right| dy \leq \left[ \int_0^1 \frac{(c(p) - G)^2}{1 - (c(p) - G(y))^2} dy \right]^{\frac{1}{2}}.$$

Therefore

$$1 + p^2 \leq \int_0^1 \frac{1}{1 - (c(p) - G(y))^2} dy. \quad (21)$$



By Holder inequality we get

$$\begin{aligned} \int_0^1 \frac{1}{1 - (c(p) - G(y))^2} dy &\leq \left[ \int_0^1 \frac{1}{[1 - (c(p) - G(y))^2]^{\frac{3}{2}}} dy \right]^{\frac{2}{3}} \\ &= \frac{\left[ \int_0^1 \frac{1}{[1 - (c(p) - G(y))^2]^{\frac{3}{2}}} dy \right]^{-\frac{1}{3}}}{c'(p)}. \end{aligned} \quad (22)$$

Again by Jensen and Holder inequalities we obtain

$$\begin{aligned} \frac{1}{\int_0^1 \sqrt{1 - (c(p) - G(y))^2} dy} &\leq \int_0^1 \frac{1}{\sqrt{1 - (c(p) - G(y))^2}} dy \\ &\leq \left[ \int_0^1 \frac{1}{[1 - (c(p) - G(y))^2]^{\frac{3}{2}}} dy \right]^{\frac{1}{3}}. \end{aligned} \quad (23)$$

Finally, recalling the definition (19) of  $A(p)$  (21), (22) and (23) give  $1 + p^2 \leq \frac{1}{A(p)}$ .

Finally we prove (20). From inequalities (21) and (22), we get

$$c'(p) \leq \left( \frac{1}{1 + p^2} \right)^{\frac{3}{2}}.$$

We define the function

$$h(p) := \left( \int_0^1 \sqrt{1 - (c(p) - G(y))^2} dy \right)^{-1}.$$

Notice that  $h'(p) = c'(p)ph^{-2}(p)$  and then  $h$  is decreasing in  $(-\infty, 0)$  and increasing in  $(0, +\infty)$ . Moreover

$$\begin{aligned} \lim_{p \rightarrow +\infty} h(p) &= \left( \int_0^1 \sqrt{1 - (1 + \underline{m} - G(y))^2} dy \right)^{-1} \\ \lim_{p \rightarrow -\infty} h(p) &= \left( \int_0^1 \sqrt{1 - (1 - \overline{m} - G(y))^2} dy \right)^{-1}. \end{aligned}$$

We define the constant

$$K_g := \max \left( \frac{1}{\int_0^1 \sqrt{1 - (1 + \underline{m} - G(y))^2} dy}, \frac{1}{\int_0^1 \sqrt{1 - (-1 + \overline{m} - G(y))^2} dy} \right).$$

Note that  $K_g > 0$  depends only on  $g$  and that it explodes as  $g \rightarrow 0$ . So we conclude, recalling (19).  $\square$

REMARK Note that if  $g \equiv 0$ , then  $c(p) = \frac{p}{\sqrt{1+p^2}}$  and

$$A(p) = \frac{c'(p)}{\int_0^1 \sqrt{1 - \frac{p^2}{1+p^2}} dy} = \frac{\sqrt{1+p^2}}{(1+p^2)^{\frac{3}{2}}} = \frac{1}{1+p^2}.$$

Moreover, it is clear that the constant  $K_g$  in (20) necessarily explodes as  $g \rightarrow 0$ .

## 5 The convergence result

In this section we study the asymptotic behaviour as  $\varepsilon \rightarrow 0$  of the solutions  $u^\varepsilon$  of the singularly perturbed equations

$$u_t^\varepsilon(t, x) = \operatorname{tr} \left[ \left( \mathbf{I} - \frac{Du^\varepsilon \otimes Du^\varepsilon}{1 + |Du^\varepsilon|^2} \right) D^2 u^\varepsilon \right] + \frac{1}{\varepsilon} g \left( \frac{x}{\varepsilon} \right) \sqrt{1 + |Du^\varepsilon|^2}, \quad (24)$$

for  $(t, x) \in (0, +\infty) \times \mathbb{R}^n$ , with initial data  $u^\varepsilon(0, x) = u_0(x)$ , where

$$u_0(x) = q \cdot x + v_0(x) \quad \text{with } v_0 \text{ bounded and uniformly continuous.} \quad (25)$$

We show that the functions  $u^\varepsilon$  converge locally uniformly to a function  $u$ , which is a continuous viscosity solution to the effective quasilinear parabolic equation

$$u_t(t, x) = \operatorname{tr} [A(Du(t, x))D^2 u(t, x)] \quad \text{in } (0, +\infty) \times \mathbb{R}^n \quad (26)$$

with initial datum  $u(0, x) = u_0(x)$ , where the differential operator  $F(p, M) = \operatorname{tr} [A(p)M]$  is the one defined in Lemma 3.2. Moreover  $u$  is the unique viscosity solution of (26) in the class

$$\mathcal{L}_q := \{u \in \mathcal{C}([0, +\infty), \mathbb{R}^n) \text{ s.t. } u(t, x) - q \cdot x \in \mathcal{C}_b([0, +\infty) \times \mathbb{R}^n)\},$$

where  $\mathcal{C}_b([0, +\infty), \mathbb{R}^n)$  is the space of bounded continuous functions in  $[0, +\infty) \times \mathbb{R}^n$ . We also discuss the geometric counterpart of this result and some consequences for a related prescribed curvature problem.

We start recalling two comparison principles for solutions of degenerate parabolic equations, which we will apply to the singularly perturbed and to the effective problem.

**Theorem 5.1** *Let  $w^\varepsilon, v^\varepsilon$  be respectively an upper semicontinuous subsolution and a lower semicontinuous supersolution to (24) in  $[0, +\infty) \times \mathbb{R}^n$ . Assume that there exists a constant  $k > 0$  such that  $\frac{w^\varepsilon(t, x)}{1+|x|^k}, \frac{v^\varepsilon(t, x)}{1+|x|^k} \rightarrow 0$  as  $|x| \rightarrow +\infty$  uniformly with respect to  $t \in [0, +\infty)$ , and that  $w^\varepsilon(0, x) \leq u_0(x) \leq v^\varepsilon(0, x)$  for every  $x \in \mathbb{R}^n$ , where  $u_0$  satisfies (25). Then  $w^\varepsilon(t, x) \leq v^\varepsilon(t, x)$  for every  $(t, x) \in [0, +\infty) \times \mathbb{R}^n$ .*

*Moreover, there exists a unique continuous viscosity solution  $u^\varepsilon$  in  $\mathcal{L}_q$  to (24), with initial datum  $u_0$ .*

The proof of this comparison principle is given in [5, Theorem 2.1], while the existence of a unique solution to (24) can be done (with easy modifications) as in [5, Cor 2.1].

Notice that, if the initial datum  $u_0$  is of class  $\mathcal{C}^{2+\alpha}(\mathbb{R}^n)$  for some  $\alpha \in (0, 1)$ , then parabolic regularity theory [21] gives that the solutions  $u^\varepsilon$  in Theorem 5.1 are uniformly of class  $\mathcal{C}^{1+\alpha/2, 2+\alpha}([0, +\infty) \times \mathbb{R}^n)$ .

**Theorem 5.2** *Let  $w, v$  be respectively a bounded upper semicontinuous subsolution and a bounded lower semicontinuous supersolution to  $u_t = \operatorname{tr} [\tilde{A}(Du)D^2 u]$  in  $[0, +\infty) \times \mathbb{R}^n$ , where  $\tilde{A}(p)$  is a symmetric  $n \times n$  matrix, which depends smoothly on  $p$  and such that  $\tilde{A}(p) \geq 0$  for any  $p$ . Assume that  $w(0, x) \leq v(0, x)$  for every  $x \in \mathbb{R}^n$ . Then  $w(t, x) \leq v(t, x)$  for every  $(t, x) \in [0, +\infty) \times \mathbb{R}^n$ .*

For the proof we refer to [13].

We will apply this result to the limiting problem (26), letting  $\tilde{A}(p) := A(p + q)$  and changing the solution  $u(t, x)$  in  $\tilde{u}(t, x) = u(t, x) - q \cdot x$ . Since we look for solutions in  $\mathcal{L}_q$ , we actually have to deal with bounded solutions  $\tilde{u}$ .

However, we need to show that the effective differential operator  $F(p, M)$  is regular and degenerate elliptic. Surprisingly the degenerate ellipticity of  $F$  is not known a priori, and we obtain it in a step of the convergence proof. We point out here that the degenerate ellipticity is expected as a consequence of results of Alvarez, Guichard, Lions and Morel [2] (see also [8]). Indeed, Equation (24) defines a monotone semi-group (in the sense that the solution  $u^\varepsilon$  depends on  $u_0$  or  $v_0$  in a monotone way by the comparison principle) and the limiting semigroup is also expected to be monotone, hence, by the results of [2] or [8], it is certainly associated to a parabolic equation. We finally remark that we can see this monotonicity as a geometric “inclusion principle” and use as well the geometric version of the above results by Souganidis and the first author [6].

We now state our main result.

**Theorem 5.3** *Let  $u^\varepsilon$  be the unique continuous viscosity solution to (24) with polynomial growth and initial datum  $u_0$ , which satisfies (25) for some  $q \in \mathbb{R}^n$ . Then  $u^\varepsilon(t, x)$  converges locally uniformly to the unique function  $u$  in the class  $\mathcal{L}_q$  which solves (26) in the viscosity sense, with initial datum  $u(0, x) = u_0(x)$ .*

**PROOF** The argument of the proof is an appropriate adaptation of the *perturbed test function method*, introduced by Evans in [15] (see also [12], [10]). As we already pointed out above, we emphasize the fact that we do not know a priori that the limiting equation is degenerate parabolic.

The proof is divided into five steps.

**Step 1 (Local equiboundedness of  $u^\varepsilon$ .)** The existence and uniqueness of  $u^\varepsilon$  is assured by Theorem 5.1. Actually, it is possible to show that  $u^\varepsilon$  inherits the same growth of the initial data  $u_0$ . We consider the solution  $\chi$  of (13) with  $p = q$  and such that  $\chi(0; q) = 0$ . Then the function  $w^\varepsilon(t, x) = \varepsilon\chi\left(\frac{x}{\varepsilon}; q\right) + q \cdot x + \|v_0\|_\infty + \varepsilon\|\chi\|_\infty$ , where  $v_0$  is the function appearing in (25), is a stationary solution to (24) with  $w^\varepsilon(0, x) \geq u_0(x)$ , for  $\varepsilon > 0$ . Analogously  $v^\varepsilon(t, x) = \varepsilon\chi\left(\frac{x}{\varepsilon}; q\right) + q \cdot x - \|v_0\|_\infty - \varepsilon\|\chi\|_\infty$  is a stationary solution to (24) with  $v^\varepsilon(0, x) \leq u_0(x)$ , for  $\varepsilon > 0$ . Then, by the comparison principle (Theorem 5.1) we obtain that  $v^\varepsilon(t, x) \leq u^\varepsilon(t, x) \leq w^\varepsilon(t, x)$ , which gives in particular that  $|u^\varepsilon(t, x) - q \cdot x| \leq \|v_0\|_\infty + 2\varepsilon\|\chi\|_\infty$  for any  $\varepsilon > 0$  and  $(t, x) \in [0, +\infty) \times \mathbb{R}^n$ .

**Step 2 (Relaxed semilimits of  $u^\varepsilon$ .)** We define the relaxed semilimits (see [13])

$$\underline{u}(t, x) := \liminf_{(\varepsilon, t', x') \rightarrow (0, t, x)} u^\varepsilon(t', x'), \quad \overline{u}(t, x) := \limsup_{(\varepsilon, t', x') \rightarrow (0, t, x)} u^\varepsilon(t', x')$$

for  $(t, x) \in [0, +\infty) \times \mathbb{R}^n$ . Observe that  $\overline{u}, \underline{u}$ , due to the previous step, satisfy  $|\overline{u}(t, x) - q \cdot x| \leq \|v_0\|_\infty$ ,  $|\underline{u}(t, x) - q \cdot x| \leq \|v_0\|_\infty$  for any  $(t, x) \in [0, +\infty) \times \mathbb{R}^n$ .

We are going to prove that  $\overline{u}$  (resp.  $\underline{u}$ ) is a viscosity subsolution (resp. supersolution) to (26).

We describe the argument just for  $\overline{u}$ , since for  $\underline{u}$  it is completely analogous. We consider a smooth function  $\phi$  and we assume that  $\overline{u} - \phi$  has a strict maximum at  $(\bar{t}, \bar{x})$ ; we have to prove that

$$\phi_t(\bar{t}, \bar{x}) \leq \text{tr}[A(D\phi(\bar{t}, \bar{x}))D^2\phi(\bar{t}, \bar{x})]. \quad (27)$$

We define the perturbed test function

$$\phi^\varepsilon(t, x) := \phi(t, x) + \varepsilon \chi\left(\frac{x}{\varepsilon}; D\phi(t, x)\right) + \varepsilon^2 \psi\left(\frac{x}{\varepsilon}; D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x})\right), \quad (28)$$

where  $\chi(y; D\phi(t, x))$  is the periodic solution to (13) with  $p = D\phi(t, x)$  and  $\chi(0; p) = 0$ , and  $\psi(y; D\phi(\bar{t}, \bar{x}), D^2\phi(\bar{t}, \bar{x}))$  is the periodic solution to (15) with  $p = D\phi(\bar{t}, \bar{x})$ ,  $M = D^2\phi(\bar{t}, \bar{x})$  and  $\psi(0; p, M) = 0$ . Observe that

$$\limsup_{(\varepsilon, t', x') \rightarrow (0, t, x)} (u^\varepsilon(t', x') - \phi^\varepsilon(t', x')) = \bar{u}(t, x) - \phi(t, x).$$

By a standard compactness argument, there exist subsequences  $\varepsilon_n \rightarrow 0$  and  $(t_n, x_n) \rightarrow (\bar{t}, \bar{x})$  such that  $u^{\varepsilon_n}(t_n, x_n) - \phi^{\varepsilon_n}(t_n, x_n) \rightarrow \bar{u}(\bar{t}, \bar{x}) - \phi(\bar{t}, \bar{x})$  and  $(t_n, x_n)$  is a strict maximum of  $u^{\varepsilon_n} - \phi^{\varepsilon_n}$ . Since  $u^{\varepsilon_n}$  is a subsolution to (24), at  $(t_n, x_n)$  we have

$$\phi_t^{\varepsilon_n} \leq \text{tr} \left[ \left( \mathbf{I} - \frac{D\phi^{\varepsilon_n} \otimes D\phi^{\varepsilon_n}}{1 + |D\phi^{\varepsilon_n}|^2} \right) D^2\phi^{\varepsilon_n} \right] + \frac{1}{\varepsilon_n} g\left(\frac{x_n}{\varepsilon_n}\right) \sqrt{1 + |D\phi^{\varepsilon_n}|^2}.$$

Recalling the definition of  $\phi^\varepsilon$  in (28) and using standard asymptotic expansion arguments, we can rewrite the r.h.s. of the previous inequality as

$$\begin{aligned} & \frac{1}{\varepsilon_n} \left\{ \text{tr} \left[ \left( \mathbf{I} - \frac{(D\phi + D\chi) \otimes (D\phi + D\chi)}{1 + |D\phi + D\chi|^2} \right) D^2\chi \right] + g\left(\frac{x_n}{\varepsilon}\right) \sqrt{1 + |D\phi + D\chi|^2} \right\} \\ & + \text{tr} \left[ \left( \mathbf{I} - \frac{(D\phi + D\chi) \otimes (D\phi + D\chi)}{1 + |D\phi + D\chi|^2} \right) (D^2\psi + D^2\phi + 2D_{py}^2\chi D^2\phi) \right] \\ & - 2(D\phi + D\chi)^T D^2\chi \left( \frac{D\psi + (D_p\chi)^T D^2\phi}{1 + |D\phi + D\chi|^2} \right) \\ & + 2(D\phi + D\chi)^T D^2\chi (D\phi + D\chi) \frac{(D\phi + D\chi) \cdot (D\psi + (D_p\chi)^T D^2\phi)}{(1 + |D\phi + D\chi|^2)^2} \\ & + g\left(\frac{x_n}{\varepsilon}\right) \frac{(D\phi + D\chi) \cdot (D\psi + (D_p\chi)^T D^2\phi)}{\sqrt{1 + |D\phi + D\chi|^2}} + R(\varepsilon_n) \end{aligned}$$

where  $R(\varepsilon_n) \rightarrow 0$  uniformly as  $\varepsilon_n \rightarrow 0$ . Using the characterization of  $\chi$  and  $\psi$  as solutions of (13) and (15) respectively, and the regularity of  $\phi, \chi, \psi, A$ , we obtain that

$$\phi_t(t_n, x_n) \leq \text{tr} [A(D\phi(t_n, x_n))] D^2\phi(t_n, x_n) + R'(\varepsilon_n)$$

for some  $R'(\varepsilon_n) \rightarrow 0$  uniformly as  $\varepsilon_n \rightarrow 0$ . Letting  $n \rightarrow +\infty$ , we obtain (27).

**Step 3 (Degenerate ellipticity of the limiting operator.)** We prove the following

**Lemma 5.1** *The differential operator  $F(p, M) = \text{tr}[A(p)M]$  defined in Lemma 3.2 is degenerate elliptic.*

**PROOF** It is sufficient to show that the matrix  $A(p)$  is nonnegative definite for any  $p \in \mathbb{R}^n$ . We consider the unique solution  $u^\varepsilon$  to (24) with polynomial growth and initial data  $u_0(x) = p \cdot x$  (see Theorem 5.1) and the solution  $\chi(y; p)$  of (13) with  $\chi(0; p) = 0$ . Then, for all  $\varepsilon > 0$ , the function  $w^\varepsilon(t, x) = \varepsilon \chi\left(\frac{x}{\varepsilon}; p\right) + p \cdot x + \varepsilon \|\chi\|_\infty$  is a stationary solution to (24) with  $w^\varepsilon(0, x) \geq u_0(x)$ . Analogously  $v^\varepsilon(t, x) = \varepsilon \chi\left(\frac{x}{\varepsilon}; p\right) + p \cdot x - \varepsilon \|\chi\|_\infty$  is a stationary solution to (24) with  $v^\varepsilon(0, x) \leq u_0(x)$ , for  $\varepsilon > 0$ . By the comparison principle

(Theorem 5.1) we then obtain that  $v^\varepsilon(t, x) \leq u^\varepsilon(t, x) \leq w^\varepsilon(t, x)$ , which gives in particular  $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t, x) = p \cdot x$ , locally uniformly.

According to Step 2, if  $(\bar{t}, \bar{x})$  is a strict maximum of  $u(t, x) - \phi(t, x) = p \cdot x - \phi(t, x)$  then (27) holds. For small  $\delta > 0$ , we choose  $\phi(t, x) := p \cdot x + (q \cdot x)^2 + \delta(|x|^2 + (t-1)^2)$  and observe that  $(1, 0)$  is a strict maximum of  $u(t, x) - \phi(t, x)$  for any  $\delta$ ; moreover (27) reads

$$-A(p)q \cdot q = -\text{tr}[A(p)(q \otimes q)] \leq o(\delta),$$

where  $o(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . Since the inequality  $A(p)q \cdot q \geq 0$  holds for any  $p$  and  $q$ , the thesis follows letting  $\delta \rightarrow 0$ .  $\square$

**Step 4 ( $\bar{\mathbf{u}}(\mathbf{x}, \mathbf{0}) = \underline{\mathbf{u}}(\mathbf{x}, \mathbf{0}) = \mathbf{u}_0(\mathbf{x})$ .)** Let  $v_0$  be the function appearing in (25). For every  $\delta > 0$ , it is possible to find  $v_+^\delta, v_-^\delta \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$  such that  $v_-^\delta \leq v_0 \leq v_+^\delta$  and that  $\|v_+^\delta - v_0\|_\infty, \|v_-^\delta - v_0\|_\infty \leq \delta$  (this can be done by using standard mollification arguments). For any fixed  $\delta > 0$ , we consider the functions

$$w_\pm^{\varepsilon,\delta}(t, x) := v_\pm^\delta(x) + q \cdot x + \varepsilon \chi_\pm \left( \frac{x}{\varepsilon} \right) \pm \left( C^\delta t + \varepsilon \|\chi_\pm\|_\infty \right)$$

where  $\chi_+(x) = \chi(x; q + Dv_+^\delta(x))$  is the solution to (13) with  $p = q + Dv_+^\delta(x)$ ,  $\chi_-(x) = \chi(x; q + Dv_-^\delta(x))$  is the solution to (13) with  $p = q + Dv_-^\delta(x)$ , and  $\chi_\pm(0) = 0$ . Choosing  $C^\delta > 0$  sufficiently large, it is easy to see that  $w_\pm^{\varepsilon,\delta}$  are respectively a super and a subsolution to (24). Moreover  $w_-^{\varepsilon,\delta}(0, x) \leq u_0(x) \leq w_+^{\varepsilon,\delta}(0, x)$ . So, by the comparison principle (Theorem 5.1), we get that  $w_-^{\varepsilon,\delta}(t, x) \leq u^\varepsilon(t, x) \leq w_+^{\varepsilon,\delta}(t, x)$ , for every  $\delta > 0$ .

Passing to the relaxed semilimits, we then obtain

$$v_-^\delta(x) + q \cdot x \leq \underline{u}(0, x) \leq \bar{u}(0, x) \leq v_+^\delta(x) + q \cdot x.$$

Letting  $\delta \rightarrow 0$ , this gives  $\bar{u}(x, 0) = \underline{u}(x, 0) = u_0(x)$ .

**Step 5 (Uniform convergence.)** Let us define  $\bar{v}(t, x) := \bar{u}(t, x) - q \cdot x$  and  $\underline{v}(t, x) := \underline{u}(t, x) - q \cdot x$ . It is easy to show that  $\bar{v}$  and  $\underline{v}$  are bounded, satisfy  $\bar{v}(0, x) = v_0(x) = \underline{v}(0, x)$  and are respectively a sub and a supersolution of

$$v_t(t, x) = \text{tr} [A(q + Dv(t, x))D^2v(t, x)] \quad \text{in } (0, +\infty) \times \mathbb{R}^n. \quad (29)$$

Then, by the comparison principle (Theorem 5.2), we obtain that  $\bar{v}(t, x) \leq \underline{v}(t, x)$  in  $(0, +\infty) \times \mathbb{R}^n$ , and therefore  $\underline{v}(t, x) = \bar{v}(t, x)$  in  $(0, +\infty) \times \mathbb{R}^n$ , since the opposite inequality holds by definition of the semilimits. In particular, we get that  $v(t, x) := \bar{v}(t, x) = \underline{v}(t, x)$  is the unique bounded viscosity solution to (29) with initial data  $v_0$ .

This implies that  $\bar{u}(t, x) = \underline{u}(t, x) = u(t, x)$  is the unique continuous viscosity solution to (26) in the class  $\mathcal{L}_q$ , with initial datum  $u_0$ . Moreover, using the definition of relaxed limits, we obtain that  $u^\varepsilon(t, x) \rightarrow u(t, x)$  locally uniformly.  $\square$

We observe that Theorem 5.3 has also a geometric counterpart. Namely, given a solution  $u^\varepsilon$  to (24) we consider its graph  $\Gamma^\varepsilon(t) := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = u^\varepsilon(x, t)\}$ . Then,  $\Gamma^\varepsilon(t)$  evolves in time accordingly to (2). In the limit  $\varepsilon \rightarrow 0$ , we then obtain the geometric evolution

$$V = \alpha(\nu)H \quad (30)$$

where the function  $\alpha$  is defined as

$$\alpha(\nu) := \frac{A(-\nu_x/\nu_{n+1})}{\nu_{n+1}^2} \geq 0 \quad \nu_{n+1} \neq 0.$$

Note that, when  $n = 1$ , due to (20), we have  $0 < \alpha(\nu) \leq K_g |\nu_2|$ , which implies in particular  $\lim_{\nu_2 \rightarrow 0} \alpha(\nu) = 0$ .

As discussed for instance in [22], the asymptotic limit of (2) is strictly related to the existence of compact embedded solutions to the following prescribed curvature problem:

$$H + g = 0. \tag{31}$$

Indeed, the existence of a compact solution to (31) implies by rescaling the existence of compact solutions to

$$H + \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}\right) = 0,$$

which can be used as barriers for the evolution (2). It would then follow that the solutions to (2) converge, as  $\varepsilon \rightarrow 0$ , to a stationary hypersurface, thus implying  $\alpha \equiv 0$  in (30), that is,  $A(p) = 0$  for all  $p \in \mathbb{R}^n$ .

However, if  $g$  has zero average and does not depend on  $x_{n+1}$ , from Remark 3 we know that  $A(0) > 0$  if  $\|g\|_{\text{Lip}} < \delta$ ; moreover we know from (20) that  $A(p) > 0$  for all  $p$ , if  $n = 1$  and  $g$  satisfies (12). As a consequence, in such cases we conclude that there are no compact embedded solutions to (31).

We point out that this result cannot be expected for a generic function  $g$ , which is periodic and of zero average, but which depends also on  $x_{n+1}$ . Indeed, in [22] it has been proved that we can always find a sequence  $g_n \rightarrow g$  in  $L^1(\mathbb{R}^n)$ , where the functions  $g_n$  are all periodic, of zero average, uniformly bounded, and satisfy (11), such that there exist compact embedded solutions to (31) with  $g$  replaced by  $g_n$ .

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