CHAPTER 1

Preliminaries: the higher-dimensional heaven

We start by recalling some basic notions of differential topology (such as manifolds, fiber bundles, transversality, immersions) and homotopy. We then introduce handle decompositions and use them quite intensively to describe Smale's proof of Poincaré Conjecture in dimension $\geqslant 6$.

The proof of Poincaré Conjecture in high dimension introduces various techniques, some of which are valid only in dimension $\geqslant 5$ or $\geqslant 6$.

1. Manifolds and fiber bundles

1.1. Manifolds. A *topological manifold* is a topological paracompact Hausdorff space M locally homeomorphic to $\mathbb{R}^n$. A *differentiable structure* on M is an open covering $\{U_\alpha\}$ of M and a set of homeomorphisms

$$f_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$$

called *charts* with open sets V_α of $\mathbb{R}^n$, that are pairwise *compatible*. That is, the map $f_\alpha \circ f_\beta^{-1}$ is a diffeomorphism wherever it makes sense. Two differentiable structures are equivalent if their union is again a differentiable structure.

If $n = 2k$ and the diffeomorphisms are actually biolomorphisms of open sets in $\mathbb{C}^k = \mathbb{R}^{2k}$, the manifold has a *complex structure*. Recall that a biolomorphism is a diffeomorphism whose differential in every point is $\mathbb{C}$-linear.

Differentiable (holomorphic) maps between smooth (complex) manifolds are defined as maps that are locally differentiable (holomorphic) when transported along charts into open sets of $\mathbb{R}^n$ ($\mathbb{C}^n$).

A (topological or smooth) *manifold with boundary* is defined as above, with a half-space $\mathbb{R}^n_+$ instead of $\mathbb{R}^n$. A *closed manifold* is a compact manifold without boundary.

1.2. Fiber bundles. A *fiber bundle* is a smooth surjective map $\pi : E^{n+k} \to B^n$ between manifolds which is locally a product. More precisely, there is another manifold F^k, a covering of B into open sets, and for each such open set U the map $\pi : E \to U$ looks locally like the projection $p : F \times U \to U$. That is, there is a diffeomorphism $\psi : \pi^{-1}(U) \to F \times U$ such that $p \circ \psi = \pi$.

A *section* of a bundle is a smooth map $\sigma : B \to E$ such that $\pi \circ \sigma$ is the identity. As the example below shows, some bundles do not have sections. A *trivial bundle* is a bundle $B \times F \to B$.

EXAMPLE 1.1. The *Hopf fibration* $\pi : S^3 \to S^2$ is defined by considering S^3 as the unit sphere $|z|^2 + |w|^2 = 1$ in $\mathbb{C}^2 = \{(z, w)\}$ and taking

$$\pi \colon (z, w) \longmapsto [z, w] \in \mathbb{CP}^1 \cong S^2.$$

This map is indeed a fiber bundle with $F^1 = S^1$. If $(z,w) \in S^3$, the fiber $\pi^{-1}(\pi(z,w))$ is the circle $(e^{i\theta}z, e^{i\theta}w)$.

The Hopf fibration has no sections. If σ were a section, the map $\pi \circ \sigma$ would be the identity and hence $\sigma_* : \pi_2(S^2) \to \pi_2(S^3)$ would be injective, which is impossible since $\pi_2(S^2) = \mathbb{Z}$ and $\pi_2(S^3)$ is trivial.

1.3. Vector bundles. A *real vector bundle* is a fiber bundle where each fiber $f^{-1}(x)$ has a structure of $\mathbb{R}$-vector space, which varies smoothly on x. That is, we require that $F = \mathbb{R}^k$ and the local trivializations $\psi : \pi^{-1}(U) \to F \times U$ restrict to isomorphisms of vector spaces on fibers. A *complex vector bundle* is defined analogously. The dimension k is the *rank* of the vector bundle.

On a vector bundle we usually identify B with the image of the *0-section*, *i.e.* the section $\sigma(x) = 0$. Some h sections are *independent* if they give h independent vectors at each point (in particular, they are nowhere zero). A *trivialization* of a vector bundle of rank k is the choice of k independent sections.

Two vector bundles $\pi : E \to B$ and $\pi' : E' \to B$ are *isomorphic* if there is a diffeomorphism $\psi : E \to E'$ with $\pi = \pi' \circ \psi$ which is an isomorphism on fibers. A trivialization is in fact an isomorphism with the trivial vector bundle $B \times \mathbb{R}^k \to B$. Many vector bundles do not have trivializations, and this leads to the definition of various *characteristic classes*[1].

1.4. Operations on vector bundles. Every operation on vector spaces extend easily to vector bundles. That is, given two vector bundles $E \to B$ and $E' \to B$, we may construct new vector bundles such as E^*, $\mathrm{Hom(E,E')}$, $E \oplus E'$, $E \otimes E'$, etc. A *subbundle* of $\pi : E \to B$ is a subset $E' \subset E$ such that $\pi|_{E'}$ is a vector bundle. A *quotient bundle* $E/E' \to B$ is then defined.

A vector bundle $E \to B$ may be *restricted* to a submanifold $B' \subset B$. More generally, for any smooth map of manifolds $f : B' \to B$, the vector bundle $E \to B$ defines a *pull-back* $E' \to B'$, denoted by f^*E. It is defined as

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\}.$$

A *riemannian metric* on a real vector bundle $E \to B$ is a positive definite scalar product on each fiber $f^{-1}(x)$, which varies smoothly on x (that is, it is a section of $E^* \otimes E^*$). A riemannian metric on a real vector bundle of rank k defines a sphere bundle (with $F = S^{k-1}$, by taking the unitary vectors) and a disc bundle (with $F = D^k$, by taking vectors of norm $\leqslant 1$). By using a partition of unity for B, it is always possible to assign a riemannian metric to a vector bundle.

An *orientation* on a real vector bundle $E \to B$ is an orientation on each $f^{-1}(x)$ which varies continuously on x. An orientation might not exist, and in this case the bundle is *not orientable*.

1.5. Structure group. A fiber bundle $\pi : E^{n+k} \to B^n$ has *structure group* G if G is a fixed group of homeomorphism of the abstract fiber F^k and there is an open covering U_α of B and trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to F \times U_\alpha$ that match along fibers via elements of G. That is, for every $x \in U_\alpha \cap U_\beta$ the map

$$\psi_\alpha \circ \psi_\beta^{-1}|_{\psi_\beta(\pi^{-1}(x))} : F \longrightarrow F$$

[1]Some of which are introduced in Section 4.

is an element of G. A vector bundle may be defined as a fiber bundle with fiber $\mathbb{R}^n$ and structure group $\mathrm{GL}_n(\mathbb{R}^n)$. A structure group may sometimes be *reduced* to a subgroup $G' < G$ by choosing more appropriate local trivializations.

PROPOSITION 1.2. *The group structure of a vector bundle E may always be reduced to $O(n)$.*

PROOF. Pick a riemannian metric for E and choose local isometric trivializations with $U \times \mathbb{R}^n$. The matching functions must preserve the metric, and thus lie in $O(n)$. □

A vector bundle is orientable if and only if its structure group may be reduced to $SO(n)$. One such reduction fixes an orientation for E.[2]

1.6. Tangent bundle. Let M^n be a smooth manifold, whose differential structure is determined by some charts $f_\alpha : U_\alpha \to V_\alpha$. The *tangent bundle* may be defined as follows. For every α, take the trivial bundle $V_\alpha \times \mathbb{R}^n$. Wherever $h = f_\alpha \circ f_\beta^{-1}$ makes sense, identify the corresponding portions of $V_\beta \times \mathbb{R}^n$ and $V_\alpha \times \mathbb{R}^n$ by gluing (x, v) to $(h(x), dh_x(v))$. The resulting object is naturally a vector bundle over M.

The tangent bundle is denoted by $TM \to M$ and the fiber over a point x is the *tangent space* T_xM. A smooth function $f : M \to N$ induces a bundle map $f : TM \to TN$.

1.7. Riemannian metric. A *riemannian metric* on M is a metric on the tangent bundle. A riemannian metric is a very powerful object, which furnishes a large series of useful geometric entities, such as geodesics, angles, lengths, volumes, and a (Levi-Civita) connection for M.

It also yields the *exponential map* at each point $x \in M$, a map $\exp_x : T_x \to M$ with $\exp_x(0) = x$ which is locally invertible in a neighborhood of 0. The tangent space thus gives a local approximation of a neighborhood of x, as one would expect[3].

1.8. Normal bundle. A *submanifold* of a smooth manifold M is a subset $X^h \subset M^m$ which is locally like a linear h-subspace S in $\mathbb{R}^m$. That is, every point $x \in X$ has an open neighborhood U in M such that $(U, U \cap X)$ is diffeomorphic to $(\mathbb{R}^m, S)$. Of course, the submanifold X inherits a structure of h-dimensional manifold.

The tangent bundle TX is naturally a subbundle of the restriction $TM|_X$ of the tangent bundle TM on X. The *normal bundle* $N_X \to X$ is the quotient $TM|_X/TX$. More concretely, a riemannian metric on M identifies N_X with the subbundle of $TM|_X$ orthogonal to TX.

Moreover, if X is compact, it is possible to extend the exponential function to the whole of TX and map (with a diffeomorphism) the ϵ-disc bundle of N_X onto an open neighborhood of X in M, called *tubolar neighborhood*. A tubolar neighborhood is therefore diffeomorphic to N_X.

[2] This may be taken as a definition for orientability.

[3] Actually, the exponential map is defined in the whole of T_x only if M is *complete*, which is always the case when M is closed. Otherwise, it is only defined on an open neighborhood of 0.

2. Transversality and immersions

2.1. Transversality. Two vector subspaces U, V in $\mathbb{R}^n$ are *transverse* if $U + V = \mathbb{R}^n$. Two smooth submanifolds $N_1, N_2 \subset M$ are *transverse* if their tangent spaces intersect transversely. That is, for every $x \in N_1 \cap N_2$, the tangent spaces T_xN_1 and T_xN_2 are transverse in T_xM. Note that two disjoint submanifolds are transverse!

PROPOSITION 2.1. *Two transverse manifolds N_1 and N_2 intersect in a smooth submanifold.*

PROOF. The exponential map (for some riemannian metric) furnishes an open neighborhood U of every point $x \in N_1 \cap N_2$ such that $(U, U \cap N_1, U \cap N_2) \cong (\mathbb{R}^n, V_1, V_2)$ for some transverse linear subspaces V_1, V_2. □

Two oriented transverse manifolds N_1 ad N_2 in an oriented manifold M intersect in points. At each such point x, compare the orientations of $T_xN_1 \oplus T_xN_2$ and T_xM. If they coincide, assign a sign $+1$. If not, assign -1. The sum of these numbers is the *algebraic intersection number* of N_1 and N_2 and we denote it by $N_1 \cdot N_2$. The *geometric* intersection number is just the number of intersection points.

More generally, vector subspaces $V_1, \ldots, V_k$ in $\mathbb{R}^n$ are transverse if

$$\mathrm{codim}(V_1 \cap \ldots \cap V_k) = \mathrm{codim} V_1 + \ldots + \mathrm{codim} V_k.$$

and the notion of transversality may be extended to an arbitrary number of manifolds.

2.2. Immersions. Let M be a compact manifold. A smooth map $f : M \to N$ is an *immersion* if the differential df_x is injective for every $x \in M$. If f is also injective, the function f is an *embedding*[4]. In that case, the image $f(M)$ is a smooth submanifold of N and $f|_M : M \to f(M)$ is a diffeomorphism.

THEOREM 2.2 (Whitney, weak immersion). *Let $f : M^m \to N^n$ be a continuous map between smooth manifolds. Let M^m be compact.*

- *There is a map $g : M^m \to N^n$ arbitrarily close to f which is smooth.*
- *If $m < 2n$, there is a map g arbitrarily close to f which is an embedding.*
- *If $m = 2n$, there is a map g arbitrarily close to f which is an immersion.*

In all cases, "arbitrarily close" means that for any riemannian metric on N and any $\epsilon > 0$ there is a g homotopic to f with $d(f(x), g(x)) < \epsilon$ for all x.

Whitney also proved a strengthened version of the theorem concerning immersions in $\mathbb{R}^N$.

THEOREM 2.3 (Whitney, strong immersion). *Every closed manifold M^n embeds in $\mathbb{R}^{2n}$ and immerses in $\mathbb{R}^{2n-1}$.*

2.3. Generic immersions. An immersion $f : M^m \to N^n$ is *generic* if for every $y \in N^n$ the set $f^{-1}(y) = \{x_1, \ldots, x_k\}$ is finite and the vector subspaces $\mathrm{Im}\, df_{x_1}, \ldots, \mathrm{Im}\, df_{x_k}$ are transverse in T_yN^n. Every immersion is arbitrarily close to a generic one. Some examples follow.

Figure.

Verificare! e figure.

2.3.1. *Surfaces.* A generic immersion of a surface in a 3-manifold self-intersects into double and triple points. Note that Theorem 2.2 does not apply here: some maps from S^2 to $\mathbb{R}^3$ cannot be perturbed to immersions.

A generic immersion of a surface in a 4-manifold self-intersects into finitely many isolated double points. Each double point is locally like $\{zw = 0\}$ in $\mathbb{C}^2 = \{(z, w)\}$.

2.3.2. *Higher-dimensional manifolds.* By Whitney's weak theorem, for every pair of manifolds M^m and N^{2m} there is a generic immersion of M^m in N^{2m}. Such an immersion self-intersects in finitely many double points. By Whitney's strong immersion theorem, every compact manifold M^n also has a generic immersion in $\mathbb{R}^{2n-1}$. If $n \geqslant 3$, this immersion self-intersects only in double points, which form compact 1-manifolds in $\mathbb{R}^{2n-1}$, *i.e.* circles.

2.4. Isotopy. In various cases, submanifolds can be put in transverse position up to arbitrarily small perturbations. Perturbation in this case means isotopy.

An *isotopy* between two embeddings $f_0, f_1 : M \to N$ is a smooth homotopy $F : M \times [0,1] \to N$ relating them such that each level $f_t = F(\cdot, f)$ is an embedding. An *ambient isotopy* is an isotopy induced by an isotopy of the ambient space N; that is, there is an isotopy between $\mathrm{id} : N \to N$ and some diffeomorphism $\psi : N \to N$ such that $\psi \circ f_0 = f_1$. Two embeddings are in fact isotopic if and only if they are ambient isotopic.

Mettere funzioni trasverse rispetto a sottovarietá

2.5. Cut and paste. Let $N \subset M$ be a closed smooth submanifold. The operation of *cutting* M *along* N consists of removing from M an open tubolar neighborhood of N.

Let M be a (possibly disconnected) smooth manifold with boundary of dimension n. Let $N_1, N_2 \subset \partial M$ be two compact $(n-1)$-submanifolds of ∂M, possibly with boundary. (For instances, these might be two boundary components of M.) Let $\varphi : N_1 \to N_2$ be a diffeomorphism. Let $M/_\varphi$ be the topological space obtained by identifying N_1 and N_2 along φ. Such a space is a topological manifold, and is also equipped with a differential structure (which depends only on M and φ). We say that $M/_\varphi$ is obtained by *gluing* M along the map φ.

If M is oriented, then N_1 and N_2 also are. If φ *reverses* orientations, then there is a canonical orientation on $M/_\varphi$ coherent with the one on M.

3. Homology and homotopy groups

We assume that the reader is familiar with basic algebraic topology, including homotopy groups and the main (co-)homology theories of topological spaces (cellular homology, singular homology, De Rham cohomology).

3.1. Homology. Integral homology $H_*(M, \mathbb{Z})$ and cohomology $H^*(M, \mathbb{Z})$ are essential tools in dimension 4. We recall some important facts.

Theorem 3.1. *Let M^n be a compact manifold, possibly with boundary.*

(1) *Every $H_i(M, \mathbb{Z})$ is a finitely generated abelian group, which thus decomposes as $F_i \oplus T_i$, with F_i free and T_i torsion. Analogously, $H^i(M, \mathbb{Z}) \cong F^i \oplus T^i$. Such groups are trivial when $i > n$.*
(2) *If M is connected then $H_0(M, \mathbb{Z}) \cong H^0(M, \mathbb{Z}) \cong \mathbb{Z}$.*
(3) *We have $F_i \cong F^i$ and $T_i \cong T^{i+1}$ for all i.*

[4] If M is not compact we also need to require that $f|_M : M \to f(M)$ is a homeomorphism.

(4) *If M is closed and oriented, there is a canonical isomorphism between $H_i(M,\mathbb{Z})$ and $H^{n-i}(M,\mathbb{Z})$.*

If there is no torsion, the groups $H_i(M,\mathbb{Z})$ and $H^i(M,\mathbb{Z})$ are thus isomorphic; the isomorphism is however not canonical[5].

The properties listed allow us to describe the (co-)homology groups of closed oriented 2-, 3-, and 4-manifolds.

Two-manifolds:

i	0	1	2
H^i	$\mathbb{Z}$	F_1	$\mathbb{Z}$
H_i	$\mathbb{Z}$	F_1	$\mathbb{Z}$

Three-manifolds:

i	0	1	2	3
H^i	$\mathbb{Z}$	F_1	F_1+T_1	$\mathbb{Z}$
H_i	$\mathbb{Z}$	F_1+T_1	F_1	$\mathbb{Z}$

Four-manifolds:

i	0	1	2	3	4
H^i	$\mathbb{Z}$	F_1	F_2+T_1	F_1+T_1	$\mathbb{Z}$
H_i	$\mathbb{Z}$	F_1+T_1	F_2+T_1	F_1	$\mathbb{Z}$

This shows in particular that the homology of a closed oriented 4-manifold is governed by three groups F_1, T_1, and F_2. If the manifold is simply connected, the first two groups vanish and all (co-)homology groups are determined by the free abelian group $F_2 \cong H_2(M,\mathbb{Z})$. Moreover, we have $\chi(M) = 2 + b_2$, where $b_2 = \dim F_2$ is the second Betti number of M.

If M is not simply connected the second homology may have torsion, in contrast with dimension three.

3.2. Homotopy. A continuous map $f : X \to Y$ between topological spaces is a *homotopy equivalence* if there is a $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are both homotopic to identities. A homotopy equivalence induces isomorphisms on all (co-)homology and homotopy groups. Conversely, we have the following (due to Whitehead).

THEOREM 3.2 (Whitehead, homotopy). *Let $f : X \to Y$ be a continuous map between CW-complexes. It is a homotopy equivalence if and only if it induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ on all homotopy groups.*

Integral homology suffices if the spaces are simply connected.

THEOREM 3.3 (Whitehead, homology). *Let $f : X \to Y$ be a continuous map between simply connected CW-complexes. It is a homotopy equivalence if and only if it induces isomorphisms $f_* : H_n(X,\mathbb{Z}) \to H_n(Y,\mathbb{Z})$ on all homology groups.*

A *homology sphere* is a closed n-manifold N having the same $\mathbb{Z}$-homology as a sphere S^n, that is having $H_i(N,\mathbb{Z}) = \{e\}$ for all $i = 1,\dots,n-1$. A *homotopy sphere* is a closed manifold N homotopically equivalent to S^n. A homotopy sphere

[5]This holds for every CW-complex M.

is also a homology sphere, while the converse does not hold[6]. The only obstruction actually lies in the fundamental group.

Proposition 3.4. *A simply connected homology sphere is a homotopy sphere.*

Proof. For every oriented manifold M^n there is a map $f : M^n \to S^n$ which induces an isomorphism $f_* : H_n(M) \to H_n(S^n)$.[7] Simply take a disc $D^n \subset M^n$ and the projection $f : D^n \to S^n$ which quotients ∂D^n to a point. Extend $f : M^n \to S^n$ by sending the rest of M to this point.

The map f_* is an isomorphism on homology groups and by Whitehead homology theorem it is a homotopy equivalence. □

The following result is also useful. A map $f : \pi_n(X) \to H_n(X, \mathbb{Z})$ is naturally defined for all n.

Theorem 3.5 (Hurewicz). *Let X be a connected CW-complex. If $\pi_1(X) = \ldots = \pi_n(X) = 0$ for some $n \geqslant 1$, then f_{n+1} is an isomorphism.*

3.3. Eilenberg-MacLane spaces. Let G be a finitely generated group and $n \geqslant 1$ a natural number. An *Eilenberg-MacLane space* $K(G, n)$ is any CW-complex X whose homotopy groups are all trivial, except that $\pi_n(X) \cong G$. If $n \geqslant 2$, we of course require G to be abelian. An Eilenberg-MacLane exists for every G and n and is unique up to homotopy equivalence.

Example 3.6. We have $K(\mathbb{Z}, 1) = S^1$. Let $\mathbb{RP}^\infty$ ($\mathbb{CP}^\infty$) denote the infinite real (complex) projective space, obtained by taking the union of all inclusions $\mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \ldots$ ($\mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \ldots$). It has a CW complex structure with one cell for each (even) dimension. We have $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$ and $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$.

If G is an abelian group, then $H^n(K(G, n), G)$ is canonically isomorphic to $G^* = \mathrm{Hom(G, G)}$ and has a preferred element u, corresponding to identity.[8] Therefore a CW complex X and a map $f : X \to K(G, n)$ yield an element $f^*(u) \in H^n(X, G)$.

Theorem 3.7. *Let G be an abelian group and X be a CW-complex. The above construction gives a bijection*

$$[X, K(G, n)] \longrightarrow H^n(X, G)$$

where $[X, K(G, n)]$ is the set of continuous maps $f : X \to K(G, n)$ up to homotopy.

Corollary 3.8. *The sets $H^1(X, \mathbb{Z}), H^1(X, \mathbb{Z}_2), H^2(X, \mathbb{Z})$ are in natural bijection with $[X, S^1], [X, \mathbb{RP}^\infty], [X, \mathbb{CP}^\infty]$.*

4. Handle decompositions

4.1. Handles. Let M^n be a compact manifold with boundary. A *k-handle* over M is the operation of gluing a n-disc $D^k \times D^{n-k}$ to M along an embedding $\varphi : S^{k-1} \times D^{n-k} \to \partial M^n$. The result is a new manifold

$$M' = M \cup_\varphi (D^k \times D^{n-k}).$$

[6]The first known example of homology sphere which is not a homotopy sphere was discovered by Poincaré, and is called the *Poincaré homology sphere*.

[7]Such a map has *degree one*. Therefore every manifold has degree-one maps onto the sphere. In general, given two manifolds, there might be no degree-one map between them.

[8]Here we are actually only interested in the case $G = \mathbb{Z}$: for such a G, this is a consequence of Hurewicz lemma.

The integer $k \geqslant 0$ is the *index* of the handle. Such a move modifies the topology of both M and ∂M. The effect on the boundary ∂M consists of substituting an embedded (along φ) copy of $S^{k-1} \times D^{n-k}$ with a copy of $D^k \times S^{n-k-1}$ (both objects share the same boundary $S^{k-1} \times S^{n-k-1}$). This operation is called a *surgery.*

The disc $D^k \times 0$ is the *core disc*, and its boundary $S^{k-1} \times 0$ is the *attaching sphere*. Dually, the disc $0 \times D^{n-k}$ is the *cocore disc* and its boundary $0 \times S^{n-k-1}$ is the *belt (or cocore) sphere*.

In particular, a 0-handle is a n-disc which is not attached to any manifold. Conversely, a n-handle is a n-disc which is attached along its entire boundary.

4.2. Handle decompositions. A *handle decomposition* of a compact manifold M (possibly with boundary) is a description of M as a result of the attaching of finitely many handles:

$$M = H_1 \cup_{\varphi_2} H_2 \cup_{\varphi_3} \ldots \cup_{\varphi_h} H_h.$$

That is, H_1 is a 0-handle, and the handle H_{i+1} is attached to the manifold

$$M_j = H_1 \cup_{\varphi_2} H_2 \cup_{\varphi_3} \ldots \cup_{\varphi_j} H_j$$

via some map φ_{j+1}. We often omit the maps for simplicity and write

$$M = H_1 \cup \ldots \cup H_h.$$

If M is closed, the last handle H_h is necessarily a n-handle.

The $(n-1)$-dimensional closed manifold ∂M_j is a *level manifold.* The attaching sphere of H_{j+1} is contained in the level manifold ∂M_j. Every level manifold ∂M_{j+1} is obtained from the previous one ∂M_j by surgery along the attaching sphere of H_i.

Morse theory shows the following.

THEOREM 4.1. *Every smooth manifold may be described via some handle decomposition.*

Such a decomposition is however not unique, and we now describe some moves that relate different decompositions of the same manifold.

4.3. Reordering handles. Let H_j, H_{j+1} be two subsequent handles of index k_j, k_{j+1}. Suppose $k_j \geqslant k_{j+1}$. These two handles can be permuted, as we now show.

The level manifold ∂M_j contains the attaching sphere of H_{j+1} and the belt sphere of H_j. These have dimensions respectively $k_{j+1} - 1$ and $n - k_j - 1$. The sum of the dimensions is strictly smaller than $n - 1 = \dim M_j$. Therefore the two spheres can be separated by a small isotopy. The handles H_j and H_{j+1} intersect the level manifolds into two tubolar neighborhoods of such spheres: if the spheres can be made disjoint by an isotopy, their neighborhoods can too.

We can therefore attach H_j and H_{j+1} to two disjoint regions of ∂M_{j-1}. In particular, we can reorder the two handles, by attaching first H_{j+1} and then H_j.

As a consequence, we can reorder the handles of any handle decomposition so that their indexes are non-decreasing. We first take all the 0-handles, then we attach (either simultaneously, or with an arbitrary ordering) all the 1-handles, and so on. We write an ordered handle decomposition as

$$M = (H^0_1 \cup \ldots \cup H^0_{i_0}) \cup \left(H^1_1 \cup \ldots \cup H^1_{i_1}\right) \cup \ldots \cup \left(H^n_1 \cup \ldots H^n_{i_n}\right)$$

where H^j_i is the i-th handle of index j. We also re-define

$$M_j = (H^0_1 \cup \ldots \cup H^0_{i_0}) \cup \left(H^1_1 \cup \ldots \cup H^1_{i_1}\right) \cup \ldots \cup \left(H^j_0 \cup \ldots H^j_{i_j}\right)$$

and the level manifold ∂M_j. All handles of index $j+1$ are attached simultaneously to ∂M_j.

4.4. Turning a decomposition upside-down. A handle decomposition

$$M = (H^0_1 \cup \ldots \cup H^0_{i_0}) \cup \big(H^1_1 \cup \ldots \cup H^1_{i_1}\big) \cup \ldots \cup \big(H^n_1 \cup \ldots H^n_{i_n}\big)$$

of a closed manifold can be turned upside-down. Every n-handle may be interpreted as a 0-handle, every $(n-1)$-handle can then be interpreted as a 1-handle attached to the former n-handles, and so on. Every k-handle $D^k \times D^{n-k}$ may be interpreted as a $(n-k)$-handle attached to the (former) higher handles by simply permuting its factors.

4.5. Intersection numbers. Consider an ordered handle decomposition

$$M = (H^0_1 \cup \ldots \cup H^0_{i_0}) \cup \big(H^1_1 \cup \ldots \cup H^1_{i_1}\big) \cup \ldots \cup \big(H^n_1 \cup \ldots H^n_{i_n}\big).$$

The level manifold ∂M_j contains all the attaching j-spheres of the $(j+1)$-handles, and the belt $(n-j-1)$-spheres of the j-handles. Attaching and belt spheres have complementary dimensions in ∂M_j. We can therefore suppose that they intersect transversely in a finite number of points.

Consider two handles H^j_a and H^{j+1}_b. The belt sphere S_a of H^j_a and the attaching sphere S_b of H^{j+1}_b are oriented. A tubolar neighborhood of the belt sphere in ∂M_j is also oriented by H^j_a. Such a neighborhood contains all the transverse intersections of the two oriented spheres. Since everything is oriented, we have a well-defined algebraic intersection $S_a \cdot S_b$. We set

$$i(H^j_a, H^{j+1}_b) = S_a \cdot S_b.$$

This is the *algebraic intersection number* of the two handles. The *geometric* intersection number is the geometric intersection number of S_a and S_b, *i.e.* simply the number of these transverse intersections.

We can construct a homology theory, similar to cellular homology. Let C_j be the $\mathbb{Z}$-module generated by the k-handles. Define a boundary map $\partial_{j+1} : C_{j+1} \to C_j$ via

$$\partial_{j+1}(H^{j+1}_b) = \sum_{a=0}^{i_j} i(H^j_a, H^{j+1}_b) H^j_a.$$

We have $\partial_j \circ \partial_{j+1} = 0$ and the resulting homology is isomorphic to the standard homology $H_*(M, \mathbb{Z})$.[9]

4.6. Canceling pair. Two handles H^j_a and H^{j+1}_b with *geometric* intersection 1 can be canceled, since they somehow annihilate each other. This can be seen as follows.

Tentare di semplificare la dimostrazione.

We have $H^j_a = D^j \times D^{n-j}$ and $H^{j+1}_b = D^{j+1} \times D^{n-j-1}$. The belt sphere $0 \times S^{n-j-1}$ and the attaching sphere $S^j \times 0$ intersect transversely in a point, say $0 \times p \sim q \times 0$. Therefore we can suppose that the handles intersect in $D^j \times D(p)$ and $D(q) \times D^{n-j-1}$ respectively, where $D(p) \subset S^{n-j-1}$ and $D(q) \subset S^j$ are discs centered in p and q. Let $D_*(p), D_*(q)$ be their complementary discs in S^{n-j-1}, S^j.

We first attach H^{j+1}_b and then H^j_a and prove that at each stage nothing changes. The handle H^{j+1}_b is attached along $D(q) \times D^{n-j-1}$ to ∂H^j_a and along $D_*(q) \times D^{n-j-1}$

[9] By "shrinking" all the handles to their cores we may construct a CW-complex homotopically equivalent to M, whose cellular homology coincides with the homology just defined.

to the rest of ∂M_j. If we first attach H_b^{j+1} we do nothing, since we attach a n-disc along a $(n-1)$-subdisc of the boundary. Next, it remains to attach H_a^j: the handle has to be attached along $S^{j-1} \times D^{n-j} \cup D^j \times D(p)$. This is again a disc, since it is the complement in $\partial(D^j \times D^{n-j})$ of the disc $D^j \times D_*(q)$. Again, we have attached a n-disc along a $(n-1)$-disc in its boundary, which has no effect.

Conversely, we may create two canceling pair of j- and $(j+1)$-handles which do not intersect geometrically any other handle.

4.7. Handle slide. We may slide a j-handle H_b^j over another j-handle H_a^j. As a result, the handle H_b^j is replaced by a new j-handle $H_b'^j$. This operation is defined as follows.

Instead of attaching them simultaneously, let us first attach H_a and then H_b. Let then N be the level manifold between them. It contains the belt sphere of H_a and the attaching sphere of H_b. These have dimension respectively $n-k-1$ and $k-1$. Note that the sum is $n-2$ and the ambient manifold N has dimension $n-1$. Choose an arc joining two points of the two spheres. We can slide the attaching sphere along this arc, so that at the end it crosses through the belt sphere.[10] By sliding a tubolar neighborhood of the attaching sphere, we in fact isotope the handle H_b. The result is a new handle H_a'.

This operation may be also encoded as follows. Recall that both H_a^j and H_b^j are attached to ∂M_{j-1}. Let S_a and S_b be their attaching spheres in ∂M_{j-1}. Take
figura
an arc in joining the two spheres, and slide S_b along it as shown in the figure. The resulting sphere S_b' is the attaching sphere of H_b'. As a result, we have

$$i(H_c, H_b') = i(H_c, H_b) \pm i(H_c, H_a)$$

for every $(j-1)$-handle H_c, with sign depending upon whether the orientations of the two spheres match along the joining arc. This depends on how we decide to
Ri-figura.
slide S_b, as the figure suggests.

4.8. Killing 0-handles. We prove the following.

Proposition 4.2. *Every compact connected manifold M has a handle decomposition with a single 0-handle.*

Proof. Let a decomposition start with some 0-handles $H_1^0, \ldots, H_{i_0}^0$. If $i_0 = 1$ we are done. If $i_0 > 1$, they form a disconnected set.

The addition of a k-handle with $k > 0$ does not modify the number of connected components of a manifold, except when $k = 1$ and the 1-handle is attached to distinct 0-handles. Since M is connected, there must be at least one such 1-handle. The geometric intersection of this 1-handle and one adjacent 0-handle is 1, so the pair may be canceled: we proceed by induction. □

Corollary 4.3. *Every closed connected manifold M^n has a handle decomposition with one 0-handle and one n-handle.*

Proof. Apply the proposition. Turn the handle decomposition upside-down. Apply the proposition again. □

[10] This makes sense, like for knots in S^3.

5. Poincaré in dimensions $n \geqslant 6$

We prove here the following result.

THEOREM 5.1 (Smale). *A smooth manifold homotopy equivalent to S^n is homeomorphic to S^n if $n \geqslant 6$.*

The proof goes roughly as follows: let M^n be homotopy equivalent to S^n. We take a handle decomposition of M^n with one 0- and one n-handle. We kill all the other intermediate handles[11]. Then M^n is obtained by gluing two discs, and is thus homeomorphic to S^n.

5.1. Trading 1-handles for 3-handles. We have shown how to kill redundant 0-handles in Corollary 4.3, so we now turn to 1-handles.

PROPOSITION 5.2. *A simply connected manifold M^n of dimension $n \geqslant 5$ has a handle decomposition with one 0-handle and no 1-handles.*

PROOF. We have a handle decomposition

$$M = H^0 \cup \left(H^1_1 \cup \ldots \cup H^1_{i_1}\right) \cup \ldots \cup \left(H^n_1 \cup \ldots H^n_{i_n}\right)$$

with a single 0-handle H^0. We have $\partial M_0 \cong S^{n-1}$. Take a 1-handle, say H^1_1. Its belt sphere S has dimension $n-2$. Since both ends of H^1_1 are attached to the same 0-handle H^0, there is a loop $\gamma \subset \partial M_1$ intersecting S transversely in one point. We can take γ disjoint from the attaching 2-spheres of the 2-handles (because $n \geqslant 3$). The curve γ thus survives in ∂M_2.

We now note that ∂M_2 is simply connected. In fact, we can turn upside-down the 0-, 1-, and 2-handles and say that M is obtained from $\partial M_2 \times [0,1]$ by attaching 3-, …, n-handles on the right side and n-, $(n-1)$-, $(n-2)$-handles on the left. Attaching handles of index $\geqslant 3$ does not modify the fundamental group of the manifold. Since $n \geqslant 5$, this holds here. Therefore $\pi_1(M_2) \cong \pi_1(M) = \{e\}$.

Since $n \geqslant 5$, the level manifold ∂M_2 has dimension $\geqslant 4$, and in such dimension a null-homotopic curve γ bounds an embedded disc D.[12]

We can now thicken D to a new canceling pair consisting of a 2- and 3-handle. By construction, the new 2-handle intersects H^1_1 geometrically once. Therefore we can cancel H^1_1 and the 2-handle, being left with a new 3-handle. □

COROLLARY 5.3. *A closed simply connected manifold M^n of dimension $n \geqslant 5$ has a handle decomposition with one 0-handle, one n-handle, and no 1- and $(n-1)$-handles.*

PROOF. Use the previous proposition. Turn it upside-down. Use the proposition again. □

5.2. Whitney trick. The Whitney trick is a move which allows (in high dimensions) to transform algebraic intersections into geometric intersections.

LEMMA 5.4. *Let P^p and Q^q be two transverse connected oriented manifolds in an oriented M^{p+q}. Suppose that $n = p+q \geqslant 5$ and $\pi_1(M \setminus (P \cup Q)) = \{e\}$. Let x, y be a pair of intersection points with opposite sign. There is an isotopy of P which cancels these two intersections.*

---

[11] We need $n \geqslant 6$ for this.

[12] See Corollary 2.2 on page 19.

Proof. Take two arcs $\alpha \subset P$, $\beta \subset Q$ joining x and y. Consider the closed curve $\alpha \cup \beta$. Since $M \setminus (P \cup Q)$ is simply connected, there is a map $f : D^2 \to M$ such that $\operatorname{Im} f \cap (P \cup Q) = \gamma$. This map can be perturbed to be an embedding because $n \geqslant 5$.[13]

We fix a riemannian metric, so that P and Q intersect orthogonally at x, y, and D is also orthogonal to both P and Q.

Trivialize the orthogonal bundle N_D over D as $D \times \mathbb{R}^{n-2}$. Let E be the $(p-1)$-dimensional sub-bundle over ∂D which is contained in P (over α) and orthogonal to Q (over β). As a subbundle, it determines an element of $\pi_1(G(p-1, n-2))$. Note that this fundamental group is always $\mathbb{Z}_2$, except for $G(1,2)$. Since $n \geqslant 5$, we have $\mathbb{Z}_2$. The element determined depends in fact on whether the algebraic intersections of P and Q in x and y coincide or not. Since they are opposite, this element is trivial.

All'inizio mettere π_1 di grassmanniana?

This implies that E can be extended to D. This trivialization over D allows one to slide P over Q.[14] □

5.3. Killing all the handles. We prove the following.

Theorem 5.5 (Smale). *A closed smooth manifold M^n homotopy equivalent to S^n has a decomposition with one 0-handle and one n-handle if $n \geqslant 6$.*

Proof. By Proposition 5.2, we can describe M^n via a handle decomposition with one 0-handle, no 1-handles, no $(n-1)$-handles, and one n-handle. We now show that, after finitely many handle slides, we end up with a decomposition in which every i-handle has non-zero algebraic intersection with exactly one $(i \pm 1)$-handle, and this intersection is ± 1.

We have the boundary maps

$$0 \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} 0.$$

Since M is homotopy equivalent to S^n, the resulting homology is trivial. That is, we have an exact sequence of free modules which split everywhere as $C_i = D_i \oplus E_i$ with $E_i = \ker \partial_i = \operatorname{Im} \partial_{i+1}$ and isomorphisms

$$\partial_i|_{D_i} : D_i \longrightarrow E_{i+1}.$$

Pick some basis for each D_i, E_i. Now note that a handle slide may be interpreted as an elementary move which changes a basis by adding (or subtracting) one element from another. The other elementary moves (permuting two elements, replacing an element by its opposite) can also be realized (permuting handles, changing the orientation of a handle). By iterating such moves we can represent the elements in the chosen basis for each D_i, E_i as handles.

The handles are now paired. That is, for every i-handle there exists exactly one handle (of index $i \pm 1$) having algebraic intersection ± 1 with it (and all others zero). We employ the Whitney trick to transform such algebraic intersections into geometric ones. The same argument used in the proof of Proposition 5.2 shows that all level manifolds are simply connected. We simply need to check that, inside each level $(n-1)$-manifold ∂M_2, the belt $(n-k-1)$-sphere of a k-handle and the

[13] If $n = 4$, we only get an immersion and the rest of the argument does not work.

[14] If $n = 4$, we have another problem here: even if γ bounds an embedded disc, the bundle E does not necessarily extend over D because $\pi_1(G(1,2)) = \mathbb{Z}$, and it might be impossible to slide P over Q.

attaching k-sphere of a $(k+1)$-handle have simply connected complement. If they both have codimension $\geqslant 3$ we are done. This always holds, except when $k=2$ or $k=n-2$ (both cannot occur). These two cases are actually symmetric, so we suppose $k=2$. So we have a belt sphere inside ∂M_2. Note however that its complement in ∂M_2 is the same as that of the attaching sphere in $\partial M_0 = S^{n-1}$: the attaching sphere is a circle, so its complement in S^{n-1} is simply connected. □

We can finally prove the topological Poincaré Conjecture in dimension $n \geqslant 6$.

COROLLARY 5.6. *A closed smooth manifold M^n homotopy equivalent to S^n is homeomorphic to S^n when $n \geqslant 6$.*

PROOF. By the previous theorem, the manifold M has a handle decomposition made of a 0- and a n-handle. That is, it may be obtained by gluing two n-discs D_1, D_2 along their boundaries. Correspondingly, represent $S^n = D_1' \cup D_2'$ as the union of two discs (the north and south emisphere). Pick a diffeomorphism $\varphi : D_1 \to D_1'$. Extend it to $\varphi : M \to S^n$ radially. That is, identify each D_2, D_2' with D^n, so that φ restricts to a diffeomorphism $\varphi|_{S^{n-1}} : S^{n-1} \to S^{n-1}$, and extend φ to the *continuous* function

$$\varphi(tx) = t\varphi(x).$$

This function is a homeomorphism from D^n to itself, which might not be a diffeomorphism! (As any continuous function, it may be perturbed into an arbitrarily close smooth function, which does not need however to be injective.) □

REMARK 5.7. Note that, if working in the PL category, the radial map is a PL map: The Poincaré Conjecture in dimension $n \geqslant 6$ holds also in the PL category, with the same proof (many concepts like handles, transversality, Whitney trick must however be adapted to the PL setting).

5.4. Connected sum. If M is an oriented manifold, we denote by $\overline{M}$ the same manifold with opposite orientation. We use the following.

LEMMA 5.8 (Cerf). *For every connected oriented manifold M^n there is a unique orientation-preserving embedding $\varphi : D^n \to M^n$ up to self-homeomorphisms of M^n.*

We can now define the *connected sum* $M_1 \# M_2$ of two compact oriented connected n-manifolds (possilbly with boundary) as follows. Pick two orientations-preserving embeddings $\varphi_1 : D^n \to M_1$ and $\varphi_2 : D^n \to \overline{M}_2$ and define $\dot{M}_i = M_i \setminus (\varphi_i(\mathrm{int}(D^n)))$ for $i = 1, 2$. Define

$$M_1 \# M_2 = \dot{M}_1 \cup_{\varphi_2 \circ \varphi_1^{-1}} \dot{M}_2.$$

Cerf's lemma implies the following.

PROPOSITION 5.9. *The connected sum $M_1 \# M_2$ is a smooth oriented manifold which depends only on M_1 and M_2. The operation # is commutative, associative, and $M \# S^n \cong M$.*

EXERCISE 5.10. Let M_1, M_2 be closed connected oriented manifolds of dimension n. We have the following.

- $\pi(M_1 \# M_2) \cong \pi(M_1) * \pi(M_2)$ if $n \geqslant 3$,
- $H_i(M_1 \# M_2) \cong H_i(M_1) \oplus H_i(M_2)$ for every $0 < i < n$.

5.5. Exotic spheres. We can also get an inverse for # if we restrict to exotic spheres.

PROPOSITION 5.11. *Oriented homotopy spheres form a group under # in dimension* $n \geqslant 6$.

PROOF. By Smale's Theorem 5.5, every oriented homotopy sphere M is constructed by gluing two discs along an orientation-reversing map φ on their boundaries. If we use φ^{-1} instead of φ we get $\overline{M}$. This easily implies that $M\#\overline{M} \cong S^n$. Therefore every homotopy sphere M has an inverse. □

We denote such group by Γ_n (defined here only for $n \geqslant 6$). Such groups are all finite. The first ones are listed below.

n	6	7	8	9	10	11
Γ_n	0	$\mathbb{Z}_{28}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_6$	$\mathbb{Z}_{992}$

5.6. Boundary-connected sum. The *∂-connected sum* of two oriented manifolds M_1^n, M_2^n with connected boundary is defined similarly. Take two embeddings $\varphi_1 : D^{n-1} \to \partial M_1$ and $\varphi_2 : D^{n-1} \to \partial M_2$ and construct

$$M_1 \natural M_2 = M_1 \cup_{\varphi_2 \circ \varphi_1^{-1}} M_2.$$

By Cerf's lemma, the operation $\natural$ is well-defined, commutative, and associative on compact manifolds with connected boundary. Of course, we have $M \natural D^n = M$.

EXERCISE 5.12. We have $\partial(M_1 \natural M_2) \cong \partial M_1 \# \partial M_2$.

CHAPTER 2

Basics on dimension 4

We now turn to dimension 4. Much topology here is controlled by an important object, which is absent in lower dimensions: this is the *intersection form*, a bilinear unimodular pairing on the (integral) second homology (unimodularity is a manifestation of Poincaré duality). After introducing the reader to this important geometric/algebraic object, we turn to study some basic facts about curves and surfaces in 4-manifolds. Curves are easily studied, while surfaces play a central rôle in dimension 4.

We end with a section on characteristic classes and introduce some basic examples of 4-manifolds: the disc bundles over surfaces.

1. Intersection forms

On a simply connected 4-manifold M, all the homology is concentrated in the free second homology module $H_2(M,\mathbb{Z})$.[1] This module is equipped with a bilinear form, induced by Poincaré duality. We introduce such forms in a more general context.

1.1. Poincaré duality. Let M be a closed oriented manifold of dimension n. The cup product yields a bilinear form

$$\begin{array}{rcl} Q : H^k(M,\mathbb{Z}) \times H^{n-k}(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ (\alpha,\beta) & \longmapsto & (\alpha\cup\beta)[M] \end{array}$$

called *intersection pairing.* Any bilinear form is necesarily zero on torsion elements. It is therefore harmless to restrict the pairing to the quotients modulo torsion.

By Poincaré duality, the intersection pairing restricted to quotients modulo torsion is *unimodular*. That is,

Definition 1.1. Let F_1, F_2 be two finitely generated free $\mathbb{Z}$-modules (*i.e* each is isomorphic to $\mathbb{Z}^r$ for some r). A bilinear form $Q : F_1 \times F_2 \to \mathbb{Z}$ is a *unimodular pairing* if the adjoint map

$$\begin{array}{rcl} \varphi^1_Q : F_1 & \longrightarrow & F_2^* = \mathrm{Hom}(F_2,\mathbb{Z}) \\ \alpha & \longmapsto & \big(\beta \mapsto Q(\alpha,\beta)\big) \end{array}$$

and the analogously defined $\varphi^2_Q : F_2 \to F_1^*$ are both isomorphisms.

As we said, we have the following.

Theorem 1.2 (Poincaré duality). *The intersection pairing Q is a unimodular pairing.*

[1]See Section 3.1 on page 5.

Actually, Poincaré duality also yields informations on the torsion part of homology, and yields a canonical isomorphism $H_i(M,\mathbb{Z}) \cong H^{n-i}(M,\mathbb{Z})$. Via this isomorphism, the intersection form is also defined on homology:

$$Q : H_k(M,\mathbb{Z}) \times H_{n-k}(M,\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Moreover, this form has a nice geometric representation (which is actually closer to Poincaré's original definition).

Theorem 1.3 (Geometric representation). *If two classes $\alpha \in H_k(M,\mathbb{Z})$ and $\beta \in H_{n-k}(M,\mathbb{Z})$ are represented by oriented transverse submanifolds S_α and S_β of dimension k and $n-k$, we have*

$$Q(\alpha,\beta) = S_\alpha \cdot S_\beta.$$

1.2. Manifolds with boundary. On an oriented n-manifold M with boundary, two types of intersection forms may be defined. On homology, they are as follows.

$$\begin{array}{rcl} Q : H_k(M,\mathbb{Z}) \times H_{n-k}(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ Q' : H_k(M,\partial M,\mathbb{Z}) \times H_{n-k}(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \end{array}$$

Both of these pairings have a geometric representation as stated in Theorem 1.3 (when α lies in $H_k(M,\partial M,\mathbb{Z})$, the manifold S_α is a properly embedded manifold possibly with boundary). Only the second one is always unimodular.

Theorem 1.4 (Lefschetz duality). *The intersection pairing Q' is a unimodular pairing.*

As above, we have a canonical identification $H_i(M,\mathbb{Z}) \cong H^{n-i}(M,\partial M,\mathbb{Z})$. The pairing Q is not necessarily unimodular.

Example 1.5. Take $M = S^1 \times [-1,1]$. The module $H_1(M,\mathbb{Z}) \cong \mathbb{Z}$ is generated by $S^1 \times 0$ and $H_1(M,\partial M,\mathbb{Z}) \cong \mathbb{Z}$ is generated by $0 \times [-1,1]$. We have $Q = [0]$ and $Q' = [1]$.

1.3. Middle homology. Let M be closed oriented of even dimension $n = 2k$. The *intersection form*

$$Q_M \colon H_k(M,\mathbb{Z}) \times H_k(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

is unimodular. Since we have the same module on both sides, unimodularity may be stated more explicitly.

A *basis* for a $\mathbb{Z}$-module F isomorphic to $\mathbb{Z}^h$ is just a set of h generators. The *rank* of a bilinear form on F is the number h. A bilinear form may be represented as a square $(h \times h)$-matrix. A square matrix is *unimodular* if its determinant is ± 1.

Proposition 1.6. *Let F be a free finitely generated $\mathbb{Z}$-module. Let $Q : F \times F \to \mathbb{Z}$ be a bilinear form. The following are equivalent.*

(1) *Q is a unimodular pairing.*
(2) *There is a basis $\mathcal{B}$ such that the matrix M associated to Q with respect to $\mathcal{B}$ is unimodular.*
(3) *For every basis $\mathcal{B}$, the matrix M associated to Q with respect to $\mathcal{B}$ is unimodular.*
(4) *Every basis $\mathcal{B} = (a_1,\dots,a_h)$ has a (unique)* dual basis, *that is a basis $(b_1,\dots,b_h)$ such that $Q(a_i,b_j) = \delta_{ij}$.*

Proof. Note that every basis $\mathcal{B} = (a_1, \dots, a_n)$ of F defines a (unique) *dual basis* $\mathcal{B}^* = (a_1^*, \dots, a_n^*)$ of F^* such that $a_i^*(a_j) = \delta_{ij}$.

(1)⇔(2)⇔(3): The matrices M and ${}^tM$ represent φ^1 and φ^2 with respect to $\mathcal{B}$ and $\mathcal{B}^*$. Both these maps are isomorphisms if and only if M is unimodular.

(1)⇒(4): Take $b_i = (\varphi^2)^{-1}(a_i^*)$.

(4)⇒(1): The adjoint φ^1 sends a_i to b_i^* and is hence an isomorphism. □

Therefore Q_M is represented by unimodular matrices.

Proposition 1.7. *The intersection form Q_M of a closed manifold M^{2k} is symmetric or antisymmetric, depending upon whether k is even or odd.*

Proof. Concerning cup products, we have $\alpha \cup \beta = (-1)^{k \cdot k} \beta \cup \alpha$. This formula can analogously be checked using the geometric representation (when it exists!) □

We deduce the following.

Corollary 1.8. *A closed oriented manifold M of dimension $4h+2$ has $\chi(M)$ even.*

Proof. We have $\chi(M) = \sum_{i=0}^{4h+2} (-1)^i b_i$, with $b_i = \dim H_i(M, \mathbb{Z})$. By Poincaré duality we have $b_i = b_{4h+2-i}$ and thus $\chi(M) = 2\big(\sum_{i=0}^{2h} (-1)^i b_i\big) + b_{2h+1}$. The intersection form Q_M on $H_{2h+1}(M, \mathbb{Z})$ is unimodular and antisymmetric, hence b_{2h+1} is even. □

1.4. Signature and parity. A symmetric unimodular bilinear form Q may be diagonalized over the real numbers by Sylvester's theorem, and has therefore a well-defined *signature* (i_+, i_-). Actually, we define here as a signature $\sigma(Q)$ the difference $i_+ - i_-$ between the two indexes.

Definition 1.9. The *signature* $\sigma(M)$ of a closed oriented manifold M^{4h} is the signature of its symmetric intersection form Q_M.

A symmetric bilinear form Q over a free module has another invariant $p(Q)$. The form Q is said to be *even* if $Q(v, v)$ is even for every v. It is *odd* otherwise. We set correspondingly $p(Q) = 0$ or 1. Note that Q is even if and only if every element in the diagonal (of a matrix representing Q) is even.

Definition 1.10. The *parity* $p(M)$ of a closed oriented manifold M^{4h} is the parity of its symmetric intersection form Q_M.

Given a form Q, we denote by $-Q$ the opposite form $(-Q)(v, w) = -Q(v, w)$. We have the following easy.

Proposition 1.11. *We have $Q_M \cong -Q_{\overline{M}}$. Therefore $\sigma(\overline{M}) = -\sigma(M)$ and $p(\overline{M}) = p(M)$.*

Given two unimodular forms Q and Q' defined on some free finitely generated $\mathbb{Z}$-module F and f', we let $Q \oplus Q'$ be the form on $F \oplus F'$ as

$$(Q \oplus Q')\big((v, v'), (w, w')\big) = Q(v, w) + Q'(v', w').$$

Proposition 1.12. *We have $\sigma(Q \oplus Q') = \sigma(Q) + \sigma(Q')$ and $p(Q \oplus Q') = p(Q) \cdot p(Q')$.*

Proof. If M, M' are matrices representing Q, Q', a matrix representing $Q \oplus Q'$ is simply $\begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}$. □

PROPOSITION 1.13. *Let M, N be closed oriented manifolds of dimension $4k$. We have $Q_{M\#N} \cong Q_M \oplus Q_N$. In particular, $\sigma(M\#N) = \sigma(M) + \sigma(N)$.*

PROOF. Let $\dot{M}$ be M with an open ball removed. Recall that $M\#N$ is obtained by gluing $\dot{M}$ and $\dot{N}$ along their boundaries. Inclusion gives an isomorphism $i_* : H_{2k}(\dot{M}) \to H_{2k}(M)$. Mayer-Vietoris sequence gives then an isomorphism between $H_{2k}(\dot{M}) \oplus H_{2k}(\dot{N})$ and $H_{2k}(M\#N)$, which preserves pairings. □

1.5. Manifolds of zero signature. Non-zero signature is an obstruction to bounding manifolds, as the following shows.

THEOREM 1.14. *Let M^{4k} be an oriented (possibly disconnected) closed $(4k)$-manifold. If $M^{4k} = \partial W^{4k+1}$ is the boundary of an oriented $(4k+1)$-manifold, then $\sigma(M) = 0$.*

PROOF. Suppose $M = \partial W$. We consider homology over $\mathbb{R}$. A piece of the long exact sequence shows

$$H_{2k+1}(W, \partial W) \xrightarrow{\ \partial\ } H_{2k}(\partial W) \xrightarrow{\ i\ } H_{2k}(W).$$

There are two unimodular pairings here, one Q between $H_{2k}(\partial W)$ and itself, and another Q' between $H_{2k+1}(W, \partial W)$ and $H_{2k}(W)$. There is a compatibility between them:

$$Q'\big(\alpha, i(\beta)\big) = Q\big(\partial\alpha, \beta\big).$$

This implies that $\operatorname{Ker} i = \operatorname{Im} \partial$ is totally isotropic with respect to Q, since

$$Q(\partial\alpha, \partial\beta) = Q'\big(\alpha, i(\partial\beta)\big) = Q'(\alpha, 0) = 0.$$

Analogously, we see that the spaces $\operatorname{Ker} \partial$ and $\operatorname{Im} i$ are Q'-orthogonal. Since Q' is non-degenerate, we get

$$\dim \operatorname{Ker} \partial + \dim \operatorname{Im} i \leqslant \dim H_{2k}(W, \partial W) = \dim H_{2k}(W).$$

This implies easily that

$$\dim \operatorname{Im} \partial + \dim \operatorname{Ker} i \geqslant \dim H_{2h}(\partial W).$$

That is, we have $\dim \operatorname{Ker} i \geqslant \dim H_{2h}(\partial W)/2$. Since Q is non-degenerate and $\operatorname{Ker} i$ is totally isotropic, equality holds and the signature of Q is zero. □

REMARK 1.15. The proof of Theorem 1.14 also shows that the image of

$$\partial : H_{2k+1}(W, \partial W; \mathbb{R}) \to H_{2k}(\partial W; \mathbb{R})$$

has dimension $b_{2k}/2$.

In dimension 4, signature is the only obstruction for bounding a 5-manifold.[2]

[2]See Theorem 2.15 on page 41.

2. Loops in 4-manifolds

We study 1-manifolds in 4-manifolds. We start with the following.

PROPOSITION 2.1. *Let M be a compact 4-manifold. Two homotopic embedded loops $\gamma_1, \gamma_2 : S^1 \to \text{int}(M)$ with disjoint images bound a smooth annulus.*

PROOF. The homotopy is a map $F : S^1 \times [0,1] \to M$. By transversality, we can suppose F is an immersion from a 2-dimensional annulus to M, which some double points. Since $S^1 \times [0,1]$ has boundary, each double point can be easily eliminated by sliding it out. □

COROLLARY 2.2. *A null-homotopic embedded loop in* $\text{int}(M)$ *bounds a smooth disc.*

This result does not extend to loops contained in the boundary! For instance, we have the following.

PROPOSITION 2.3. *Every knot in S^3 bounds an immersed disc in D^4. The trefoil knot does not bound embedded discs in D^4.*

PROOF. The loop is homotopically trivial and hence bounds an immersed disc. The assertion concerning the trefoil knot is proved below. □ Dove?

A knot in S^3 which bounds a smooth embedded disc in D^4 is called *slice*. More generally, the *slice genus* of the knot is the minimum genus of a surface in D^4 spanning it. Of course, the slice genus is smaller or equal then the (usual) genus (defined as the minimum genus of a surface spanning it in S^3, *i.e.* a Seifert surface).

Note that smoothness is important here.

PROPOSITION 2.4. *Every knot in S^3 bounds a (piecewise-linear) topologically embedded disc in D^4.*

PROOF. Take the cone of the knot, with center the center of D^4. □

In the piecewise-linear category, to avoid objects of this kind one usually requires them to be *locally flat*. A submanifold $M^m \subset N^n$ is locally flat if it is locally embedded as an m-plane in $\mathbb{R}^n$. In the smooth category, local flatness is guaranteed. In the PL and topological categories, it is not. A cone on a non-trivial knot is not PL locally flat.

Every known example of slice knot belongs to the following category.

DEFINITION 2.5. A *ribbon knot* is a knot in S^3 which bounds an immersed disc $i : D^2 \to S^3$ self-intersecting in arcs, whose pre-images in D^2 are arcs whose endpoints are either both in ∂D^2 or both in its interior.

PROPOSITION 2.6. *Every ribbon knot is a slice knot.*

PROOF. The pre-image of every self-intersecting arc consists of two arcs α and β in D^2, one (say α) with both endpoints in ∂D^2 and β with both endpoints in the interior of ∂D^2. Eliminate the self-intersecting arc by pushing down towards the interior of D^4 a small subdisc of D^2 containing β. □

It it still unknown whether every slice knot must be ribbon.

3. Surfaces in 4-manifolds

We state some general facts about surfaces in 4-manifolds. We show in particular that the intersection form has a useful geometric representation.

3.1. Isolated double points. We want to show that the geometric realization of Poincaré duality is possible in dimension 4. To do this, we introduce a basic move which will also be used later at various points.

Every map $f : \Sigma^2 \to M^4$ from a surface to a 4-manifold can be perturbed to be a generic immersion.[3] Such an immersion is everywhere injective, except at finitely many isolated double points. Each such isolated intersection looks locally like two transverse discs in $\mathbb{R}^4$, that is like the set $C = \{zw = 0\} \cap D^4$.

This set may be perturbed by slightly modifying the equation defining it as $C_\varepsilon = \{zw = \varepsilon\} \cap D^4$ for some small real number $\varepsilon > 0$. Now C_ε is a smooth complex curve, *i.e.* a surface without singularities.

Proposition 3.1. *We have*

$$C_\varepsilon = \left\{ \left(z, \frac{\varepsilon}{z} \right) \;\Big|\; m < |z| < M \right\}$$

for some $0 < m < M$ *and therefore* C_ε *is an annulus.*

Proof. The set C_ε consists of all points $(z, \varepsilon/z)$ with $|z|^2 + \varepsilon^2/|z|^2 \leqslant 1$, that is $|z|^4 - |z|^2 + \varepsilon^2 \leqslant 0$. Therefore

$$\frac{1 - \sqrt{1 - 4\varepsilon^2}}{2} \leqslant |z|^2 \leqslant \frac{1 + \sqrt{1 - 4\varepsilon^2}}{2}.$$

In particular, we get $m \to 0$ and $M \to 1$ as $\varepsilon \to 0$. □

The boundary $\partial C_0 \subset S^3$ is a link with two components, called *Hopf link*. The boundary ∂C_ε is equivalent to ∂C_0 up to a small isotopy: we can thus modify C_ε with a small isotopy so that $\partial C_0 = \partial C_\varepsilon$.

Every double point can be removed by substituting the two trasverse discs C_0 with an annulus C_ε. Note however that the topology of the immersed surface has changed: before we had an immersion $f \colon \Sigma^2 \to M^4$, now we have an immersion $f_* \colon \Sigma^2_* \to M^4$ with one double point less, and the surface Σ_* is obtained from Σ by substituting two discs with an annulus. In particular, we have $\chi(\Sigma_*) = \chi(\Sigma) - 2$.

If Σ is oriented, the surface Σ_* also is.[4] Therefore f and f_* represents elements of $H_2(M, \mathbb{Z})$.

Proposition 3.2. *The maps f and f_* represent the same element in $H_2(M, \mathbb{Z})$.*

Proof. The two functions define cycles that differ only in the interior of a ball. Their difference is represented by a cycle whose image is entirely in a ball, and is thus homologous to zero. □

Summing up, we can eliminate double points and remain within the same homology class. The price to pay is to modify the topology of the surface, by decreasing its Euler characteristic. If the surface is connected, this is equivalent to raising its genus.

[3]See Section 2.3 on page 4.

[4]The surfaces C_0 and C_ϵ are oriented as complex curves, and they induce the same orientation on the boundary.

3.2. Geometric realizations. We first show that the geometric realization of Poincaré duality is possible in dimension 4.

PROPOSITION 3.3. *Let M be a compact 4-manifold (possibly with boundary). Every element in $H_2(M^4,\mathbb{Z})$ can be realized as an embedding of an oriented (possibly disconnected) surface.*

PROOF. A class α is a sum of maps from triangles to M. These maps can be glued in pairs along edges, and the result is a continuous map from a (possibly disconnected) surface to M representing α. By transversality, α is homotopic to an immersion. Every double point can in turn be perturbed as described above. Note that the genus of the surface increases by one at each desingularization. (More precisely, its Euler characteristic decreases by -2.) □

The embedded surface representing a cycle is not necessarily a sphere. However, if we drop embeddability, we have the following.

PROPOSITION 3.4. *If M is a simply connected 4-manifold, every element in $H_2(M^4,\mathbb{Z})$ can be realized as an immersed sphere.*

PROOF. By Hurewicz Theorem the map $\pi_2(M) \to H_2(M,\mathbb{Z})$ is an isomorphism. Therefore every element in $H_2(M,\mathbb{Z})$ is realized as a map $f : S^2 \to M$ which can be perturbed to an immersion. □

3.3. Examples. The intersection forms of S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, $S^2\times S^2$ are respectively

$$\emptyset, \quad [1], \quad [-1], \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote the latter form by H. Note that the two latter forms have the same signature but are not isomorphic, since they have distinct parity: therefore $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ is not homeomorphic to $S^2\times S^2$. We study these manifolds more closely. We start with a preliminary remark on complex manifolds.

REMARK 3.5. Every complex manifold M is naturally oriented, and two complex submanifolds intersecting transversely in points always intersect positively (*i.e.* with sign $+1$). Therefore their algebraic intersection equals the geometric intersection. As a consequence, if two homology classes $\alpha \in H_k(M,\mathbb{Z})$ and $\beta \in H_{n-k}(M,\mathbb{Z})$ can be represented by two complex submanifolds intersecting transversely, then $Q(\alpha,\beta) \geqslant 0$.

3.3.1. $\mathbb{CP}^2$. The even homology $H_{2i}(\mathbb{CP}^n,\mathbb{Z})$ of a complex projective space is identified with $\mathbb{Z}$, generated by the class $[S]$, represented by any i-dimensional linear subspace S. Odd homologies are trivial.

In particular, the group $H_2(\mathbb{CP}^2,\mathbb{Z}) = \mathbb{Z}$ is generated by $[l]$ for any complex line $l \subset \mathbb{CP}^2$. Two distinct lines l, l' intersect in a point and represent the same class $[l]=[l']$. We thus have $Q([l],[l]) = l\cdot l' = 1$. Therefore the intersection pairing of $\mathbb{CP}^2$ is $[1]$.

A projective curve C_d of degree d is defined as the zero-set of a homogeneous polynomial of degree d. The space of such polynomials is a big projective space $\mathbb{CP}^N$. Smooth curves (*i.e.* curves without singularities) form an open Zariski set. In particular, every two such curves are connected with a path in this open set, which parametrizes an isotopy between them inside $\mathbb{CP}^2$.

A smooth C_d defines a class $[C_d] \in H_2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$. Not surprisingly, we have $[C_d] = d$. This holds because C_d intersects a generic line in d points, and hence $Q([C_d], [l]) = C_d \cdot l = d$.

Corollary 3.6 (Bézout Theorem, easy case). *Two smooth curves C_d and $C_{d'}$ intersecting transversely have $d \cdot d'$ intersection points.*

Proof. We have $C_d \cdot C_{d'} = Q\big([C_d], [C_{d'}]\big) = d \cdot 1 \cdot d' = d \cdot d'$ and the algebraic intersection number equals the geometric one by Remark 3.5. □

3.3.2. $S^2 \times S^2$. The homology group $H_2(S^2 \times S^2, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ is generated by $\alpha = [S^2 \times q]$ and $\beta = [p \times S^2]$. The two surfaces intersect positively in one point, hence $Q(\alpha, \beta) = 1$. Moreover, two representatives $S^2 \times q_1$ and $S^2 \times q_2$ of α with $q_1 \neq q_2$ do not intersect, hence $Q(\alpha, \alpha) = 0$. Similarly, we have $Q(\beta, \beta) = 0$.

The intersection form is therefore $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Exercise 3.7. Show that $S^2 \times S^2 \cong \overline{S^2 \times S^2}$ and $\mathbb{CP}^2 \not\cong \overline{\mathbb{CP}}^2$. (The symbol "$\cong$" indicates an orientation-preserving diffeomorphism.)

4. Characteristic classes

Before proceeding in the study of 4-manifolds, we introduce some fundamental objects in differential topology. A vector bundle $E \to M$ defines a numer of elements in the (co-)homology groups of M, called *characteristic classes*. In the simplest cases, these classes describe the bundle completely. We quickly recall some basic facts on these classes.

4.1. Euler class. Let $E \to M$ be an oriented vector bundle of rank r on an oriented closed n-manifold M. This object defines a *Euler class*

$$e(E) \in H^r(M, \mathbb{Z}).$$

The Poincaré dual of $e(M)$ lies in $H_{n-r}(M, \mathbb{Z})$. It may be defined as (the homology class of) the oriented $(n-r)$-manifold obtained by intersecting of the zero-section of E with any other transverse section.

When $n = r$, the two sections intersect in points with sign ± 1. The Euler number $e(M) \in \mathbb{Z}$ is the algebraic intersection number of the two sections. We single out a couple of useful results in that case, when E is the tangent and normal bundle to N.

Proposition 4.1. *We have $e(T_M) = \chi(M)$.*

Proof. This is Poincaré-Hopf theorem on vector fields. □

Proposition 4.2. *Let X^r be a smooth closed oriented submanifold of a smooth manifold M^{2r}. Let N_X be the normal bundle over X. We have $e(N_X) = Q([X], [X])$.*

Proof. The normal bundle may be embedded as a tubolar neighborhood of X in M, such that X corresponds to the zero-section and any transverse section is another manifold $X' \subset M$ isotopic to X and transverse to it. Therefore $e(N_X) = X \cdot X' = Q([X], [X']) = Q([X], [X])$. □

Proposition 4.3. *If $X^n \subset \mathbb{R}^{n+r}$ then $e(N_X) = 0$.*

PROOF. Compactify $\mathbb{R}^n$ to S^n. Inclusion $i : X^n \hookrightarrow S^{n+r}$ gives a map

$$i^* \colon H^r(S^{n+r}, \mathbb{Z}) \longrightarrow H^r(X^n, \mathbb{Z}).$$

By definition of Euler number, this map sends the Poincaré dual to $[M] \in H_n(S^{n+r}, \mathbb{Z})$ to the Euler class $e(N_X)$. However, the group $H_n(S^{n+r}, \mathbb{Z})$ is trivial and so also $e(N_X)$ is.[5]

□

Spostare dopo.

PROPOSITION 4.4. *Let $E \to M$ be a rank-r bundle with $e(E) = 0$. The restriction of E on any r-skeleton of M is trivial.*

Sistemare queste due proposizioni.

4.2. Chern classes. Let $E \to M$ be a complex vector bundle of rank r over a closed connected oriented n-manifold M. This object defines some *Chern classes*

$$c_i(E) \in H^{2i}(M, \mathbb{Z})$$

for all $i = 0, 1 \ldots, r$. We will actually use few properties of Chern classes. The lowest Chern class $c_0(E) = 1$ is constant and therefore not interesting. The top-most one coincides with the Euler class.

PROPOSITION 4.5. *The top-most Chern class $c_r(E)$ equals the Euler number $e(E)$ of E, seen as a rank-$2r$ real vector bundle.*

In particular, on a complex line bundle (*i.e.* a complex bundle of rank $r = 1$) we have $c_1(E) = e(E)$. The first Chern class (and hence the Euler number) classifies completely complex line bundles up to isomorphism.

PROPOSITION 4.6. *The first Chern class c_1 yields a bijection between isomorphism classes of (complex) line bundles over M and $H^2(M, \mathbb{Z})$.*

PROOF. For a complete proof, see the book of Griffiths-Harris. Complex line bundles correspond to elements in the first cohomology $H^1(M, C^\infty(\mathbb{C}^*))$ of the sheaf of smooth functions with values in $\mathbb{C}^*$. The exact sequence of groups

$$\mathbb{Z} \xrightarrow{\ i\ } \mathbb{C} \xrightarrow{\ \exp\ } \mathbb{C}^*$$

yields a long exact sequence on sheaves cohomology

$$H^1(M, C^\infty(\mathbb{C})) \xrightarrow{\exp_*} H^1(M, C^\infty(\mathbb{C}^*)) \xrightarrow{\ c_1\ } H^2(M, \mathbb{Z}) \xrightarrow{\ i_*\ } H^1(M, C^\infty(\mathbb{C}))$$

where c_1 is the first Chern class. The sheaf $C^\infty(\mathbb{C})$ has no cohomology, hence c_1 is an isomorphism. □

Trovare inglese per fine

The dual L^* of a line bundle is another line bundle. The tensor product $L \otimes L'$ of two line bundles is another line bundle.

PROPOSITION 4.7. *We have $c_1(L^*) = -c_1(L)$ and $c_1(L \otimes L') = c_1(L) + c_1(L')$.*

A fundamental example is the *tautological vector bundle* E over $\mathbb{CP}^n$. It is a sub-bundle of the product bundle $\mathbb{C}^{n+1} \times \mathbb{CP}^n$, defined as follows: the fiber of $p \in \mathbb{CP}^n$ is the line of $\mathbb{C}^{n+1}$ corresponding to p.

PROPOSITION 4.8. *We have $c_1(E) = -1 \in H_2(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}$.*

Proof?

Finally, note that every oriented real plane bundle may be given a complex line bundle structure.

[5]We have actually proved a stronger theorem: if $X^n \subset Y^{n+r}$ are closed oriented manifolds and X^n is homologically trivial in Y^{n+r}, then $e(N_X) = 0$.

PROPOSITION 4.9. *Let $E \to M$ be an oriented real plane bundle. It is possible to assign to each fiber a complex structure which makes E a complex line bundle.*

PROOF. The structure group of E may be reduced to $SO(2)$ by fixing a riemannian metric on E.[6] Take the usual identification of $\mathbb{R}^2$ with $\mathbb{C}$ on each trivialization. A rotation in $SO(2)$ is a complex-linear transition functions of type $z \mapsto e^{\alpha\pi i}z$. Therefore the whole E has a complex structure. $\square$

4.3. Line bundles over surfaces. Our discussion on Chern classes implies that complex line bundles on an oriented surface Σ are in 1-1 correspondence with the elements of $H_2(\Sigma, \mathbb{Z}) = \mathbb{Z}$, *i.e.* with integers. Every line bundle yields (via a metric) a disc-bundle over Σ. The boundary of this disc-bundle in turn gives a S^1-bundle over Σ. All such objects are determined by an integer, the Euler number (or equivalently the first Chern class).

We already know some bundles over the sphere S^2. The trivial bundle has Euler number zero, the principal vector bundle over $S^2 \cong \mathbb{CP}^1$ has Euler number -1, while the tangent bundle has Euler number $\chi(S^2) = 2$ (and the cotangent one -2). Topologically, we may state the following.

PROPOSITION 4.10. *The S^1-bundle over S^2 with Euler number e is homeomorphic to the lens space $L(e, 1)$.*

In particular, when $e = 0, 1, 2$ we get $S^2 \times S^1$, S^3, and $\mathbb{RP}^3$.

[6]See Section 1.5 on page 2.

CHAPTER 3

Topological 4-manifolds

A topological 4-manifold is a manifold without smooth structure. As shown by Freedman in 1981, the topological and smooth categories differ radically in dimension 4. There are topological manifolds that admit no smooth structure, while others admit infinitely many.

This chapter is devoted to topological 4-manifold. The main results of Freedman are highly non-trivial and are thus only stated here, without a proof. They furnish a beautiful and simple description of the world of topological 4-manifolds.

Some important tools in the smooth setting are introduced here. We start with Kirby diagrams, which encode handle decompositions in dimension 4 (as Heegaard diagrams do in dimension 3). Then we turn to contractible 4-manifolds. Finally, we state Freedman's theorems.

1. Kirby diagrams

A handle decomposition of a smooth 4-manifold can be encoded via a combinatorial object, called *Kirby diagram.* We introduce it here.

1.1. One-handles. Let M be a connected compact smooth 4-manifold, possibly with boundary. The manifold M has a handle decomposition with one 0-handle and at most one 4-handle. The boundary of the 0-handle is S^3. Every 1-handle is attached to S^3, more specifically it glues $D^3 \times D^1$ to two 3-discs in S^3 along $D^3 \times S^0$. We can therefore encode each 1-handle by drawing couples of 3-discs in S^3.

Using 0- and 1-handles we can actually construct only few manifolds. These manifolds are the generalizations of the 3-dimensional handlebodies. We denote by $\#_k M^n$ the connected sum of k copies of M^n: when $k = 0$ we set $\#_0 M^n = S^n$. Analogously, let $\natural_k M^n$ be the ∂-connected sum of k copies of M^n; when $k = 0$ we set $\#_0 M^n = D^n$.[1]

PROPOSITION 1.1. *Let a connected M^4 decompose into 0-handle and 1-handles. Then $M = \natural_k(D^3 \times S^1)$ and $\partial M = \#_k(S^2 \times S^1)$ for some k.*

PROOF. First, we can suppose there is one 0-handle only.[2] Now we claim that if M^n has connected boundary and N^n is obtained from M^n by attaching a 1-handle, then $N^n = M^n \natural (D^{n-1} \times S^1)$. This assertion proves the proposition.[3]

We prove the assertion. The 1-handle is attached along two $(n-1)$-discs $D_1, D_2 \subset \partial M^n$. By Cerf's Lemma[4] these two discs are contained in a bigger

[1] See Section 5.6 on page 14.

[2] See Proposition 4.2 on page 10.

[3] Together with Exercise 5.12 on page 14.

[4] See Lemma 5.8 on page 13.

$(n-1)$-disc $D \subset \partial M^n$. If we cut N^n along D we get M^n and $D^{n-1} \times S^1$, as required. □

1.2. Two-handles. A 2-handle is a product $D^2 \times D^2$ attached along the solid torus $S^1 \times D^2$. Different 2-handles are attached along disjoint solid tori. The attaching sphere is a circle $S^1 \times 0$. It can be drawn as a circle in S^3, which may sometimes pass over some 1-handle: it disappears in one 3-disc and reappear on the corresponding 3-disc of the 1-handle.

The gluing of the solid torus $D^2 \times S^1$ is not determined solely by the gluing of its attaching circle. Gluing such a solid torus corresponds to fixing a trivialization of the normal bundle of the attaching circle S^1. A trivialization of the normal bundle is sometimes called a *frame*. Fixing a frame means fixing two independent sections of the normal bundle. Actually, one suffices: the other one is then determined up to homotopy and up to sign, but homotopy and sign do not change the way the 2-handle is attached.

Dire di più in dimensione qualsiasi?

Framings may be encoded by drawing ribbons instead of knots. If there are no 1-handles, a framing can be encoded as an integer: orient each knot, and take the algebraic intersection of the section with any (transverse) Seifert surface of the knot oriented coherently. The resulting integer is actually independent of the chosen orientation for the knot. In particular, a 0-framing corresponds to the framing induced by any Seifert surface. It is also possible to extend this convention and use integers to encode framings when 1-handles are present.

A diagram as described, which contains pairs of 3-discs and disjoint knots (jumping occasionally over the discs) decorated with integers, is a *Kirby diagram*. A Kirby diagram encodes a decomposition with 0-, 1-, and 2-handles of a manifold N. Note that ∂N is always connected.

1.3. Handles of index 3 and 4. To know whether we can attach 3- and 4-handles to close up a manifold N, we use the following.

LEMMA 1.2. *Let N decompose into 0-, 1-, and 2-handles. It is possible to get a closed manifold by attaching 3- and 4-handles to N if and only if $\partial N \cong \#_k(S^2 \times S^1)$ for some $k \geqslant 0$.*

PROOF. Handles of index 3 and 4 may be turned upside down. They become 1- and 0-handles. Therefore, together they form a manifold as in Proposition 1.1, whose boundary is diffeomorphic to $\#_k(S^2 \times S^1)$. □

If ∂N is homeomorphic to $\#_k(S^2 \times S^1)$, we can close the 4-manifolds by adding 3- and 4-handles. There are many ways to attach them, but they luckily all lead to the same closed manifold, thanks to the following.

PROPOSITION 1.3 (Laudenbach-Poenaru). *If M and M' are two closed manifolds obtained by attaching 3- and 4-handles to the same manifold N with boundary, then $M \cong M'$.*

SKETCH. They prove that every self-homeomorphism of $\#_k(S^2 \times S^1)$ extends to a self-homeomorphism of $\natural_k(D^3 \times S^1)$. □

We can therefore forget about 3- and 4-handles. Note that a random Kirby diagram does *not* describe a closed 4-manifold: it describes a random 4-manifold

N made of 0-, 1-, 2-handles, whose boundary ∂N is a random 3-manifold[5], which is unlikely to be $\#_k(S^2 \times S^1)$.

1.4. Only 0- and 2-handles. Many interesting manifolds admit a decomposition without 1- and 3-handles. That is, we have one 0-handle, some 2-handles, and maybe one 4-handle to close the manifold up. A manifold having a decomposition of this type is necessarily simply connected; conversely, we still do not know if every closed simply connected 4-manifold may be described in this way.

QUESTION 1.4 (open). Does every simply connected compact 4-manifold have a decomposition without 1-handles?

Let N be a manifold made of one 0-handle and n 2-handles. The corresponding Kirby diagram is a link $L = K_1 \cup \ldots \cup K_n$ in S^3, each component K_i being decorated by an integer. Orient each K_i arbitrarily. This defines an *intersection matrix* M as follows. The diagonal element M_{ii} is the integer decorating K_i. When $i \neq j$, the element M_{ij} is the intersection number of the knots K_i and K_j. This number is defined as the sum of all intersections (with signs) between K_i and a Seifert surface spanning K_j and transverse to K_i (knots are oriented, and their Seifert surfaces are also oriented coherently). Cambiare lettera M?

PROPOSITION 1.5. *The matrix M describes the intersection form Q_N of $H_2(N, \mathbb{Z})$.*

PROOF. For each i take a Seifert surface $\Sigma_i \subset S^3$ for K_i, oriented coherently with K_i. By attaching to Σ_i the core disc of the handle attached to K_i we get a closed oriented surface $\overline{\Sigma}_i \subset N$. We show that the classes $[\overline{\Sigma}_1], \ldots, [\overline{\Sigma}_n]$ form a basis for $H_2(N, \mathbb{Z})$, and that the intersection form with respect to this basis is M.

By shrinking every 2-handle to its core disc, and then the 0-handle to its core point, we can build a deformation retract of N onto a wedge $S^2 \vee \ldots \vee S^2$ of n spheres contained in N. These spheres generate $H_2(S^2 \vee \ldots \vee S^2, \mathbb{Z}) \cong H_2(N, \mathbb{Z})$. In the deformation retraction, the surface Σ_i is shrinked to a point. Therefore $\overline{\Sigma}$ is sent to the i-th sphere generating $H_2(N, \mathbb{Z})$. This shows that the classes $[\overline{\Sigma}_i]$ form a basis.

It remains to show that $M_{ij} = Q([\overline{\Sigma}_i], [\overline{\Sigma}_j])$. In order to take two transverse representatives, we push both Σ_i and Σ_j towards the center of D^4. We push Σ_i more than Σ_j. The transverse intersections between the new surfaces correspond to the one between K_i and Σ_j, with the same signs. Therefore we are done. □ Figura.

REMARK 1.6. The proof of Proposition 1.5 shows that (if N has a decomposition without 1-handles) there is a basis $H_2(N, \mathbb{Z})$ of elements represented by *topologically* embedded spheres. Note however that such spheres cannot in general be perturbed to smooth embedded ones.

COROLLARY 1.7. *Every symmetric bilinear (not necessarily unimodular) form is the intersection form of some compact simply connected 4-manifold N with boundary.*

PROOF. It suffices to prove that every symmetric integer matrix M is the intersection matrix of some link. Let n be the rank of M. Take n discs in S^3. Connect each pair of discs with an arc (which connects their boundaries and does

[5]Every connected closed 3-manifold can be constructed in this way, as we will see later.

not intersect any other disc). For every pair $i \neq j$, slide K_j along the arc and make it wind M_{ij} times around K_j. Decorate each K_i with the integer M_{ii}.

The manifold N described by the resulting Kirby diagram has intersection form M and is simply connected, because it is made of 0- and 2-handles. □

Remark 1.8. In the proof of Corollary 1.7, the manifold N is constructed via a Kirby diagram in which every knot is trivial. With a manifold N constructed in this way, the homology $H_2(N, \mathbb{Z})$ has a basis whose elements are represented by *smoothly* embedded spheres.

2. Topological manifolds

This section is devoted to topological manifolds. We wil also introduce the fundamental form E_8, which is even more important in the smooth category.

2.1. The E_8 form. There are finitely many isomorphism types of unimodular forms on each rank r. Every form of rank $r \leqslant 8$ is constructed by summing the elementary forms [1], [−1], H, except the following form of rank $r = 8$.

$$E_8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}.$$

We see immediately that E_8 is even (there are only even numbers on the diagonal). It is also quite easy to see that E_8 is positive definite: the top-left sub-matrices of rank $1, 2, \ldots, 8$ have positive determinants $2, 3, 4, 5, 6, 7, 8, 1$. Actually, the only non-immediate thing to prove is that the determinant of the whole matrix is 1. This is done by calculating the determinant on the last row and then on the last column.

Cerca inglese per sviluppo per riga

$$\det E_8 = 2 \cdot 8 - \begin{pmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & & \\ & & & & 2 & 1 \\ & & & & 1 & 2 \end{pmatrix} = 2 \cdot 8 - 5 \cdot 3 = 1.$$

The form E_8 is thus even and has signature $\sigma(E_8) = 8$. We let nQ denote the n-times sum $Q \oplus \ldots \oplus Q$ if $n > 0$, the empty (zero rank) form if $n = 0$, and the $(-n)$-times sum $(-Q) \oplus \ldots \oplus (-Q)$ if $n < 0$.

The only even forms we may obtain with $[1], [-1]$, and H are nH, and they all have signature zero. Therefore E_8 is a new form. By using E_8 and H, we may construct more even forms, with signature $\sigma(mE_8 \oplus nH) = 8m$ any number divisible by 8.

Exercise 2.1. Some exercises on intersection forms.

(1) Show that $H = -H$, but $E_8 \neq -E_8$.
(2) Show that $H \oplus [1] \cong 2[1] \oplus [-1]$.

(3) Show that $E_8 \oplus (-E_8) \cong 8H$.
(4) Let $Q = [-1] \oplus 8[1]$. Sia $W \subset \mathbb{Z}^9$ be the submodule orthogonal (with respect to Q) to the vector $v = (3, 1, \ldots, 1)$.
- Show that $Q|_W$ is even and has signature 8.
- Show that $Q|_W \cong E_8$.
- Deduce that $E_8 \oplus [-1] \cong 8[1] \oplus [-1]$. On the other hand, show that $E_8 \oplus [1] \ncong 9[1]$. (Hint: count the number of elements with norm 1 in both forms.)

2.2. Plumbing. Proposition 1.7 indicates that there are 4-manifolds with boundary having intersection form $Q = E_8$. We now introduce another technique for building such manifold.

Take two oriented disc bundles E_1, E_2 with Euler numbers e_1, e_2 over two oriented surfaces Σ_1, Σ_2. A *plumbing* consists in the following operation. Take two discs $D_1 \subset \Sigma_1$ and $D_2 \subset \Sigma_2$, and identify each (orientation-preservingly) with D^2. The bundles trivialize as $D_1 \times D^2$ and $D_2 \times D^2$. We glue together the two 4-manifolds E_1, E_2 by identifying $D_1 \times D^2$ with $D_2 \times D^2$ along the map $(x, y) \mapsto (y, x)$ which switches the factors.

The result is a new manifold N. The surfaces Σ_1 and Σ_2 still live in N and have the same Euler number as before. They intersect transversely in one point, with positive sign. The intersection matrix of N is therefore $\begin{pmatrix} e_1 & 1 \\ 1 & e_2 \end{pmatrix}$.

Consider a connected graph, with an integer labeling each vertex. This defines a manifold, obtained by taking a disc bundle over S^2 for each vertex (with Euler number determined by the label), and performing a plumbing for each edge.

A manifold with intersection form E_8 can be constructed from the following plumbing. We denote it by $\mathcal{P}_{E_8}$. Figura piombaggio

2.3. Homology 3-spheres. We can construct compact 4-manifolds M with arbitrary intersection pairing. It turns out that unimodularity is strictly linked with the absence of homology on ∂M.

Proposition 2.2. *Let M be an oriented compact simply connected 4-manifold with boundary. Let ∂M be connected. The intersection form Q on $H_2(M, \mathbb{Z})$ is unimodular if and only if ∂M is a homology sphere.*

Proof. Consider the exact sequence (over the integers)

$$H_3(M, \partial M) \xrightarrow{\partial} H_2(\partial M) \xrightarrow{i} H_2(M)$$
$$\xrightarrow{j} H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i} H_1(M)$$

Since M is simply connected, the first and last modules are trivial.[6] Therefore we have

$$0 \xrightarrow{\partial} H_2(\partial M) \xrightarrow{i} H_2(M) \xrightarrow{j} H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i} 0$$

The module $H_2(M, \partial M)$ is canonically identified with $H^2(M)$, which can in turn be identified with the dual module $H_2(M)^*$ since there is no torsion (because M is simply connected). The map j can therefore be interpreted as a map

$$j : H_2(M) \to H_2(M)^*$$

[6]Actually, triviality of $H_1(M)$ suffices.

and this map is in fact the adjunction of the intersection form on $H_2(M)$. By definition, the intersection form is unimodular if and only if j is an isomorphism. This holds if and only if both $H_2(\partial M)$ and $H_1(\partial M)$ vanish, as required. □

Descrivere qui meglio la sfera di Poincaré?

In particular, the 3-manifold $\partial\mathcal{P}_{E_8}$ is a homology sphere.

2.4. Contractible 4-manifolds. A 4-manifold M with boundary is contractible if it is homotopically equivalent to a point. Note that ∂M does not need to be a sphere. The following result is a particular case of Proposition 2.2.

Proposition 2.3. *The boundary ∂M of a contractible 4-manifold M is a homology sphere.*

There are actually plenty of contractible smooth 4-manifolds. Note that we need 1-handles to construct them: by using 0- and 2-handles we only get D^4 (because there cannot be any 2-handles, since the Euler characteristic must be 1).

The following proposition shows a technique to construct them, by taking a 0-handle, a 1-handle, and a 2-handle which cancels algebraically the 1-handle.

Proposition 2.4. *Let $M = H^0 \cup H^1 \cup H^2$ be a manifold decomposed into handles such that $i(H^1, H^2) = 1$. It is a contractible manifold.*

Esercizio su presentazioni?

Proof. The manifold M is easily seen to be simply connected. It has the homology type of a point because H^1 and H^2 cancel. Use Whitehead Theorem.[7] □

If H^1 and H^2 have *geometric* intersection 1, then they cancel and we get D^3. To get other manifolds, we can take a (sufficiently knotted) core sphere of H^2 which crosses the belt sphere of H^1 three times (with signs $+, +, -$). To be certain that the result is not D^3, it suffices to check somehow that the boundary is a homology sphere distinct from S^3.

The only smooth contractible manifold currently known bounded by S^3 is D^4: whether this is the only one or not, the question is equivalent to the (still open) smooth 4-dimensional Poincaré conjecture.

Question 2.5 (Smooth Poincaré Conjecture in dimension 4). Let M^4 be smooth and homotopy equivalent to S^4. Is M^4 diffeomorphic to S^4?

2.5. Freedman's theorems. A converse of Proposition 2.3 holds in the topological setting.

Theorem 2.6 (Freedman). *Every homology sphere bounds a contractible 4-manifold.*

Super-sketch. Let Σ be a homology 3-sphere. Take $\Sigma \times [0, 1]$. Two steps:

(1) By doing some topological surgery, transform $\Sigma \times [0, 1]$ into a manifold S with the same boundary and homology, but simply connected.
(2) Take countably many copies $S_1, S_2, \ldots, S_k, \ldots$ of S, glue them altogether, and compactify with one point. The resulting object is clearly contractible. Much less clearly, it is a topological manifold: the cone point has indeed a neighborhood homeomorphic to a 4-ball.

□

[7] Theorem 3.3 on page 6.

Theorem 2.6 is not valid in the smooth category. For instance, as we will see below, Poincaré homology sphere does not bound any smooth contractible 4-manifold.

Corollary 2.7 (Freedman). *Every symmetric unimodular bilinear form is the intersection form of a simply connected closed topological 4-manifold.*

Proof. By Corollary 1.7 every symmetric unimodular bilinear form Q is the intersection form of a simply connected 4-manifold M with boundary. By Proposition 2.2 the boundary ∂M is a homology sphere. It therefore bounds a contractible topological manifold N. Glue M and N together. Since N is contractible, the resulting manifold has the same fundamental group and 2-homology as M. □

This result is far from being true in the smooth setting, as we will see below. This leads to plenty of topological 4-manifolds having no smooth structure. This existence result is actually strengthened by an (almost) uniqueness theorem.

Theorem 2.8 (Freedman). *Every even (odd) symmetric unimodular bilinear form is the intersection form of precisely one (two) simply connected topological closed 4-manifold, up to homeomorphism.*

The two manifolds in the even case are distinguished by the *Kirby-Siebenmann invariant*, an obstruction to admitting smooth structures which is also defined in higher dimensions. Indeed, one of the two manifolds sharing the same form does not admit a smooth structure (and in many cases, both do not). In particular, we get the following.

Corollary 2.9. *Two smooth closed oriented simply connected 4-manifolds are homeomorphic if and only if they have isomorphic fundamental form.*

Since the empty form is even, we have the following.

Corollary 2.10 (Poincaré conjecture in dimension 4). *A closed topological manifold homotopically equivalent to S^4 is homeomorphic to S^4.*

Some examples follow.

2.5.1. [1]. The form [1] is odd. There are therefore two topological manifolds with this form. One is of course $\mathbb{CP}^2$, while the other one is denoted by $*\mathbb{CP}^2$. Both such manifolds can be constructed as follows: take a knot $K \subset S^3$ and label it with 1. This is a Kirby diagram of a manifold with one 0-handle and one 2-handle, with intersection form 1. Its boundary is a homology sphere, which can be closed up via a contractible topological manifold.

Depending upon the knot K, the resulting manifold is homeomorphic to either $\mathbb{CP}^2$ or $*\mathbb{CP}^2$. The trivial knot yields $\mathbb{CP}^2$, the trefoil knot gives $*\mathbb{CP}^2$.

2.5.2. H. The forms H is even. The only topological manifold with form H is $S^2 \times S^2$, which of course is also smooth.

2.5.3. E_8. The form E_8 is even. There is only one topological (closed, simply connected) manifold with form E_8, and is just named "the E_8-manifold". As we will see below, the E_8-manifold does not admit any smooth structure.

CHAPTER 4

Classic theorems on smooth 4-manifolds

We turn back to the smooth 4-dimensional category. We state and prove here some important "classic" theorems about 4-manifolds, which were stated and proved before 1970. They include:

- Whitehead theorem, which says that the homotopy type of a closed simply connected 4-manifold is determined by its intersection form only,
- Wall's theorem, which says that two homotopic simply-connected smooth manifolds become diffeomorphic after summing with some copies of $S^2 \times S^2$,
- Rohlin theorem, which says that an oriented 4-manifold with zero signature bounds a 5-manifold.

1. Whitehead theorem

We prove in this section Whitehead's theorem, which says that the fundamental form determines the homotopy type of a closed simply connected 4-manifold. This result was proved in 1940. Much later, Freedman proved in 1980 that it also determines the homeomorphism type of the manifold[1].

A nice geometric way to prove this theorem mimics a famous construction of Thom-Pontryagin, which allowed in 1950 to determine various homotopy groups of spheres.

1.1. The Thom-Pontryagin construction.

A *framed submanifold* $Y^k \subset X^{m+k}$ of a smooth closed manifold X is a smooth submanifold equipped with a trivialization of the normal bundle, *i.e.* an identification of N_Y with $Y \times \mathbb{R}^m$.

A *cobordism* of two framed k-manifolds $Y_0, Y_1 \subset X$ is a properly embedded framed $(k+1)$-manifold $Z \subset X \times [0,1]$, whose intersection with $X \times 0$ and $X \times 1$ coincides with $Y_0 \times 0$ and $Y_1 \times 1$ as framed manifolds (that is, framings match).

Scrivere qualcosa su transversalitá di mappe, o mettere valore regolare.

Fix a point $p \in S^m$. Let f be a smooth map

$$f\colon X^{m+k} \longrightarrow S^m$$

which is transverse to p. Its counterimage $f^{-1}(p)$ is a smooth submanifold $Y^k \subset X$. Take a small disc D^m around p. Over this disc the map looks like a projection $Y^k \times D^m \to D^m$, and this equips the manifold Y with a framing.

Proposition 1.1 (Thom-Pontrjagin construction). *This operation defines a bijection*

$$[X^{m+k}, S^m] \longrightarrow \Omega_k^{\text{framed}}(X^{m+k})$$

[1]This is actually true only for *smooth* manifolds. For topological ones, we also need the Kirby-Siebenman invariant, see Theorem 2.8 on page 31. Whitehead theorem holds for topological manifolds, but we prove it only in the smooth category for simplicity.

between the set $[X^{m+k}, S^m]$ of maps from X^{m+k} to S^m seen up to homotopy, and the set $\Omega_k^{\text{framed}}(X^{m+k})$ of k-dimensional framed submanifolds in X^{m+k} seen up to cobordism.

Proof. We prove that the function

$$\Psi : [X^{m+k}, S^m] \longrightarrow \Omega_k^{\text{framed}}(X^{m+k})$$

introduced above is well-defined. Given f, the trivialization on $Y = \Psi(f)$ is well-defined only up to homotopy; however, homotopic trivializations are easily seen to be cobordant, so this is not a problem. Let f_0 and f_1 be two functions, both transverse to p, linked by a homotopy $F : X \times I \to S^m$. They define two framed manifolds Y_1 and Y_2. We can perturb F so that it is also transverse to p. The pre-image $F^{-1}(p)$ is thus a framed manifold $Z \subset X \times I$ which connects Y_1 and Y_2: these are thus cobordant, as required.

We define an inverse

$$\Phi : \Omega_k^{\text{framed}}(X^{m+k}) \longrightarrow [X^{m+k}, S^m]$$

as follows. Let $Y^k \subset X^{m+k}$ be a framed manifold. A tubolar neighborhood is identified with $Y^k \times D^m$. Let $D^m \to S^m$ be the surjective map which sends 0 to p and collapses ∂D^m to the antipodal point q. By projecting $Y^k \times D^m$ onto its second factor we get

$$Y^k \times D^m \longrightarrow D^m \longrightarrow S^m.$$

Extend this map to the whole of X by sending every point in $X^{k+m} \setminus (Y^k \times D^m)$ to q. We get a map $X \to S^m$, as required. If Y^k changes by cobordism, the resulting map changes by homotopy. This defines Φ.

The map $\Psi \circ \Phi$ is clearly the identity. We prove that $\Phi \circ \Psi$ also is. A map f_0 induces a framed $Y = \Psi(f_0)$, which in turn induces another map $f_1 = \Phi(Y)$. The maps f_0 and f_1 coincide (up to homotopy) on a fixed tubolar neighborhood $Y \times D^m$, which is sent to a disc D in S^m, and may differ a lot on the complement $X \setminus (Y \times D^m)$. However, such a complement is sent by both f_0 and f_1 into the complementary disc $S^m \setminus \text{int}(D)$. Two maps with values in a disc are homotopic (relative to their boundary), and hence we are done. □

The set $\Omega_k^{\text{framed}}(S^{m+k})$ has a natural group structure. Define the sum of two framed manifolds as the disjoint union of such, embedded in two disjoint n-discs in S^{m+k}.

Corollary 1.2. *The bijection above induces an isomorphism of groups*

$$\pi_{m+k}(S^m) \longrightarrow \Omega_k^{\text{framed}}(S^{m+k}).$$

Pontrjagin used this isomorphism to calculate the homotopy groups of spheres for $k = 1, 2$.

Proposition 1.3. *We have $\pi_m(S^m) = \mathbb{Z}$.*

Proof. A framing of a connected 0-manifold, *i.e.* a point, is the choice of a basis in the tangent bundle. It has a sign ± 1 depending on the orientation of the basis (compared with the orientation of S^n). The sign of a framed 0-manifold is the sum of the signs of its components. Sign gives an isomorphism between $\Omega_0^{\text{framed}}(S^m)$ and $\mathbb{Z}$. □

Proposition 1.4. *We have $\pi_3(S^2) = \mathbb{Z}$ and $\pi_{m+1}(S^m) = \mathbb{Z}_2$ for all $m \geqslant 3$.*

Proof. Fix a framing on a connected 1-manifold in S^m. Every other framing is determined by a curve in $O(m-2)$. □ Finire dimo.

Proposition 1.5. *We have $\pi_{m+2}(S^m) = \mathbb{Z}_2$ for all $m \geqslant 3$.* Check!

Later, Rohlin attempted to extend these techniques to $k = 3$. The following result is misteriously related to another theorem of Rohlin on 4-manifolds which we will explore in the next sections.

Proposition 1.6. *We have $\pi_{m+3}(S^m) = \mathbb{Z}_{24}$ for all $m \geqslant 5$.*

Later on, Serre introduced more powerful techniques to calculate the homotopy groups of spheres, and the Pontjagin-Thom construction was then used the other way round, to get more informations on cobordisms. We mention some results obtained with Serre's techniques.

Proposition 1.7. *We have $\pi_{m+4}(S^m) = 0, \pi_{m+5}(S^m) = 0, \pi_{m+6}(S^m) = \mathbb{Z}_2$ when m is sufficiently big.*

1.2. Wedge product of spheres. The following generalization of Pontryagin-Thom construction will be needed in the proof of Whitehead's theorem below. Let $\vee_h S^2$ be a wedge product of h spheres. Fix points $p_1, \ldots, p_h$ in distinct spheres, disjoint from the vertex v of the wedge. Let f be a continuous map

$$f\colon X^{m+k} \longrightarrow \vee_h S^2$$

which is everywhere smooth except at $f^{-1}(v)$, and is transverse to $p_1, \ldots, p_h$. The counterimages $f^{-1}(p_1), \ldots, f^{-1}(p_h)$ define h disjoint (not necessarily connected) framed k-submanifolds of X.

If $X \cong S^3$, the h framed 1-submanifolds define an intersection matrix Q as described in Section 1.4. Vedere caso sconnesso. Cercare dimostrazione decente.

Proposition 1.8. *This operation defines an isomorphism of groups*

$$\pi_3(\vee_h S^2) \longrightarrow S(h, \mathbb{Z}) \cong \mathbb{Z}^{\frac{(h+1)h}{2}}$$

where $S(h, \mathbb{Z})$ is the group of all symmetric integer matrices of rank h.

1.3. Whitehead theorem. We prove here the following.

Theorem 1.9 (Whitehead). *Let M_1 and M_2 be two closed smooth oriented simply connected 4-manifolds. They are homotopically equivalent if and only if their intersection forms Q_{M_1} and Q_{M_2} are isomorphic.*

Proof. Two homotopically equivalent manifolds have the same cohomology ring and thus the same intersection form. Conversely, let M_1 and M_2 have isomorphic intersection forms. Take $M = M_1$. Let $\dot{M}$ be M with the interior of a n-disc removed. The only non-trivial homologies of $\dot{M}$ are $H_0 = \mathbb{Z}$ and $H_2 = \mathbb{Z}^h$. Since M is simply connected, Hurewicz theorem guarantees that every element in H_2 is represented by an immersed sphere.[2]

In particular, a basis $\alpha_1, \ldots, \alpha_h$ is represented by immersions $f_i : S^2 \to M$ with $i = 1, \ldots, h$. We can form a bouquet of these immersions (we homotope them so that they all touch a fixed point in M) and get a map

$$f : \vee_h S^2 \longrightarrow \dot{M}.$$

[2]See Proposition 3.4 on page 21.

This map induces isomorphisms on homologies H_0 and H_2, and is thus a homotopy equivalence by Whitehead's Homology Theorem 3.3. The manifold M is obtained from $\dot{M}$ by attaching a 4-cell. The attaching map translates via the homotopy equivalence to an attaching map $\psi : \partial D^4 \to \vee_h S^2$, well-defined up to homotopy. The homotopy equivalence extends to an equivalence between M and the CW-complex $\vee_h S^2 \cup_\psi D^4$.

The map ψ defines an element in $\pi_3(\vee_h S^2)$. By Proposition 1.8, the map ψ is determined up to homotopy by the corresponding matrix Q. It remains to show that Q represents the intersection form Q_M. Following Thom-Pontryagin construction, take a point p_i in each 2-sphere. This defines a framed link $Y_i \subset S^3$.

Let F_i be an oriented surface properly embedded in D^4 bounding Y_i. By collapsing ∂F_i to a point we get an oriented surface $\bar{F}_i$ in $\vee_h S^2 \cup_\varphi D^4$ and hence a homology element in $H_2(\vee_h S^2 \cup_\varphi D^4) \cong H_2(M)$. The homology elements we get are dual to $\alpha_1, \ldots, \alpha_h$, thus they form a basis. The way they intersect (transversely) in D^4 transports to M: therefore Q represents Q_M. □

Sistemare connessione F_i

2. Cobordism groups

2.1. Definitions. Two oriented closed smooth manifolds M^n and N^n are *cobordant* if there is an oriented manifold W^{n+1} such that $\partial W^{n+1} = M^n \sqcup \overline{N}^n$.

PROPOSITION 2.1. *Cobordism is an equivalence relation on the set of all closed oriented n-manifolds.*

PROOF. A manifold M^n is cobordant to itself: take $W^{n+1} = M^n \times [0,1]$. If M^n is cobordant to N^n via W^{n+1}, then N^n is cobordant to M^n via $\overline{W}^{n+1}$. To prove transitivity, simply glue two cobordisms. □

We denote the set of cobordism classes of closed oriented smooth n-manifolds by Ω_n. Two cobordant manifolds are denoted as $M^n \sim N^n$. We denote the cobordism class of M^n by $[M^n]$. Given two oriented closed manifolds M^n and N^n, we define the sum $[M^n] + [N^n]$ via the disjoint union $[M^n \sqcup N^n]$.

PROPOSITION 2.2. *The sum is well-defined and makes Ω_n an abelian group.*

PROOF. The sum is well-defined since cobordisms of M^n and N^n trivially yield cobordisms of $M^n \sqcup N^n$. Disjoint union is an abelian operation. □

Note that $[M^n] = 0$ if and only if there is an oriented W^{n+1} with $\partial W^{n+1} = M^n$.

EXERCISE 2.3. We have $[M^n \sqcup N^n] = [M^n \# N^n]$.

The cobordism groups Ω_n form in fact a graded ring Ω_* with the product $[M^m] \cdot [N^n] = [M^m \times N^n]$. The unit is the point in Ω_0, with sign $+1$.

2.2. General results. An oriented point is a point with a sign ± 1.

EXERCISE 2.4. The map which associates to an oriented 0-manifold the sum of the signs of its components gives an isomorphism $\Omega_0 \to \mathbb{Z}$.

Every 1-manifold and 2-manifold is easily shown to bound a 2- and 3-manifold. Therefore $\Omega_1 = \Omega_2 = 0$. The same result holds for 3-manifolds and is proved below. Concerning 4-manifolds, we already know[3] that the signature induces a surjective

[3] See Theorem 1.14 on page 18.

map $\sigma : \Omega_4 \to \mathbb{Z}$. We prove below that such a map is in fact an isomorphism. The first cobordism groups are shown below.

n	0	1	2	3	4	5	6	7	8	9	10	11
Ω_n	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

A generator for Ω_4 is $[\mathbb{CP}^2]$. Generators for Ω_8 are $[\mathbb{CP}^4]$ and $[\mathbb{CP}^2 \times \mathbb{CP}^2]$. This kind of picture extends to higher dimensions and fully describe the free part of Ω_n. Let $\Omega_*/_{\rm torsion}$ denote the quotient of Ω_* by the subring consisting of all torsion elements.

THEOREM 2.5 (Thom). *The ring $\Omega_*/_{\rm torsion}$ is freely generated by complex spaces of even dimension.*

Proiettivi di dimensione dispari bordano? Si vede facile?

2.3. Surgery. A simple way to relate two cobordant oriented manifolds M^n and N^n is to construct one from the other via *surgery*. In its most general meaning, a surgery consists of removing some n-dimensional submanifold with boundary $P^n \subset \mathrm{int}(M^n)$ and substituting it with an oriented manifold Q^n. The substitution is made by fixing an orientation-preserving diffeomorphism $\varphi : \partial P^n \to \partial Q^n$ and taking

$$N^n = (M^n \setminus \mathrm{int}(P^n)) \cup_\varphi Q^n.$$

We can also define the manifold $P^n \cup_\varphi Q^n$ by identifying the two boundaries ∂P^n and ∂Q^n along the same φ. It is a closed n-manifold. We assign it an arbitrary orientation (coinciding with either P^n or Q^n).

PROPOSITION 2.6. *If $[P^n \cup_\varphi Q^n] = 0$ then M^n and N^n are cobordant.*

PROOF. By hypothesis there is an oriented manifold W^{n+1} with $\partial W^{n+1} = P^n \cup_\varphi Q^n$. A cobordism between M^n and N^n is constructed by taking $M^n \times [-1, 1]$ and attaching W^{n+1} to $M^n \times 1$ along the map which sends P^n to $P^n \times 1$ identically. □

Semplice figura.

EXAMPLE 2.7 (Handles). We have already encountered the simplest example of surgery when adding a k-handle $W^{n+1} = D^k \times D^{n-k+1}$. The level manifold changes by substituting a $P^n = S^{k-1} \times D^{n-k+1}$ with $Q^n = D^k \times S^{n-k}$. We have $P^n \cup Q^n = \partial W^{n+1} \cong D^{n+1}$.

EXAMPLE 2.8 (Round handles). Let M^n be an oriented manifold containing a copy of $S^{n-2} \times S^1$. Since M^n is oriented, this submanifold has a product tubolar neighborhood $P^n = S^{n-2} \times S^1 \times D^1$. A *round handle* on M^n consists of attaching $W^{n+1} = D^{n-1} \times S^1 \times D^1$ to this tubolar neighborhood. The manifold M^n changes by surgery, and P^n is replaced by $Q^n = D^{n-1} \times S^1 \times S^0$.

EXAMPLE 2.9 (Twisted round handles). Suppose $n \geqslant 3$. We take $W^{n+1} = D^n \times S^1$ as in the previous example, but we choose a slightly more elaborate subdivision of $\partial W^{n+1} = S^{n-1} \times S^1$ as $P^n \cup Q^n$. Subdivide S^{n-1} as $S^{n-1} = A \cup B$ where B is the union of two small discs centered at the poles and A is the closure of $S^{n-1} \setminus B$, *i.e.* a tubolar neighborhood of the equator S^{n-2}.

If we take $P^n = A \times S^1$ and $Q^n = B \times S^1$ we get the previous example. Here, when traveling along S^1, we rotate A and B so that the two components of B get interchanged after one complete turn. That is, we fix an axis intersecting the equator and define R_θ as the counterclockwise rotation of S^{n-1} along this axis of angle θ. Then we take

$$P^n = \bigcup_{\theta \in [0,\pi]} (R_\theta(A), 2\theta)$$

and define Q^n similarly. The manifold P^n is diffeomorphic to $D^{n-2} \times S^1$. The manifold Q^n is the orientable D^1-bundle over a non-orientable S^{n-2}-bundle over S^1![4]

2.4. Smooth cycles. We want to prove that every closed 3-manifold bounds a 4-manifold, and that every closed 4-manifold of zero signature bounds a 5-manifold. We do this by mimicking the construction of a Seifert surface for a knot in S^3. These are the steps:

(1) Embed M^n in $\mathbb{R}^{n+k}$, or equivalently in S^{n+k}.
(2) Let T be the tubolar neighborhood of M^n in S^{n+k}. Find a section of M in ∂T which is homologically trivial in the complement $S^{n+k} \setminus \dot{T}$.
(3) The section bounds a cycle: try to represent this cycle by a manifold.

Step (3) works if $k \leqslant 3$. As a general rule, cycles of codimension at most 2 can always be smoothen.

LEMMA 2.10. *Let Z^{n+k} be a compact smooth manifold and $M^n \subset \partial Z^{n+k}$ be a closed oriented connected submanifold which is homologically trivial,* i.e. $[M^n] = 0$ *in $H_n(Z^{n+k}, \mathbb{Z})$. If $k \leqslant 3$ there is a properly embedded smooth oriented submanifold $W^{n+1} \subset Z^{n+k}$ such that $\partial W^{n+1} = M^n$.*

PROOF. The long exact sequence (over the integers)

$$\dots \longrightarrow H_{n+1}(Z, \partial Z) \xrightarrow{\partial} H_n(\partial Z) \xrightarrow{i_*} H_n(Z) \longrightarrow \dots$$

says that $[M] = \partial\alpha$ for some cycle $\alpha \in H_{n+1}(Z, \partial Z)$. This group is canonically isomorphic to $H^{k-1}(Z)$. If $k = 1$ this implies that $\partial Z = M$ and we are done.

Suppose $k \in \{2, 3\}$. Recall that there is a canonical bijection

$$[Z, K(\mathbb{Z}, k-1)] \longrightarrow H^{k-1}(Z, \mathbb{Z}),$$

and since $k - 1 \in \{1, 2\}$ we can either take $K(\mathbb{Z}, 1) = S^1$ or $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$.[5] The Eilenberg-MacLane space has a preferred class $u \in H^{k-1}$ and the bijection sends a map f to the class $f^*(u)$.

Let f be a map such that $f^*(u)$ is dual to α. Consider the simpler case $k = 2$. Up to homotopy, we can suppose f is smooth and transverse to a point $p \in S^1$. The counterimage $f^{-1}(p)$ is thus a proper submanifold $W^{n+1} \subset Z^{n+2}$. The point p is Poincaré dual to u and hence $f^{-1}(p)$ is dual to α. Actually, we may start with an f such that $f^{-1}(p) \cap \partial Z^{n+2} = M^n$, so that $\partial W^{n+1} = M^n$ as required.

Spiegare come si fa?

If $k = 3$, the map f has compact image and thus lie in $\mathbb{CP}^N \subset \mathbb{CP}^\infty$ for some N. We substitute p with a complex hyperplane H in $\mathbb{CP}^N$, which is dual to u here: we make f smooth transverse to H and take $f^{-1}(H)$. □

The same proof shows in fact that all cycles of codimension at most 2 may be represented by smooth manifolds.

[4] The equator is mirrored along the axis, and thus yield a non-orientable S^{n-2}-bundle. When $n = 3$ the manifold Q^3 is the orientable line bundle over the Klein bottle.

[5] See Section 3.3 on page 7.

Lemma 2.11. *Let M be a smooth manifold of any dimension and $i \in \{1,2\}$. The Poincaré (or Lefschetz) dual to any element in $H^i(M,\mathbb{Z})$ may be represented as a smooth codimension-i orientable submanifold.*

2.5. Embedding in $\mathbb{R}^N$.

Point (2) of our program also works if $k \leqslant 2$.

Theorem 2.12. *Every oriented connected smooth manifold $M^n \subset \mathbb{R}^{n+2}$ bounds a smooth oriented "Seifert" manifold $W^{n+1} \subset \mathbb{R}^{n+2}$*

Proof. First, we prove that the normal bundle N_M of M^n in $\mathbb{R}^{n+2}$ is trivial. The normal bundle is an oriented $\mathbb{R}^2$-bundle, and is hence completely determined by its Euler number (or equivalently its Chern class, if seen as a complex line bundle). However, the Euler number is zero because the manifold is embedded in $\mathbb{R}^{n+2}$. Therefore N_M must be the trivial bundle.[6]

Let T be an open tubolar neighborhood of M^n inside $\mathbb{R}^{n+2}$. Fix an identification $\partial T \cong M^n \times S^1$. Künneth formula gives an isomorphism

$$H_n(M^n,\mathbb{Z}) \oplus H_{n-1}(M^n,\mathbb{Z}) \xrightarrow{\ \varphi\ } H_n(M\times S^1,\mathbb{Z}).$$

Thanks to Lemma 2.11, every cycle $\alpha \in H_{n-1}(M^n,\mathbb{Z})$ may be represented by a closed oriented manifold $\Sigma^{n-1} \subset M^n$. The isomorphism sends

$$\begin{array}{ccc} \big([M^n],0\big) & \longmapsto & [M^n \times \{p\}] \\ \big(0,[\Sigma^{n-1}]\big) & \longmapsto & [\Sigma^{n-1}\times S^1]. \end{array}$$

Mayer-Vietoris sequence for $\mathbb{R}^{n+2} = T \cup (\mathbb{R}^{n+2}\setminus \dot{T})$ gives an isomorphism

$$0 \longrightarrow H_n(\partial T,\mathbb{Z}) \xrightarrow{i_*\times j_*} H_n(T,\mathbb{Z}) \oplus H_n(\mathbb{R}^{n+2}\setminus\dot{T},\mathbb{Z}) \longrightarrow 0.$$

Summing up, we get an isomorphism

$$\Psi : H_n(M^n,\mathbb{Z}) \oplus H_{n-1}(M^n,\mathbb{Z}) \xrightarrow{(i_*\times j_*)\circ\varphi} H_n(T,\mathbb{Z}) \oplus H_n(\mathbb{R}^{n+2}\setminus\dot{T},\mathbb{Z}).$$

The restriction to the first factors

$$H_n(M^n,\mathbb{Z}) \xrightarrow{i_*\circ\varphi} H_n(T,\mathbb{Z})$$

is induced by inclusion and is in fact an isomorphism. We identify these two spaces. Let $([M^n],[\Sigma^{n-1}])$ be the element which is sent to $([M^n],0)$ by Ψ. It is represented as

$$(M^n\times 0)\cup(\Sigma^{n-1}\times S^1)$$

inside $\partial T \cong M^n\times S^1$. To get a manifold, we perform the move suggested in the picture. The surface Σ^{n-1} is orientable and hence has a trivial normal bundle in M^n. A tubolar neighborhood of $\Sigma^{n-1}\times S^1$ in $M^n\times S^1$ is thus diffeomorphic to $\Sigma^{n-1}\times S^1\times[-1,1]$. The cycle $(M^n\times 0)\cup(\Sigma^{n-1}\times S^1)$ intersects this neighborhood in Fare figura.

$$\Sigma^{n-1}\times\Big(\big(0\times[-1,1]\big)\cup\big(S^1\times 0\big)\Big).$$

We can substitute this portion with $\Sigma^{n-1}\times C$, where C is the graphic of the function $\gamma:[-1,1]\to S^1$, $\gamma(t) = -e^{\pi i t}$. graphic?

The result is another manifold in $M^n\times S^1$, still diffeomorphic to M^n: the diffeomorphism in the surgered region is given by $(t,0)\mapsto(t,\gamma(t))$. In fact, the new manifold is another section of the bundle, which yields a different trivialization[7]. □

[6] See Section 4.2 on page 23.

[7] The trivializations of a (trivial) $\mathbb{R}^2$-bundle over a manifold M^n are indeed in 1-1 correspondence with $H^1(M^n,\mathbb{Z}) \cong H_{n-1}(M^n,\mathbb{Z})$.

2.6. Embedding in $\mathbb{R}^N$ up to cobordism. We are left with the point (1) of our program. If we can embed M^n in $\mathbb{R}^{n+2}$, we are done. However, Whitney's theorem only provides embeddings in the much larger $\mathbb{R}^{2n}$. When $n = 3$, we get an embedding of a 3-manifold in $\mathbb{R}^6$ instead of the required $\mathbb{R}^5$.

Whitney's theorem provides however an immersion of M^n in $\mathbb{R}^{2n-1}$. There are manifolds that do not embed in $\mathbb{R}^{2n-1}$. Up to cobordism, they embed.

THEOREM 2.13. *Every closed oriented manifold M^n is cobordant to a closed oriented manifold embedded in $\mathbb{R}^{2n-1}$.*

PROOF. If $n \leqslant 2$ the assertion is trivial, so we suppose $n \geqslant 3$. Let $f : M^n \to \mathbb{R}^{2n-1}$ be a generic immersion. It is injective excepts at double points, which form circles in $\mathbb{R}^{2n-1}$.[8] The counterimage of one circle C is a 1-manifold double-covering it: that is, it consists of either one or two circles in M^n.

The normal bundle on $f^{-1}(C)$ inside M^n is trivial because M^n is orientable. Fix a trivialization $\mathbb{R}^{n-1} \times f^{-1}(C)$. Let N_C be the normal bundle over C in $\mathbb{R}^{2n-1}$. The trivialization induces an identification of each fiber of N_C with $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. Actually, this identification is well-defined only up to permuting the two factors: in fact, when carrying the dentification along C, the factors may be preserved (if $f^{-1}(C)$ is disconnected) or interchanged (if $f^{-1}(C)$ is connected). In the first case, we get a global trivialization of N_C as $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times C$. Since N_C is in fact always trivial (because $\mathbb{R}^{n-1}$ is orientable), the second case may occur only if n is odd.

Project the normal bundle onto a tubolar neighborhood of $f(C)$ in $\mathbb{R}^{2n-1}$. On each fiber the manifold M^n self-intersects as

$$\big(D^{n-1} \times 0\big) \cup \big(0 \times D^{n-1}\big).$$

Perform at each fiber a generalization of the desingularization used to destroy nodal points in complex curves. The desingularization for curves is elegantly described as a perturbation from $zw = 0$ to $zw = \epsilon$.[9] In general dimension, we can describe it as follows.

Let R_θ denote the counterclockwise rotation of angle θ on $\mathbb{R}^2$. Let W_i be the plane generated by e_i and e_{n+i-1}. We have $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} = W_1 \oplus \ldots \oplus W_n$. Let $S_\theta = R_{-\theta} \oplus R_\theta \oplus \ldots \oplus R_\theta$ be the endomorphism of $\mathbb{R}^{2n}$ which rotates W_1 by the angle $-\theta$ and all the other planes $W_i, i > 1$ by the angle θ. Note that $S_{\pi/2}$ sends $(x_1, \ldots, x_n, 0, \ldots, 0)$ to $(0, \ldots, 0, -x_1, x_2, \ldots, x_n)$ and thus interchanges the two discs via an orientation-reversing map.

Define $A \subset S^{2n-1}$ as the union of $S_\theta(S^{n-1} \times 0)$ when θ varies from 0 to $\pi/2$. The subset A is diffeomorphic to $S^{n-1} \times [0, \pi/2]$, a manifold. We have

$$\partial A = \partial(D^n \times 0) \cup \partial(0 \times D^n).$$

By construction, we can give A an orientation so that the orientations on ∂A and $\partial(D^n \times 0) \cup \partial(0 \times D^n)$ also match. Note that A is symmetric with respect to interchanging the two factors in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, and hence the move is really well-defined.

Figura unidimensionale che spiega l'orientazione

If we do this operation for each component C of self-intersections, we get a new manifold N^n embedded in $\mathbb{R}^{2n-1}$. The new manifold N^n is obtained from M^n by surgery, *i.e.* by substituting a submanifold P with some other maniflold Q, see Section 2.3. Both P and Q fiber over $C \cong S^1$. The fiber in P consists of two

[8] See Section 2.3.2 on page 5.

[9] See Section 3.1 on pager 20.

$(n-1)$-discs D, D'. The fiber in Q is $A \cong S^{n-2} \times [-1,1]$. Together, they form a $(n-1)$-sphere. It is easy to conclude that $P \cup Q = S^{n-2} \times S^1$ and hence M^n and N^n are cobordant by Proposition 2.6: in fact, the surgery corresponds to the attaching of a round handle, which is twisted if $f^{-1}(C)$ is connected, see Examples 2.8 and 2.9. □

We are ready to prove the following.

COROLLARY 2.14. *Every oriented 3-manifold bounds an oriented 4-manifold. (That is, $\Omega_3 = 0$.)*

PROOF. A closed oriented 3-manifold is cobordant to a closed oriented 3-manifold embedded in $\mathbb{R}^5$. Such a manifold bounds an oriented 4-manifold by Theorem 2.12. □

2.7. 4-manifolds. We turn to the more difficult problem of determining Ω_4. We already know that if M^4 bounds then it has zero signature $\sigma(M^4) = 0$.[10] The signature is additive on disjoint union and thus gives a surjective homomorphism $\sigma : \Omega_4 \to \mathbb{Z}$. The class $[\mathbb{CP}^2]$ is sent to 1. We now prove that it is injective.

THEOREM 2.15 (Rohlin). *The signature $\sigma : \Omega_4 \to \mathbb{Z}$ is an isomorphism.*

Dimostrazione da ampliare in alcuni punti.

PROOF. The main idea is that all the arguments used in the 3-dimensional case may be adapted to the 4-dimensional one up to summing up with some copies of $\mathbb{CP}^2$. Actually, we prove that every closed oriented M^4 is cobordant to some $\#_k\mathbb{CP}^2$. We necessarily get $\sigma(M^4) = k\sigma(\mathbb{CP}^2) = k$, and hence σ is injective, as required.

Theorem 2.13 says that M^4 is cobordant to a closed oriented 4-manifold embedded in $\mathbb{R}^7$, which we still call M^4 for simplicity. Theorem 2.12 does not hold in general for codimension-3 embeddings. We try however somehow to mimic its proof.

The normal bundle N_{M^4} is not necessarily trivial in codimension 3. Let however T be a tubolar neighborhood of M^4 in $\mathbb{R}^7$. The Euler class of N_{M^4} is zero (because M^4 lies in $\mathbb{R}^7$). Since the Euler class is an element in $H^3(M^4, \mathbb{Z})$, there is a section of N_{M^4} on a 3-skeleton of M^4. Up to adding some $\mathbb{CP}^2$ to M^4, we can suppose the section extends.

Mettere sezione su teoria dell'ostruzione e spiegare meglio.

Let T be an open tubolar neighborhood of M^4 in $\mathbb{R}^7$. Its boundary ∂T is a S^2-bundle over M. It is not necessarily trivial, but it has a section, which we denote by $s(M)$. A section suffices for the Künneth formula to hold. Thus, as in the proof of Theorem 2.12 we have an isomorphism

$$H_4(M^4, \mathbb{Z}) \oplus H_2(M^n, \mathbb{Z}) \xrightarrow{\ \varphi\ } H_4(\partial T, \mathbb{Z}).$$

which sends

$$\begin{array}{ccc} \big([M^n], 0\big) & \longmapsto & [s(M^n)] \\ \big(0, [\Sigma^{n-1}]\big) & \longmapsto & [\pi^{-1}(\Sigma^{n-1})] \end{array}$$

where $\pi : \partial T \to M$ is the projection. Mayer-Vietoris sequence for $\mathbb{R}^7 = T \cup (\mathbb{R}^7 \setminus \dot{T})$ gives an isomorphism

$$0 \longrightarrow H_4(\partial T, \mathbb{Z}) \xrightarrow{i_* \times j_*} H_4(T, \mathbb{Z}) \oplus H_4(\mathbb{R}^7 \setminus \dot{T}, \mathbb{Z}) \longrightarrow 0.$$

[10] See Theorem 1.14 on page 18.

and the same arguments used in the proof of Theorem 2.12 define a surface $\Sigma^2 \subset M^4$ such that

$$s(M) \cup p^{-1}(\Sigma)$$

inside ∂T bounds a cycle in $\mathbb{R}^7 \setminus \dot{T}$. We now smoothen this cycle to a manifold M' and conclude using Lemma 2.10 that M' bounds a 5-manifold contained in $\mathbb{R}^7$. To smoothen this cycle we will maybe add some other projective planes, so that $M' = M\#_k(\mathbb{CP}^2)$. □

Ampliare qui.

3. Wall theorem

We know from Whitehead theorem that two closed simply connected smooth oriented 4-manifolds with isomorphic intersection forms are homotopy equivalent. Actually, we know from Freedman theorem that they are homeomorphic. A theorem of Wall moreover shows that they become diffeomorphic after summing with some copies of $S^2 \times S^2$.

3.1. Sphere bundles. We have the following.

Proposition 3.1. *The $\mathbb{R}^k$-bundles over S^h up to isomorphism are in natural 1-1 correspondence with $\pi_{h-1}(SO(k))$.*

Proof. Let E be such a bundle. Fix a riemannian metric on E. Consider the equator S^{h-1} inside S^h. The bundle is trivial on each emisphere, *i.e.* isometric to $D^h \times \mathbb{R}^k$. The bundle E is thus determined by the isometric gluing of these two trivializations over the equator, which we may suppose orientation-preserving up to switch the orientation of one of them. The gluing is encoded by a map $f : S^{h-1} \to SO(k)$.

Homotopic maps yield isomorphic bundles. The converse also holds. □

Precisare

Since $\pi_1(SO(3)) = \mathbb{Z}_2$, there are two $\mathbb{R}^3$-bundles over S^2, which yields two S^2-bundles over S^2. One is the trivial one $S^2 \times S^2$, and the other is denoted by $S^2 \widetilde{\times} S^2$. Actually, we get nothing really new.

Proposition 3.2. *We have $S^2 \widetilde{\times} S^2 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.*

We will need the following.

Lemma 3.3. *Let N^5 be obtained by adding a 2-handle to D^5. Then ∂N^5 is either $S^2 \times S^2$ or $S^2 \widetilde{\times} S^2$.*

Proof. The attaching sphere of the 2-handle is a loop, and all loops are isotopic in S^4. Represent D^5 as $D^2 \times D^3$ and take $S^1 \times 0$ as a loop. Attach the handle along $S^1 \times D^3$. The result is the attaching of two copies of $D^2 \times D^3$ which extends to a D^3-fibering over S^2. Its boundary is a S^2-fibering over S^2, as required. □

3.2. Wall theorem. Wall's theorem says the following.

Theorem 3.4. *Let M^4 and N^4 be two closed simply connected smooth oriented 4-manifolds with isomorphic intersection forms. There is a natural number h such that $M^4\#_h(S^2 \times S^2)$ is diffeomorphic to $N^4\#_h(S^2 \times S^2)$.*

PROOF. Since they have the same signature, the two manifolds are cobordant. We thus get a five-dimensional cobordism W^5 with $\partial W^5 = M^4 \cup \overline{N}^4$. Take a handle decomposition of this cobordism. Note that all the steps in Smale's proof of Poincaré Conjecture can be applied in dimension 5, except Whitney trick. In particular, we may remove 0- and 5-handles, and we may also trade 1-handles for 3-handles, using the fact that M^4 is simply connected. Analogously, we trade 4-handles for 3-handles.

We end up with a handle decomposition of W^5 with 2- and 3-handles only. Let Z^4 be the level manifold between 2- and 3-handles. We show that the attaching of a 2-handle changes the level manifold by a connected sum with either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. Every (five-dimensional) 2-handle is attached along a loop in the (four-dimensional) level manifold. This four-dimensional level manifold is simply connected, and hence the loop is isotopic to the trivial one. Therefore the loop is contained in a 4-disc, and the level manifold is changed via a connected sum with the manifold of Lemma 3.3, which is indeed either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. Therefore we get

$$Z^4 \cong M^4 \#_h (S^2 \times S^2) \#_k (S^2 \tilde{\times} S^2) \cong N^4 \#_l (S^2 \times S^2) \#_m (S^2 \tilde{\times} S^2).$$

□

Decidere se mettere enunciato debole o concludere.

4. Classifications of intersection forms

The classification of all forms that arise as intersection forms of smooth 4-manifolds is not yet complete. However, much is known, thanks to three important results:

- Serre's classification of indefinite unimodular forms, which is a general algebraic result,
- Rohlin's signature theorem, which excludes "half" of even forms in all dimensions $4k$,
- Donaldson's theorem, which deals with definite forms and is intimately linked to dimension 4.

A form Q is *definite* if it is either positive or negative definite. It is *indefinite* otherwise.

4.1. Indefinite forms. Quite suprisingly, indefinite forms behave in a simple way.

THEOREM 4.1 (Serre). *Let Q be an indefinite unimodular form. Then either $Q \cong m[1] \oplus n[-1]$ or $Q \cong \pm mE_8 \oplus nH$.*

The heavy part of this theorem relies on the following result, which says that null vectors exist on integers (or equivalently, on rational numbers).

LEMMA 4.2 (Meyer's lemma). *Let Q be indefinite unimodular on some $\mathbb{Z}$-module A. There exists a vector $v \in A$ such that $Q(v, v) = 0$.*

We use this lemma to prove Serre's theorem for odd forms. Given a submodule $B \subset A$, we define its Q-orthogonal $B^\perp$ as usual. In general, we cannot split A as $B \oplus B^\perp$ as we do with vector spaces. In fact, we can split precisely when $Q|_B$ is unimodular, as the following shows.

LEMMA 4.3. *We have $A = B \oplus B^\perp$ if and only if $Q|_B$ is unimodular. If this holds, we have $Q = Q|_B \oplus Q|_{B^\perp}$.*

PROOF. We always have $B \cap B^\perp = \{0\}$. We show that $B + B^\perp = A$ if and only if $Q|_B$ is unimodular. Actually, if $B \oplus B^\perp = A$ then $Q = Q|_B \oplus Q|_{B^\perp}$ and $\det Q = \det Q|_B \cdot \det Q|_{B^\perp}$, so necessarily $\det Q|_B = \pm 1$.

Conversely, let $Q|_B$ be unimodular. We show that $B + B^\perp = A$. Let $x \in A$ be any element. We show that it lies in $B + B^\perp$. The unimodular form Q defines an adjoint $x^* \in A^*$ as $x^*(a) = Q(x, a)$. The restriction $x^*|_B$ is an element of B^*. By unimodularity of $Q|_B$, such an element is dual to some $x_B \in B$. That is, we have $Q(x_B, b) = Q(x, b)$ for all $b \in B$. Write $x = x_B + (x - x_B)$. Since $Q(x - x_B, b) = 0$ for all $b \in B$, we have $x - x_B \in B^\perp$ and we are done. □

We now prove Serre's classification (assuming Meyer's lemma) for odd forms.

PROPOSITION 4.4 (Odd Serre). *Let Q be an odd indefinite unimodular form. Then $Q \cong m[1] \oplus n[-1]$.*

PROOF. Let Q be defined over some free module A. We prove our assertion by induction on $\dim A$. By Meyer's lemma there is an element $v \in A$ with $Q(v, v) = 0$. We can suppose v is primitive. There is an element w with $Q(v, w) = 1$: it suffices to complete v to a basis for A and take $w = v^*$ in a dual basis.[11]

Consider the submodule B generated by v and w. We have $Q|_B \cong \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}$ for some integer x. Thus $\det Q|_B = -1$ and Lemma 4.3 gives $Q = Q|_B \oplus Q|_{B^\perp}$. We want an odd x. If x is even, then $Q|_B$ is even, and thus $Q|_{B^\perp}$ is odd. Therefore there is an odd element $u \in B^\perp$, and by substituting w with $w + u$ we get an odd integer x.

It is now easy to construct a change of basis that transforms $\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}$ into $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and then finally into $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This in particular proves our assertion when $\dim A = 2$. If $\dim A > 2$ we argue by induction. We have $Q \cong [1] \oplus [-1] \oplus Q|_{B^\perp}$. Both $[1] \oplus Q|_{B^\perp}$ and $[-1] \oplus Q|_{B^\perp}$ are odd. One of these is certainly indefinite. By induction, it is isomorphic to $m[1] \oplus n[-1]$ and we are done. □

4.2. Characteristic elements. Let Q be an unimodular form on a free module A. An element $\underline{w} \in A$ is *characteristic* if

$$Q(\underline{w}, z) \equiv Q(z, z) \pmod 2.$$

That is, the parity of any element z in A is obtained by pairing z with $\underline{w}$. In some sense, a characteristic element controls the parity of all elements in A.

EXERCISE 4.5. The trivial element $0 \in A$ is characteristic if and only if Q is even. If $A = \mathbb{Z}^{n+m}$ and $Q = n[1] + m[-1]$ then $(1, \ldots, 1)$ is a characteristic element.

A strange but easy lemma shows that $Q(\underline{w}, \underline{w})$ equals the signature of Q modulo 8.

LEMMA 4.6 (Van der Blij). *We have $Q(\underline{w}, \underline{w}) \equiv \sigma(Q) \pmod 8$.*

[11] See Proposition 1.6 o page 16.

PROOF. First, take two characteristic elements $\underline{w}, \underline{w}'$. Thus $Q(\underline{w} - \underline{w}', z)$ is even for all $z \in A$. Since Q is unimodular, this implies that $\underline{w} - \underline{w}' = 2v$ for some $v \in A$. Therefore

$$\begin{aligned} Q(\underline{w}', \underline{w}') &= Q(\underline{w} - 2v, \underline{w} - 2v) = Q(\underline{w}, \underline{w}) - 4Q(v, \underline{w}) + 4Q(v, v) \\ &= Q(\underline{w}, \underline{w}) + 4\big(Q(v, v) - Q(v, \underline{w})\big). \end{aligned}$$

Since $Q(v, v) - Q(v, \underline{w})$ is even, we get

$$Q(\underline{w}, \underline{w}) \equiv Q(\underline{w}', \underline{w}') \pmod 8.$$

Every characteristic element thus yields the same number in $\mathbb{Z}_8$. It remains to prove that it the same number determined by $\sigma(Q)$. There are two cases.

(1) If Q is odd and indefinite, we have $Q \cong m[1] \oplus n[-1]$ by Serre's theorem. Take $\underline{w} = (1, \dots, 1)$. We get $Q(\underline{w}, \underline{w}) = m - n = \sigma(Q)$.
(2) In all other cases, the form $Q' = Q \oplus [1] \oplus [-1]$ is odd and indefinite. If $\underline{w}$ is a characteristic element for Q, then $\underline{w}' = (\underline{w}, 1, 1)$ is characteristic for Q'. By the previous point the theorem holds for $\underline{w}'$. Since $Q(\underline{w}, \underline{w}) = Q(\underline{w}', \underline{w}')$ and $\sigma(Q) = \sigma(Q')$, it also holds for $\underline{w}$.

□

COROLLARY 4.7. *If Q is an even form, then $\sigma(Q)$ is divisible by 8.*

PROOF. The trivial element is characteristic. □

COROLLARY 4.8. *Every even form has the same rank and signature of some $\pm mE_8 \oplus nH$.*

Contents