Existence of isoperimetric sets in periodic media

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In Euclidean spaces, it is well known that hyperplanes are local minimizers of the perimeter and that balls are the (unique) solutions to the isoperimetric problem i.e. they have the least perimeter among all the sets having a given volume. The situation of course changes for interfacial energies which are no longer homogeneous nor isotropic but it is still natural to investigate the existence of local minimizers which are plane-like and of compact isoperimetric sets in this context. More precisely, for an open set $\Omega \subseteq \mathbb{R}^d$ and a set of finite perimeter E (see [1]), we will consider interfacial energies of the form

$$\mathcal{E}(E,\Omega) := \int_{\partial^* E\cap\Omega} F(x,\nu^E)d\mathcal{H}^{d-1}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, ν^E is the internal normal to E, $\partial^* E$ is the reduced boundary of E, and the function $F(x,p)$ is Lipschitz continuous and periodic in x, convex and one-homogeneous in p, and satisfies

$$c_0|p| \le F(x,p) \le c_0^{-1}|p| \qquad \forall (x,p)\in\mathbb{R}^d\times\mathbb{R}^d \tag{1}$$

for some $c_0 \in (0,1]$. When $\Omega = \mathbb{R}^d$, we will simply denote by $\mathcal{E}$, the functional $\mathcal{E}(\cdot,\mathbb{R}^d)$. Given a volume $v > 0$, we are interested in the following isoperimetric problem

$$\min_{|E|=v} \mathcal{E}(E). \tag{2}$$

We will show the existence of compact minimizers of problem (2). The idea is to use the Direct Method of the calculus of variations together with a kind of concentration compactness argument to deal with the invariance by translations of the problem. Notice that a similar strategy has been used to prove existence of minimal clusters (see [2, Th. 29.1]).
We first recall Almgren's Lemma (for the proof see [2, Lem. II.6.18]).

Lemma 0.1. *If E is a set of finite perimeter and A is an open set of $\mathbb{R}^d$ such that $\mathcal{H}^{d-1}(\partial^* E\cap A) > 0$ then there exists $\sigma_0 > 0$ and $C > 0$ such that for every $\sigma \in (-\sigma_0,\sigma_0)$ there exists a set F such that*

- $F\Delta E \Subset A$,
- $|F| = |E| + \sigma$,

- $|\mathcal{E}(F,A) - \mathcal{E}(E,A)| \leq C|\sigma|$.

We now prove that any minimizer (if it exists) is necessarily compact.

Proposition 0.2. *For every $v > 0$, every minimizer E of (2) has bounded diameter.*

Proof. The proof follows the classical method to prove density estimates for minimizers of isoperimetric problems (see for instance [1]). Fix $v > 0$ and let E be a minimizer of (2). Let then $f(r) := |E \backslash B_r|$. Let us assume that the diameter of E is not finite, so that $f(r) > 0$ for every $r > 0$. Without loss of generality, we can also assume that $\mathcal{H}^{d-1}(\partial^* E \cap B_1) > 0$. Let σ_0 and C be given by Lemma 0.1 with $A = B_1$, and fix $R > 1$ such that $f(R) \leq \sigma_0$ then for every $r > R$ there exists F such that

- $F \Delta E \Subset B_1$,
- $|F| = |E| + f(r)$,
- $|\mathcal{E}(E, B_r) - \mathcal{E}(F, B_r)| \leq C f(r)$.

Letting $G := F \cap B_r$ we have $|G| = |E|$ thus, by minimality of E, we find

$$\mathcal{E}(E) \leq \mathcal{E}(G) \leq \mathcal{E}(F, \overline{B}_r) + c_0^{-1}\mathcal{H}^{d-1}(\partial B_r \cap F) \leq \mathcal{E}(E, \overline{B}_r) + C f(r) + c_0^{-1}\mathcal{H}^{d-1}(\partial B_r \cap E)$$

and thus

$$c_0 \mathcal{H}^{d-1}(\partial^* E \backslash \overline{B}_r) \leq \mathcal{E}(E, \overline{B}_r^c) \leq C f(r) + c_0^{-1}\mathcal{H}^{d-1}(\partial B_r \cap E).$$

Recalling that $f'(r) = -\mathcal{H}^{d-1}(\partial B_r \cap E)$ and $\mathcal{H}^{d-1}(\partial^* E \cap \partial B_r) = 0$ for a.e. $r > 0$, we get

$$c_0 \mathcal{H}^{d-1}(\partial^*(E \backslash B_r)) = c_0 \mathcal{H}^{d-1}(\partial^* E \backslash \overline{B}_r) - c_0 f'(r) \leq C f(r) - \left(c_0 + c_0^{-1}\right) f'(r)$$

for a.e. $r > 0$. By the isoperimetric inequality [1] it then follows

$$c_1 f(r)^{\frac{d-1}{d}} \leq c_2 f(r) - f'(r)$$

for some constants $c_1, c_2 > 0$. If now $R_1 > R$ is such that $f(r)^{\frac{1}{d}} \leq \frac{c_1}{2c_2}$, we get

$$\frac{c_1}{2} f(r)^{\frac{d-1}{d}} \leq -f'(r)$$

and thus $\left(f^{1/d}\right)' \leq -\frac{c_1}{2}$, which leads to a contradiction. □

We now prove the existence of compact minimizers for every volume $v > 0$.

Theorem 0.3. *For every $v > 0$ there exists a compact minimizer of (2).*

Proof. To simplify the notations, let us assume that $v = 1$. Let E_k be a minimizing sequence meaning that $|E_k| = 1$ and $\mathcal{E}(E_k) \to \inf_{|E|=1} \mathcal{E}(E)$. For every $k \in \mathbb{N}$, let $\{Q_{i,k}\}_{i \in \mathbb{N}}$ be a partition of $\mathbb{R}^d$ into disjoint cubes of equal volume larger than 2, such that the sets $E_k \cap Q_{i,k}$

are of decreasing measure, and let $x_{i,k} = |E_k \cap Q_{i,k}|$. By the isoperimetric inequality, there exist $0 < c < C$ such that

$$\begin{aligned} c\sum_i x_{i,k}^{\frac{d-1}{d}} &= c\sum_i \min\left(|E_k \cap Q_{i,k}|, |Q_{i,k} \backslash E_k|\right)^{\frac{d-1}{d}} \\ &\le \sum_i P(E_k, Q_{i,k}) \\ &\le \sum_i c_0 \mathcal{E}(E_k, Q_{i,k}) \\ &\le c_0 \mathcal{E}(E_k) \le C \end{aligned}$$

hence

$$\sum_{i=1}^{\infty} x_{i,k} = 1 \qquad \text{and} \qquad \sum_{i=1}^{\infty} x_{i,k}^{\frac{d-1}{d}} \le \frac{C}{c}. \tag{3}$$

Since $x_{i,k}$ is nonincreasing with respect to i, from (3) it follows that

$$\sum_{i=N}^{\infty} x_{i,k} \;\le\; \frac{C}{c}\frac{1}{N^{1/d}} \qquad \text{for any } N \in \mathbb{N}. \tag{4}$$

Indeed, for all $N \in \mathbb{N}$ we have

$$\frac{1}{N} \;\ge\; \frac{1}{N}\sum_{i=1}^{N} x_{i,k} \;\ge\; \frac{1}{N}\sum_{i=1}^{N} x_{N,k} \;\ge\; x_{N,k}$$

which implies

$$\sum_{i=N}^{\infty} x_{i,k} \;=\; \sum_{i=N}^{\infty} x_{i,k}^{\frac{1}{d}}\, x_{i,k}^{\frac{d-1}{d}} \;\le\; x_{N,k}^{\frac{1}{d}} \sum_{i=N}^{\infty} x_{i,k}^{\frac{d-1}{d}} \;\le\; \frac{C}{c}\frac{1}{N^{1/d}}$$

and proves (4).

Up to extracting a subsequence, we can suppose that $x_{i,k} \to \alpha_i \in [0,1]$ as $k \to +\infty$ for every $i \in \mathbb{N}$, so that by (4) we have

$$\sum_i \alpha_i = 1. \tag{5}$$

Fix $z_{i,k} \in Q_{i,k}$. Up to extracting a further subsequence, we can suppose that $d(z_{i,k}, z_{j,k}) \to c_{ij} \in [0, +\infty]$, and

$$(E_k - z_{i,k}) \to E_i \quad \text{ in the } L^1_{\text{loc}}\text{-convergence}$$

for every $i \in \mathbb{N}$. And it is not very difficult to check that E_i are minimizers of (2) under the volume constraint $v_i := |E_i|$. Notice that by Proposition 0.2, each E_i is bounded.

We say that $i \sim j$ if $c_{ij} < +\infty$ and we denote by $[i]$ the equivalence class of i. Notice that E_i equals E_j up to a translation, if $i \sim j$. We want to prove that

$$\sum_{[i]} v_i = 1, \tag{6}$$

where the sum is taken over all equivalence classes. For all $R > 0$ let $Q_R = [-R/2, R/2]^d$ be the cube of sidelength R. Then for every $i \in \mathbb{N}$,

$$|E_i| \geq |E_i \cap Q_R| = \lim_{k\to+\infty} |(E_k - z_{i,k}) \cap Q_R| \,.$$

If j is such that $j \sim i$ and $c_{ij} \leq \frac{R}{2}$, possibly increasing R we have $Q_{j,k} - z_{i,k} \subseteq Q_R$ for all $k \in \mathbb{N}$, so that

$$\lim_{k\to+\infty} |(E_k - z_{i,k}) \cap Q_R| \geq \lim_{k\to+\infty} \sum_{c_{ij}\leq \frac{R}{2}} |E_k \cap Q_{j,k}| = \sum_{c_{ij}\leq \frac{R}{2}} \alpha_j.$$

Letting $R \to +\infty$ we then have

$$|E_i| \geq \sum_{i\sim j} \alpha_j$$

hence, recalling (5),

$$\sum_{[i]} |E_i| \geq 1,$$

thus proving (6) (since the other inequality is clear).
Let us now show that

$$\sum_{[i]} \mathcal{E}(E_i) \leq \inf_{|E|=1} \mathcal{E}(E). \tag{7}$$

Choosing a representative in each equivalence class $[i]$ and reindexing, from now on we shall assume that $c_{ij} = +\infty$ for all $i \neq j$. Let $I \in \mathbb{N}$ be fixed. Then for every $R > 0$ there exists $K \in \mathbb{N}$ such that for every $k \geq K$ and i, j less than I, we have

$$d(z_{i,k}, z_{j,k}) > R.$$

For $k \geq K$ we thus have

$$\begin{aligned}
\mathcal{E}(E_k) &\geq \sum_{i=1}^{I} \int_{\partial E_k \cap (B_R + z_{i,k})} F(x, \nu^{E_k})\, d\mathcal{H}^{d-1} \\
&= \sum_{i=1}^{I} \int_{\partial (E_k - z_{i,k}) \cap B_R} F(x, \nu^{E_k})\, d\mathcal{H}^{d-1} \\
&= \sum_{i=1}^{I} \mathcal{E}(E_k - z_{i,k}, B_R)
\end{aligned}$$

From this, and the lower-semicontinuity of $\mathcal{E}$, we get

$$\inf_{|E|=1} \mathcal{E}(E) \geq \sum_{i=1}^{I} \liminf_{k\to\infty} \mathcal{E}(E_k - z_{i,k}, B_R) \geq \sum_{i=1}^{I} \mathcal{E}(E_i, B_R).$$

Letting $R \to \infty$ and then $I \to \infty$ (if the number of equivalence classes is finite then just take I equal to this number), we find (7). Let finally $d_i := \mathrm{diam}(E_i)$ and $F := \bigcup_i (E_i + 2d_i e_1)$ where e_1 is a unit vector then $|F| = 1$ and

$$\mathcal{E}(F) = \sum_i \mathcal{E}(E_i) \leq \inf_{|E|=1} \mathcal{E}(E)$$

and thus F is a minimizer of (2) (notice that by Proposition 0.2, we must have $E_i = \emptyset$ for i large enough). $\square$

References

[1] E. Giusti, *Minimal Surfaces and functions of Bounded Variation*, Monographs in Mathematics, vol. 80, Birkhäuser, 1984.

[2] F. Maggi, *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*, Cambridge Studies in Advanced Mathematics, 135, Cambridge University Press, 2012.