Ideal Simplicial Volume

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RELATIVE SIMPLICIAL VOLUME & EXAMPLES

Definition. *Given a chain $c = \sum_{j=0}^{k} a_j \cdot \sigma_j \in C_*(M, \partial M; \mathbb{R})$, we define the ℓ^1-norm of c as $|c|_1 := \sum_{j=0}^{k} |a_j|$.*
Given a connected, compact and oriented manifold with boundary M, we define the **relative simplicial volume (RSV)** *of M, $\|M\|$, to be $\|M\| = \inf\{|c|_1 \mid c \in C_n(M, \partial M; \mathbb{R})$ such that $[c] = [M, \partial M]\} \in \mathbb{R}_{\geq 0}$, where $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$ is the relative fundamental class of M.*

Theorem ([1]). *If M is either a handlebody of genus $g \geq 2$ or it is the product of a surface with an interval, then its relative simplicial volume $\|M\|$ is proportional to $\|\partial M\|$.*

Theorem ([4]). *If M is the natural compactification of a complete finite-volume hyperbolic n-manifold, then $\|M\| = Vol(M)/v_n$.*

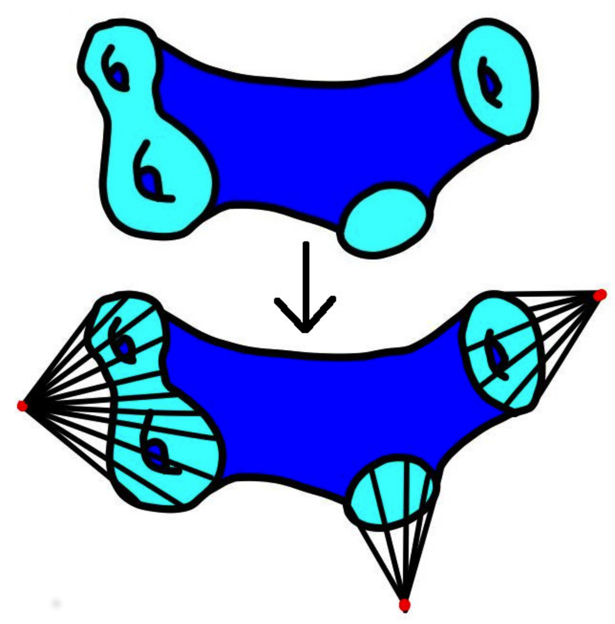
MARKED SPACES & IDEAL TRIANGULATIONS

Definition. *A* **marked space** *is a topological pair (X, B) such that B is a closed and discrete subset of X in which each point $b \in B$ has a closed neighbourhood in X homeomorphic to a topological cone.*

Definition. *Given a manifold with boundary M, we define the* **marked space associated to** *M, $(X, B)_M$, as follows: X is the topological quotient obtained from M by collapsing separately each connected component of ∂M, while B is the finite subset of X corresponding to the connected components of ∂M.*

Definition. *An* **ideal triangulation** *of M is the realization of $(X, B)_M$ as a Δ-complex whose set of vertices is equal to B.*

EXAMPLE



MARKED HOMOLOGY

Definition. *Given a marked space (X, B), a singular simplex $\sigma \colon \Delta^n \to X$ is said to be* **admissible** *if $\sigma^{-1}(B)$ is a (possibly empty) subcomplex of Δ^n. The set of* admissible *simplices in X defines a subcomplex $\widehat{C}_*^{\mathcal{M}}(X, B; \mathbb{R})$ of the singular complex $C_*(X; \mathbb{R})$ which contains $C_*(B; \mathbb{R})$. We define the* **marked chain complex** *of (X, B) to be $C_*^{\mathcal{M}}(X, B; \mathbb{R}) = \widehat{C}_*^{\mathcal{M}}(X, B; \mathbb{R})/C_*(B; \mathbb{R})$. The* **marked homology** *of (X, B), $H^{\mathcal{M}}(X, B; \mathbb{R})$, is the homology of the marked chain complex $(C_*^{\mathcal{M}}(X, B; \mathbb{R}), \partial_*)$.*

Theorem 1 ([2]). *There exists an isomorphism $\Psi_n \colon H_n(M, \partial M; \mathbb{R}) \xrightarrow{\cong} H_n^{\mathcal{M}}(X, B; \mathbb{R})$.*

IDEAL SIMPLICIAL VOLUME

Definition. *Let M be a connected, compact, oriented n-manifold with boundary. We define the* **ideal simplicial volume (ISV)** *of M, $\|M\|_{\mathcal{I}}$ to be $\|M\|_{\mathcal{I}} = \inf\{|c|_1 \mid c \in C_n^{\mathcal{M}}(M, \partial M; \mathbb{R})$ such that $[c] = [M, \partial M]^{\mathcal{M}}\} \in \mathbb{R}_{\geq 0}$, where $[M, \partial M]^{\mathcal{M}} \in H_n^{\mathcal{M}}(M, \partial M; \mathbb{R})$ is the marked fundamental class of M, that is $[M, \partial M]^{\mathcal{M}} = \Psi_n([M, \partial M])$.*

FUNDAMENTAL PROPERTIES

Theorem 2 ([2]). *The minimal number of simplices in an ideal triangulation of a compact manifold M, $c(M)$, provides an upper bound of the ideal simplicial volume:*

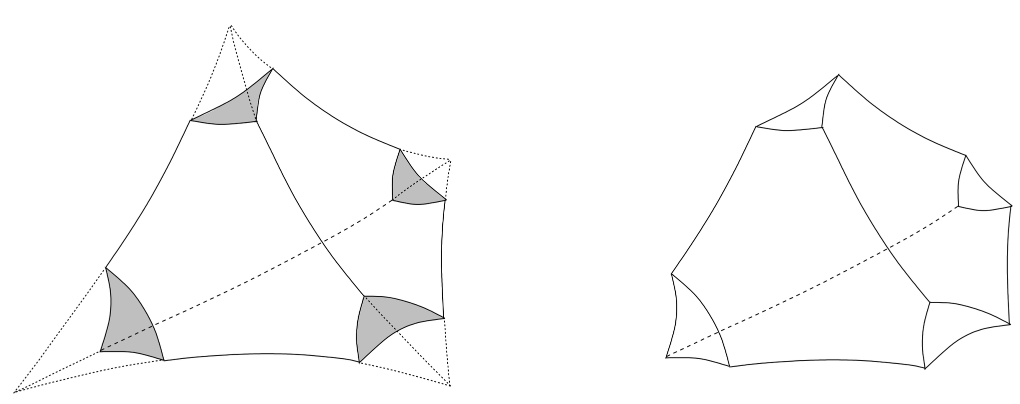
$$\|M\|_{\mathcal{I}} \leq c(M).$$

Theorem 3 ([2]). *Let $f \colon (M, \partial M) \to (N, \partial N)$ be a map of pairs between compact connected oriented manifolds of the same dimension. Then, the following holds:*

$$\|M\|_{\mathcal{I}} \geq |deg(f)| \|N\|_{\mathcal{I}}.$$

In particular, the ideal simplicial volume is a **homotopy invariant** *of manifolds with boundary (where homotopies are understood to be homotopies of pairs).*

A HYPERBOLIC TRUNCATED TETRAHEDRON



ISV VERSUS RSV

Theorem 4 ([2]). *There exists a positive constant K_n only depending on the dimension n of M such that the following holds:*

$$\|M\|_{\mathcal{I}} \leq \|M\| \leq K_n \|M\|_{\mathcal{I}}.$$

In particular, $\|M\|_{\mathcal{I}} = 0$ if and only if $\|M\| = 0$.

Theorem 5 ([2]). *Let M be an n-dimensional compact manifold such that each of its boundary components has* **amenable** *fundamental group. Then,*

$$\|M\|_{\mathcal{I}} = \|M\|.$$

In particular, when M is the compactification of a complete finite-volume hyperbolic n-manifold, then $\|M\|_{\mathcal{I}} = \frac{Vol(M)}{v_n}$.

NEW COMPUTATIONS

Theorem 6 ([3]). *Let $\ell \leq cosh^{-1}((3 + \sqrt{3})/4)$. Then, regular hyperbolic truncated tetrahedra of edge length ℓ maximize the volume among hyperbolic truncated tetrahedra whose edge lengths are all not smaller than ℓ.*

The previous technical result is a fundamental step in the computation of the ISV of an infinite family of hyperbolic 3-manifolds with geodesic boundary, for which the exact value of the RSV is not known:

Theorem 7 ([2]). *For any hyperbolic 3-manifolds M with connected geodesic boundary of genus $\mathrm{g} \geq 2$, we have*

$$\|M\|_{\mathcal{I}} \geq \mathrm{g}.$$

Moreover, the equality holds if and only if M admits an ideal triangulation with **g** *ideal simplices.*

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