DIFFERENTIABILITY AND AREA FORMULA ON STRATIFIED LIE GROUPS

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Abstract. We prove the Area Formula for Lipschitz maps between stratified nilpotent Lie groups. The main tool is the differentiablity of Lipschitz maps, proved by P. Pansu in Ann. of Math. '89. We extend this result to the case of measurable domains with non trivial technical modifications. A suitable notion of jacobian is given for differential maps, called G-linear maps, finding relations with the classical definition of jacobian. Consider two stratified nilpotent Lie groups $\mathbb{G}$, $\mathbb{P}$ and a Lipschitz map $f : A \longrightarrow \mathbb{P}$, where A is a measurable set of $\mathbb{G}$. The symbols $\mathcal{H}^Q_d$, $\mathcal{H}^Q_\rho$ denote the Q-Hausdorff measures respectively defined on the metric spaces $(\mathbb{G}, d)$ and $(\mathbb{P}, \rho)$. The jacobian of a G-linear map $L : \mathbb{G} \longrightarrow \mathbb{P}$ is defined as $J_Q(L) = \mathcal{H}^Q_\rho(L(B_1))/\mathcal{H}^Q_d(B_1)$, where B_1 is the unit metric ball of $\mathbb{G}$. Thus, the Area formula is stated as follows

$$\int_A J_Q(d_x f)\; d\mathcal{H}^Q_d(x) = \int_{\mathbb{P}} N(f, A, y)\; d\mathcal{H}^Q_\rho(y)\,,$$

where $N(f, A, y)$ is the multiplicity function of f, relative to the set A.

Contents

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1. Introduction

In the last years a considerable attention has been devoted to Carnot-Carathéodory spaces, characterized by a geometry quite far from the euclidean one, see [5], [9], [17], [18], [19], [31], [32], [34], and other authors. Since these spaces are not locally bi-Lipschitz equivalent to euclidean ones, the development of the classical tools of Geometric Measure Theory in these spaces (area and co-area formulas, sets of finite perimeter, currents) is still a largely open problem, nevertheless some important recent contributions have been given [1], [6], [8], [13], [14], [15], [16], [20], [22], [24], [25], [29], [30], [31], [35], but the list could be enlarged.

In this paper we establish the area formula for Lipschitz maps $f : A \subset \mathbb{G} \longrightarrow \mathbb{P}$, where $\mathbb{G}$ and $\mathbb{P}$ are stratified groups (with the terminology of [12]) and A is a measurable subset of $\mathbb{G}$. As in the euclidean case, the proof of this result strongly depends on the a.e. differentiability of f (that makes sense in this setting, due to the homogeneous structure of the domain and of the target space). In a fundamental work [28] Pansu proved that the Rademacher theorem still holds, i.e. any Lipschitz map $f : A \longrightarrow \mathbb{P}$ is a.e. differentiable in A, provided that A is an open subset of $\mathbb{G}$. In this paper we extend the Pansu result to a slightly more general situation, dropping the assumption that A is open and proving the differentiability property at a.e. density point of A. This generalization requires some effort, since no Lipschitz extension theorem is presently known in this general setting. Although we follow essentially the Pansu approach, our proof involves some nontrivial technical adjustments due to the fact that the interior of A could be empty (see Section 3). In the case when $\mathbb{G} = \mathbb{P}$ (and the area formula reduces to the change of variables formula), the same problem has been considered by Vodop'yanov and Ukhlov in [35], but with a different approach, involving a different definition of jacobian. However, the proof of differentiability is independent of [35], where, in the author's opinion, a technical difficulty in the extension of Pansu's technique has been overlooked.

The area formula states that

$$\int_A J_Q(d_x f)\ d\mathcal{H}_d^Q(x) = \int_{\mathbb{P}} N(f, A, y)\ d\mathcal{H}_\rho^Q\,.$$

The function $N(f, A, y)$ is the standard multiplicity function and $J_Q(d_x f)$ is the jacobian of $d_x f$, which is defined on all G-linear maps and can be computed by

$$J_Q(d_x f) = \frac{\mathcal{H}_\rho^Q\,(d_x f(B_1))}{\mathcal{H}_d^Q(B_1)}$$

(by translation invariance, any bounded open set in place of B_1 can be considered). A G-linear map $L : \mathbb{G} \longrightarrow \mathbb{P}$ is a homogeneous homomorphism with the contact property of sending horizontal elements of $\mathbb{G}$ in horizontal elements of $\mathbb{P}$. Thus, a more manageable formula for the jacobian of G-linear maps can be obtained, noticing that L induces, through the exponential map, a linear map $\tilde{L}$ between the corresponding Lie algebras $\mathcal{G}$, $\mathcal{P}$ (see Corollary 3.14). Indeed, we prove that

$$J_Q(L) = \frac{\mathcal{H}^Q_\rho \llcorner P(B_1^\rho)\, \mathcal{L}^q(B_1)}{\mathcal{H}^q \llcorner \tilde{P}(B_1^\rho)\, \mathcal{H}^Q_d(B_1)}\, J^a(\tilde{L}),$$

where $J^a(\tilde{L})$ is the classical algebraic jacobian of $\tilde{L}$, $P = L(\mathbb{G})$ and $\tilde{P} = \ln P$ (see Proposition 3.18). The other factor $(\mathcal{H}^Q_\rho \llcorner P(B_1^\rho)\, \mathcal{L}^q(B_1))/(\mathcal{H}^q \llcorner \tilde{P}(B_1^\rho)\, \mathcal{H}^Q_d(B_1))$ plays the role of a *distortion factor* which takes into account the different measures $\mathcal{H}^Q_d$, $\mathcal{H}^Q_\rho$. Notice that in the case that $\mathbb{P} = \mathbb{G}$ this factor is exactly equal to one (see Remark of subsection 3.2), so our definition of jacobian coincides with the classical one, according to the results of [35].

Our proof of the area formula follows a classical path, but the definition of jacobian allows to avoid the decomposition of the differential as a product of a symmetric linear map and a rotation, following a bit more intrinsic computation. We get a decomposition of almost all of

$$A' = \{x \in A \mid \exists d_x f,\ \ J_Q(d_x f) > 0\}$$

in countably many measurable sets A_i on which f is close to a G-linear map L_i. We have adopted a different approach from the classic one in the proof that $\mathcal{H}^Q(f(A \setminus A')) = 0$: indeed, instead of the usual approximation of $f(x)$ by $f_\varepsilon(x) = (f(x), \varepsilon x)$ (see [11]) we follow a purely intrinsic approach adopted, in the euclidean case, in [2].

2. Notation and definitions

We consider a stratified simply connected Lie group $\mathbb{G}$ and its Lie algebra $\mathcal{G}$, which is the direct sum of the spaces V_i, with the generating conditions $[V_i, V_1] = V_{i+1}$ and $V_i = 0$ for $i > n$, [12]. The least integer n is called the *degree of nilpotency* or the *step of* $\mathbb{G}$. The stratified structure of the algebra allows us to define a one parameter group of dilations $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$ as $\delta_r(v) = \sum_{i=1}^n r^i u_i$, where $v = \sum_{i=1}^n u_i$, $u_i \in V_i$ and r is any positive number. So $\delta_r \circ \delta_s = \delta_{rs}$ holds for r, $s > 0$ and δ_r is a homomorphism of the algebra $\mathcal{G}$. We can also define dilations for negative factors as follows: $\delta_t v = \delta_{|t|} v^{-1}$ for $t < 0$. Thanks to the generating hypothesis we find a basis of vectors $(v_i)_{i=1,\dots,k}$ for the space V_1 such that the set of all finite linear combinations of v_i and all their commutators $[v_i, v_j]$, $[v_i, [v_j, v_l]], \ldots$, generate $\mathcal{G}$. An important property of the basis $(v_i) \subset \mathcal{G}$ is the following: there exists a bounded neighbourhood of the unit element E, such that any $z \in E$ can be represented as a product with a fixed number of factors of the form $\exp(a_1 v_{i_1}) \cdots \exp(a_\gamma v_{i_\gamma})$, $i_j \in \{1, \ldots, k\}$ with (a_j) belonging to a bounded set of $\mathbb{R}^\gamma$. In all the subsequent we will call a basis (v_i) with the above property a *generating basis*.

The exponential map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism because $\mathbb{G}$ is simply connected, so we denote the inverse map of exp as $\ln : \mathbb{G} \longrightarrow \mathcal{G}$. The well known Baker-Campbell-Hausdorff formula translates the law group of $\mathbb{G}$ into the algebra $\mathcal{G}$, allowing explicit computations, [33]. We can interpret the exponential map as a unique chart which allows us to think objects of the group as elements of a vector space with a suitable metric. We define on $\mathbb{G}$ a left invariant distance as follows. The subspace $V_1 \subset \mathcal{G}$ can be translated over all fibers of the group, so we obtain a uniform distribution of subspaces of the tangent bundle, the so called *horizontal bundle*. Now consider all curves which are absolutely continuous with derivative a.e. in the horizontal bundle, the so called *horizontal curves*. The generating condition on the group implies that the space is connected by horizontal curves [7], [17]. We fix a scalar product on $\mathcal{G}$, then a Riemannian metric is defined on all of $\mathbb{G}$ such that all translations are isometries of the group ($\mathbb{G}$ is also called sub-Riemannian group). Thus, given any pair of points x, y of $\mathbb{G}$, we define the infimum among the lengths of all horizontal curves joining the points as the Carnot-Carathéodory distance or sub-Riemannian distance between them, denoted by $\mathrm{d}(x, y)$, [17]. This definition fits into the more general framework of Carnot-Carathéodory spaces [23]. The fact that translations are isometries implies that the distance we have defined is left invariant under translations of the group, that is, $\mathrm{d}(x, y) = \mathrm{d}(zx, zy)$ for any x, y, $z \in \mathbb{G}$. We define a distance on $\mathcal{G}$ by making the exponential map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ an isomentry, so we essentially identify $\mathcal{G}$ with $\mathbb{G}$ throughout the paper and we will not use different symbols to denote the distance both on $\mathcal{G}$ and $\mathbb{G}$. A group of dilations on $\mathbb{G}$ is defined as $\exp \circ \delta_r \circ \ln : \mathbb{G} \longrightarrow \mathbb{G}$, for any $r > 0$ and still denoted with tha same symbol. The analogous group properties of dilations in $\mathcal{G}$ hold. The self-similarities δ_r have the important compatibility property with the metric $\mathrm{d}(\delta_r x, \delta_r y) = r\, \mathrm{d}(x, y)$, whenever $x, y \in \mathbb{G}$, $r > 0$. If e is the unit element of $\mathbb{G}$, we will write $\mathrm{d}(x) = \mathrm{d}(x, e)$.

The topological dimension of $\mathbb{G}$ will be denoted by q, throughout the paper. It turns out that the Lie algebra $\mathcal{G}$ is a vector space of dimension q and a Lebesgue measure $\mathcal{L}^q$ on $\mathcal{G}$ is defined in the fixed scalar product. In view of the Baker-Campbell-Hausdorff formula, the jacobian of any translation is an upper triangular matrix, with 1's along the diagonal, this implies the left invariance of $\mathcal{L}^q$ under translations of $\mathcal{G}$. The metric balls of $\mathcal{G}$ have Lebesgue measure which scales with a power $Q = \sum_1^n i \dim V_i$ of the radius. This comes from algebraic definition of dilations, because r^Q is just the jacobian of $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$. So, $\mathcal{G}$ as a metric space has finite Q-Hausdorff measure $\mathcal{H}_d^Q$, which is left invariant. Thus, the Lebesgue measure $\mathcal{L}^q$ and $\mathcal{H}_d^Q$ are Haar measure on the group, so $\mathcal{L}^q$ is a constant multiple of $\mathcal{H}_d^Q$. A more general result about the Hausdorff dimension holds in Carnot-Carathéodory spaces [23]. To simplify notation we will denote by $\mathcal{H}_d^Q$ as the Q-Hausdorff measure both on $\mathcal{G}$ and $\mathbb{G}$, being the exponential map an isometry.

In this paper we consider maps between two not necessarily equal Carnot groups, so we have to define another stratified simply connected Carnot group $\mathbb{P}$ with an algebra $\mathcal{P}$. The algebra $\mathcal{P}$ is the direct sum of subspaces W_j, with $W_j = 0$ for $j > m$. We have dilations Δ_r both on $\mathbb{P}$ and $\mathcal{P}$, there is a distance ρ and all of their properties are stated analogously as for $\mathbb{G}$.

Definition 1 (Lipschitz functions)**.** Let (X, d), (Y, ρ) be two metric spaces and $f : X \longrightarrow Y$. If there exists a constant $L \geq 0$ such that

$$\rho(f(u), f(v)) \leq \; L\, d(u, v) \qquad \text{for any } u, v \in X$$

we say that f is *L-Lipschitz* and L is a *Lipschitz constant* of f. We denote with $Lp(f)$ as the infimum among all Lipschitz constants of f.

Definition 2 (Doubling spaces)**.** Consider a Borel measure μ on a metric space X. The couple (X, μ) is called a *Doubling space* if there is a constant C such that for any ball $B \subset X$ it follows

$$\mu(2B) \leq C\, \mu(B)\,,$$

where $2B$ denotes the ball with the same center and double radius.

Definition 3 (Density points)**.** Given a metric measure space (X, d, μ) and a measurable set $A \subset X$, we define $\mathcal{I}(A)$ as the set of points $x \in A$ such that

$$\frac{\mu(A \cap B_{x,r})}{\mu(B_{x,r})} \longrightarrow 1 \quad \text{as} \quad r \longrightarrow 0\,,$$

where $B_{x,r}$ denotes the open ball in X with center x and radius r.

We will call *density points*, the elements of $\mathcal{I}(A)$. If (X,d,μ) is a doubling space it can be proved that $A \setminus \mathcal{I}(A)$ has μ-measure zero, see for example [11]; moreover $\mathcal{I}(A)$ is bi-Lipschitz invariant. The next lemma states an elementary and well known property of density points in doubling spaces.

Lemma 2.1. *Let (X,d,μ) be a doubling space. Then for any measurable set A and $x \in \mathcal{I}(A)$ we have $d(y,A) = o\,(d(y,x))$ as $y \to x$.*

Definition 4. We define $B_{x,r} = \{y \in \mathbb{G} \mid \mathrm{d}(y,x) < r\}$ and $B_r = B_{e,r}$, where e is the unit element of $\mathbb{G}$. We distinguish balls of $\mathbb{P}$ adding the symbol ρ as $B^{\rho}_{x,r}$. We will use the same symbol to denote balls respectively of $\mathcal{G}$ and $\mathcal{P}$. In fact, the Lie groups $\mathbb{G}$, $\mathbb{P}$ are considered isometric to their Lie algebras $\mathcal{G}$ and $\mathcal{P}$.

The Carnot-Carathéodory distance and the Euclidean distance in a stratified Lie group $\mathbb{G}$ are related by the following estimate

$$|x-y| \le \mathrm{d}(x,y) \le C\,|x-y|^{1/n} \text{ for any } x,y \in K \subset \mathbb{G}\,, \tag{1}$$

where n is the degree of nilpotency of $\mathbb{G}$, K is a compact subset and C is a dimensional constant depending on K. This estimate is true in more general Carnot-Caratheodory spaces, see paragraph 0.5 of [17] and [26].

Definition 5. A map $L : \mathbb{G} \longrightarrow \mathbb{P}$ is called *homogeneous* if $\Delta_r(Lx) = L(\delta_r x)$ for any $r > 0$. A map $\tilde{L} : \mathcal{G} \longrightarrow \mathcal{P}$ is homogeneous if the map $\exp\circ\tilde{L}\circ\ln : \mathbb{G} \longrightarrow \mathbb{P}$ is.

Definition 6 (G-linear maps)**.** We say that a map $L : \mathbb{G} \longrightarrow \mathbb{P}$ is *G-linear* if it is a homogeneous Lie group homomorphism. A map $\tilde{L} : \mathcal{G} \longrightarrow \mathcal{P}$ is called G-linear if $\exp\circ L\circ\ln : \mathbb{G} \longrightarrow \mathbb{P}$ is G-linear.

Remark. Notice that our definition of homomorphism between the Lie algebras $\mathcal{G}$, $\mathcal{P}$ here is not the conventional one. In fact, in the Lie group theory Lie homomorphism of Lie algebras are assumed to be linear with respect to the linear structure of the algebra and homomorphism with respect to the Lie product $[\cdot,\cdot]$, [36]. In our setting the exponential map is a diffeomorphism, so we have an associative operation on $\mathcal{G}$ which cames from that of $\mathbb{G}$, then there is a natural structure of Lie group on the Lie algebra. Thus, our definition of G-linear map on Lie algebras is referred to the Lie group structure of the algebra. In the Subsection 3.1 the conventional properties assumed for Lie algebra homomorphisms will be proved starting from Definition 6.

Definition 7. Given a measurable set $A \subset \mathbb{G}$, $x \in \mathcal{I}(A)$ and a map $f : A \longrightarrow \mathbb{P}$, we say that f is differentiable at x if there exists a G-linear map $L : \mathbb{G} \longrightarrow \mathbb{P}$ such that

$$\lim_{y \in A,\ y \to x} \frac{\rho\big(f(x)^{-1} f(y), L(x^{-1}y)\big)}{\mathrm{d}(x,y)} = 0. \tag{2}$$

A map $f : A \subset \mathcal{G} \longrightarrow \mathcal{P}$ is differentiable at $x \in \mathcal{I}(A)$ if $\exp \circ f \circ \ln : \exp(A) \longrightarrow \mathbb{P}$ is differentiable at $\ln x$, (which is a density point, in view of Remark 2.1).

Notice that Definition 7 becomes the Pansu definition of differentiability [28] when A is an open set. The following proposition shows that the differential is unique and essentially independent of the domain A; the proof is a straightforward consequence of Lemma 2.1, so we omit it.

Proposition 2.2. *Let $f : A \subset \mathbb{G} \longrightarrow \mathbb{P}$, $g : B \subset \mathbb{G} \longrightarrow \mathbb{P}$ be Lipschitz maps, with $x \in \mathcal{I}(A \cap B)$, $f = g$ on $A \cap B$ and f satisfies (2). Then we have*

(1) *the map g is differentiable at x and*

$$\lim_{y \in B,\, y \to x} \frac{\rho\big(g(x)^{-1} g(y), L(x^{-1}y)\big)}{\mathrm{d}(x,y)} = 0.$$

(2) *If g is differentiable at x with differential T, then $T = L$.*

In particular, the differential is unique.

Motivated by Proposition 2.2 we denote by $d_x f$ the differential of f at x, wherever it exists.

3. Differentiability on measurable sets

In this section we prove the a.e. differentiability of a Lipschitz map $f : A \longrightarrow \mathbb{P}$, where $A \subset \mathbb{G}$ is a measurable set and $\mathbb{G}$, $\mathbb{P}$ are stratified groups. Since the target metric space $\mathbb{G}$ is complete and f is a Lipschitz function, if it is not otherwise stated, we will assume in this section that A is a closed subset of $\mathbb{G}$. In view of Proposition 2.2 we note that this assumption does not modify the differential map. We have already seen that the notion of differential is well posed at density points. But the question of differentiability is also related to the shape of the domain around the point where we consider the differential approximation. Moreover, we will see that the definition of differential (G-linear map) requires some further properties of the domain around the point. This issue is explained in the following proposition.

Proposition 3.1. *Consider a summable function $g : \mathbb{G} \longrightarrow \mathbb{R}$ and $z \in \mathbb{G}$. Then*

$$\int_{\mathbb{G}} |g(y\delta_t z) - g(y)|\, d\mathcal{H}_d^Q(y) \longrightarrow 0 \qquad as \qquad t \to 0\,.$$

Proof. By an isometric change of variable, the map g can be read on $\mathcal{G}$ where it is $\mathcal{L}^q$-measurable. Then we can use the standard density arguments to achieve the theorem. The density argument works because the Lebesgue measure is preserved under translations of the group. The isometric change of variable does not change the value of the integral. □

Corollary 3.2. *Let $A \subset \mathbb{G}$ be a compact set and let (τ_j) be an infinitesimal sequence. Then there exists a subsequence (t_l) such that, $\lim_{t_l \to 0} \mathbf{1}_A(y\delta_{t_l} z) = 1$, for $\mathcal{H}_d^Q$-a.e. $y \in A$.*

Proof. It is enough to apply Proposition 3.1 to $g = \mathbf{1}_A$. □

The following Lemma is a particular case of Theorem 2.10.1 in [33].

Lemma 3.3. *Let Z_1, Z_2 be two subspaces whose direct sum gives $\mathcal{G}$ and Z_1 with dimension 1. Then there are open neighbourhoods of the origin $\Omega_1 \subset Z_1$, $\Omega_2 \subset Z_2$ and an open $U \in \mathbb{G}$, $U \ni e$, such that the map $\phi : \Omega_2 \times \Omega_1 \longrightarrow U$, defined as $\phi(\omega, z) = \exp \omega \exp z$, is a diffeomorphism.*

Proposition 3.4 (Linear density)**.** *Let $v \in \mathcal{G}$ and $T_{x,v} = \{s \in \mathbb{R} \mid x \exp(sv) \in A\}$, then $0 \in \mathcal{I}(T_{x,v})$ for $\mathcal{H}_d^Q$-a.e. $x \in A$.*

Proof. Consider the map $\phi : \Omega_2 \times \Omega_1 \longrightarrow U$ of Lemma 3.3, where Z_1 is the space spanned by v and Z_2 is the complement factor. Covering A with a countable family of translated neighbourhoods $\{y_i U\}$ it is not restrictive to assume that $A \subset U$. Thus, identifying $\Omega_2 \times \Omega_1$ with $\mathbb{R}^q$, by 3.1.3(5) of [11] applied to the measurable set $\phi^{-1}(A) \subset \Omega_2 \times \Omega_1$ we obtain that for $\mathcal{L}^q$-a.e. $(\omega, z) \in \Omega_2 \times \Omega_1$ the set $\{\tau \mid (\omega, \tau v) \notin \phi^{-1}(A)\}$ has density zero at t. Then the set

$$\begin{aligned} T_{\phi(\omega,tv),v} &= \{s \mid \phi(\omega, tv) \exp(sv) \notin A\} \\ &= \{s \mid \phi\,(\omega, (t+s)v) \notin A\} = \{\tau \mid \phi(\omega, \tau v) \notin A\} - t \end{aligned}$$

has density zero at $s = 0$. □

Remark. It is important to observe that only when $v \in V_1$ (v is a horizontal vector) we have $T_{x,v} = \{s \in \mathbb{R} \mid x \exp(sv) \in A\} = \{s \in \mathbb{R} \mid x\,\delta_s(\exp v) \in A\}$. This fact will be useful in the proof of Theorem 3.9, for the construction of the approximating path (see discussion before the Theorem).

Lemma 3.5 (Horizontal extension). *Consider $v \in V_1$ and a Lipschitz function $f : A \subset U \longrightarrow \mathbb{P}$, with U as in the Lemma 3.3. Then there exists a function $f^v : U \longrightarrow \mathbb{P}$ extending f, which is $Lp(f)$-Lipschitz on any segment $\{y \exp(tv) \mid tv \in \Omega_1\} \subset U$ for any $y \in \exp(\Omega_2) \subset U$.*

Proof. Let $\phi : \Omega_2 \times \Omega_1 \longrightarrow U$ be as in the Lemma 3.3. For any $\omega \in \Omega_2$ we will extend the map $\phi(\omega, \cdot)$ to all of Ω_1. The set $Z_\omega = \{tv \in \Omega_1 \mid \phi(\omega, tv) \in A\}$ is closed in Ω_1, so $Z_\omega^c \cap \Omega_1$ is a countable disjoint union of open intervals. Thus, we can define $f^v(\omega, \cdot)$ on any bounded interval of $Z_\omega^c \cap \Omega_1$ joining with a geodesic the values of $f(\omega, \cdot)$ on the boundary of the interval (Carnot groups are geodesically complete metric spaces, [17]) and putting constant values on the unbounded intervals, if they exist. This extension of $f^v(\omega, \cdot)$ is $Lp(f)$-Lipschitz on the segment $\phi(\omega, \Omega_1)$, because we are using the Carnot-Carathéodory metric (length metric) and $\phi(\omega, tv) = \exp \omega \exp(tv)$ is a radial geodesic in $(\mathbb{G}, \mathrm{d})$, being $v \in V_1$. □

Remark. Under the hypotheses of Lemma 3.5 we make the following two observations: the extension f^v is not necessarily continuous on U and if $u = \delta_a v$, for some $a \in \mathbb{R}$, we have $f^u = f^v$. The map $\ln \circ \phi : \Omega_2 \times \Omega_1 \longrightarrow \mathcal{G}$, being differentiable, is locally Lipschitz with respect to the Euclidean metric on $\Omega_2 \times \Omega_1$ and the scalar product on $\mathcal{G}$. This implies a *Lusin property* for the map $\exp \circ \phi$, that is, $\mathcal{L}^q$-negligible sets of $\Omega_2 \times \Omega_1$ are mapped into $\mathcal{L}^q$-negligible sets of $\mathcal{G}$. But $\mathcal{L}^q$ is proportional to $\mathcal{H}_d^Q$ on $\mathcal{G}$, so the Lusin property holds for ϕ.

Using the extension lemma and the differentiability of rectifiable curves proved in [28] we get the existence of partial derivatives along horizontal directions.

Proposition 3.6 (Horizontal derivatives). *Under the hypotheses of Lemma 3.5, for $\mathcal{H}_d^Q$-a.e. $x \in U$ there exists*

$$\lim_{t \to 0} \Delta_{1/t} \left(f^v(x)^{-1} f^v(x \exp(tv)) \right) = \partial_x f^v \left(\exp(v) \right) \in \exp(W_1).$$

In particular f has partial derivative along v for $\mathcal{H}_d^Q$-a.e. $x \in A$.

Proof. Consider $f^v : U \longrightarrow \mathcal{P}$ and define the Lipschitz curve

$$J_\omega(t) = f^v \left(\phi(\omega, tv) \right) \quad \text{for any } tv \in \Omega_1 .$$

The Proposition 4.2 of [28] gives the differentiability of J_ω for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ (in the sense of definition 7) and moreover the derivative is in W_1. So the derivative is an horizontal direction of $\mathcal{P}$. Now by a Fubini argument we get the partial differentiability of f for $\mathcal{L}^q$-a.e. $(\omega, t) \in \Omega_2 \times \Omega_1$ and by Remark 3 the $\mathcal{H}_d^Q$-a.e. partial differentiability follows. □

Proposition 3.7. *Define* $T_{x,v} = \{t \in \mathbb{R} \mid x\exp(tv) \in A\}$*, with* $v \in \mathcal{G}$*. Then for any* $\tau \in \mathbb{R}$ *the map* $\beta : \mathbb{G} \longrightarrow \mathbb{R} \cup \{+\infty\}$ *defined as* $\beta(x) = \inf_{s \in T_{x,v}} |s - \tau|$ *is lower semicontinuous (where is assumed* $\inf \emptyset = +\infty$*).*

Proof. Choose $\sigma > 0$ and $x \in A$ such that $\beta(x) > \sigma$. Fix σ_1 such that $\beta(x) > \sigma_1 > \sigma$, so $x\exp(tv) \notin A$ for any $t \in [\tau - \sigma_1, \tau + \sigma_1]$. By the closure of A together with the continuity of the map $x\exp(tv)$ with respect to the variables (x, t), there exists $\varepsilon > 0$ such that $y\exp(tv) \notin A$ for any $y \in B_{x,\varepsilon}$ and any $t \in [\tau - \sigma_1, \tau + \sigma_1]$. Then for any $y \in B_{x,\varepsilon}$ it follows $\beta(y) \geq \sigma_1 > \sigma$. $\square$

Corollary 3.8. *The map* β *is finite for* $\mathcal{H}_d^Q$*-a.e.* $y \in A$ *and* $y\exp(\beta(y)v) \in A$*.*

Proof. This is a straightforward consequence of Proposition 3.4. $\square$

The following theorem is one of the two main results of this paper. As explained in the introduction, we have not in general a Lipschitz extension of the map $f : A \longrightarrow \mathbb{P}$, so when we fix a point $x \in A$ and a direction $w \in \mathcal{G}$ it might happen that $x\exp(tw) \notin A$ for many $t > 0$ and so we are not able to consider the difference quotient of f in that direction. Thus, we start to fix the attention on the generating basis of horizontal directions (v_i), selecting all density points x whose curves $J_i(t) = x\exp(tv_i)$ intersect A in one dimensional sets which have density 1 at 0, getting a set of full measure on A. At these points we are able to approximate any curve $c(t) = \exp(\delta_t z)$, $z \in \mathcal{G}$, with a path which is basically a projection of the line on the set A with controlled distance with respect to the horizontal directions of the generating basis. Then the function f is defined along this path and it is possible to evaluate the difference quotient of f.

Theorem 3.9 (Differentiability)**.** *Let* $f : A \longrightarrow \mathbb{P}$ *be a Lipschitz map, where* A *is a measurable subset of* $\mathbb{G}$*. Then* f *is differentiable* $\mathcal{H}_d^Q$*-a.e.*

Proof. *Step 1, (Existence and uniform convergence of partial derivatives)*
Consider a bounded neighbourhood $E \subset \mathbb{G}$ containing $\{y \in \mathbb{G} \mid \mathrm{d}(y) \leq 1\}$. Fix a generating basis $\{v_i \mid i = 1, \dots k\} \subset \mathcal{G}$, so there exist an integer γ and a bounded neighbourhood M of the origin of $\mathbb{R}^\gamma$ such that $E = \{\prod_{s=1}^{\gamma} \exp(a_s v_{i_s}) \mid (a_s) \subset M\}$, where the products of the elements are understood in ordered sense. By the σ-compactness of $\mathbb{G}$, $\mathcal{H}_d^Q$ being a Radon measure on $\mathbb{G}$, we can assume that A is compact. Thus, considering U as in Lemma 3.3 we cover A with a finite open covering $\{y_i U\}$ and translating f on $(y_i U) \cap A$, the invariance of differentiability under translations allows to assume $A \subset U$. Applying Proposition 3.6 we have the partial derivatives $\partial_y f^{v_i}(\exp(v_i)) \in \exp(W_1)$ of the extension f^{v_i} for $\mathcal{H}_d^Q$-a.e.

$y \in A \subset U$, for $i = 1, \ldots, k$. Thus, for any $\varepsilon > 0$, Egorov theorem and the partial differentiability of f give a closed subset $A_{1\varepsilon} \subset A$ such that $\mathcal{H}_d^Q(A \setminus A_{1\varepsilon}) \leq \varepsilon/3$ and the limits

$$\lim_{t\to 0} \Delta_{1/t}\left(f(y)^{-1} f^{v_{i_s}}\left(y \exp(tv_{i_s})\right)\right) = \partial_y f^{v_{i_s}}\left(\exp(v_{i_s})\right),$$

with $s \in \{1, \ldots, \gamma\}$, $y \in A_{1\varepsilon}$, are uniform. Defining $u_{i_s} = \exp(a_s v_{i_s})$ and $s = 1, \ldots, \gamma$, we observe that the existence of $\partial_y f^{v_{i_s}}(u_{i_s})$ is equivalent to that of $\partial_y f^{v_{i_s}}\left(\exp(v_{i_s})\right)$ and

$$\lim_{t\to 0} \Delta_{1/t}(f(y)^{-1} f^{v_{i_s}}(y\delta_t u_{i_s})) = \partial_y f^{v_{i_s}}(u_{i_s}) = \Delta_{a_s} \partial_y f^{v_{i_s}}\left(\exp(v_{i_s})\right),$$

for any $s \in \{1, \ldots, \gamma\}$, $y \in A_{1\varepsilon}$. The uniformity of the limit holds even when $a \in M$. In fact, the following equality holds

$$\rho\Big(\Delta_{1/t}\left(f(y)^{-1} f(y\delta_t u_{i_s})\right), \partial_y f^{v_{i_s}}(u_{i_s})\Big)$$
$$= a_s \rho\Big(\Delta_{1/(a_s t)}\left(f(y)^{-1} f(y\delta_{a_s t} v_{i_s})\right), \partial_y f^{v_{i_s}}(\exp(v_{i_s}))\Big).$$

For any $\zeta \neq 0$ and any $s = 1, \ldots, \gamma$ we define the map

$$\beta(y, \zeta, v_{i_s}) = \inf_{t \in T_{y, v_{i_s}}} |t - \zeta|,$$

by Proposition 3.7 this map is a measurable function. Proposition 3.4 and Lemma 2.1 imply that the quotient $|\zeta - \beta(y, \zeta, v_{i_s})|/\zeta$ tends to zero as $\zeta \to 0$ for $\mathcal{H}_d^Q$-a.e. $y \in A$. Then, by Egorov theorem we get a uniform convergence, for $s = 1, \ldots, \gamma$, in another closed subset $A_{2\varepsilon} \subset A$ such that $\mathcal{H}_d^Q(A \setminus A_{2\varepsilon}) \leq \varepsilon/3$. Define the measurable map

$$\theta_t(y) = \sup_{u \in B_{y,t} \setminus \{y\}} \left(\mathrm{d}(u, A)/\mathrm{d}(u, y)\right)$$

for $t > 0$ and use again Lemma 2.1 to obtain that $\theta_t(y) \to 0$ as $t \to 0^+$ for $\mathcal{H}_d^Q$-a.e. $y \in A$. Using Egorov theorem we are able to find a closed set $A_{3\varepsilon} \subset A$ such that $\mathcal{H}_d^Q(A \setminus A_{3\varepsilon}) \leq \varepsilon/3$ and $\theta_t(y)$ goes to zero uniformly on $A_{3\varepsilon}$ as $t \to 0$. Now consider $A_\varepsilon = A_{1\varepsilon} \cap A_{2\varepsilon} \cap A_{3\varepsilon}$ and $x \in \mathcal{I}(A_\varepsilon)$. Notice that A_ε does not depend on the vector $a = (a_s) \in M$, moreover $\mathcal{H}_d^Q(A \setminus A_\varepsilon) \leq \varepsilon$. We want to prove the convergence of the following limit

(3)
$$\lim_{x\delta_t z \in A,\, t\to 0} \Delta_{1/t}(f(x)^{-1} f(x\delta_t z)) = \prod_{s=1}^{\gamma} \partial_x f^{v_{i_s}}(u_{i_s}) = \prod_{s=1}^{\gamma} \Delta_{a_{i_s}}\left(\partial_x f^{v_{i_s}}\left(\exp(v_{i_s})\right)\right),$$

uniformly with respect to $a \in M$ and $z = \prod_{s=1}^{\gamma} \exp(a_s v_{i_s}) = \prod_{s=1}^{\gamma} u_{i_s}$. By Lemma 2.1 with $A = A_\varepsilon$ and $y = x\delta_t u_{i_1}$, we can choose $u^t \in A_\varepsilon$ such that

$$\mathrm{d}(x\delta_t u_{i_1}, u^t) \leq \mathrm{d}(x\delta_t u_{i_1}, x)\, \theta_{ct}(x)\,, \tag{4}$$

where $c = \sup_{a\in M,\, l=1,\dots,\gamma} \mathrm{d}\left(\exp(a_1 v_{i_1}) \cdots \exp(a_l v_{i_l})\right)$. Representing $u^t = x\delta_t u^t_{i_1}$, the left invariance and the homogeneity of the distance give

$$\mathrm{d}\left(u_{i_1}, u^t_{i_1}\right) = \frac{\mathrm{d}\left(x\delta_t u_{i_1}, x\delta_t u^t_{i_1}\right)}{t} \leq c\,\theta_{ct}(x) \to 0 \qquad \text{as} \qquad t \to 0$$

Then the convergence of $u^t_{i_1}$ to u_{i_1} is uniform with respect to $a \in M$. Now by induction suppose that vectors $(w^t_{i_j})$ are defined for any $j \leq s < \gamma$ such that $x\delta_t u^t_{i_1} \cdots u^t_{i_j} \in A_\varepsilon$ and $\mathrm{d}(u^t_{i_j}, u_{i_j}) \to 0$, uniformly with respect to $a \in M$ (for simplicity of notation we have omitted the parenthesis after the symbol of dilation δ_t, being understood that all subsequent terms are considered dilated). Again from Lemma 2.1 with $A = A_\varepsilon$ and $y = x\delta_t u^t_{i_1} \cdots u^t_{i_s} u_{i_{s+1}}$, we find another family of points in A_ε, which can be represented as $x\delta_t u^t_{i_1} \cdots u^t_{i_s} u^t_{i_{s+1}}$ for a suitable $u^t_{i_{s+1}}$ and with the property

$$\mathrm{d}(x\delta_t u^t_{i_1} \cdots u^t_{i_s} u_{i_{s+1}}, x\delta_t u^t_{i_1} \cdots u^t_{i_s} u^t_{i_{s+1}}) \leq 3c\,t\,\theta_{3ct}(x)\,, \tag{5}$$

for t small enough, depending on s. The estimate (5) is independent of $a \in M$. From inequality (5), by the left invariance and the homogeneity of the distance, we deduce

$$\mathrm{d}(u_{i_{s+1}}, u^t_{i_{s+1}}) = \frac{\mathrm{d}(x\delta_t u^t_{i_1} \cdots u^t_{i_s} u_{i_{s+1}}, x\delta_t u^t_{i_1} \cdots u^t_{i_s} u^t_{i_{s+1}})}{t} \leq 3c\,\theta_{3ct}(x) \longrightarrow 0$$

as $t \to 0^+$ and uniformly on $a \in M$. Finally we consider

$$\Delta_{1/t}\left(f(x)^{-1} f(x\delta_t u_{i_1} \cdots u_{i_\gamma})\right) = \Big(\prod_{s=1}^{\gamma} D^t_s B^t_s\Big) G^t$$

where $z = u_{i_1} \cdots u_{i_\gamma} = \prod_{s=1}^{\gamma_1} \exp(a_s v_{i_s})$ and we have defined :

$$D^t_s = \Delta_{1/t}\left(f(x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}})^{-1} f^{v_{i_s}}(x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}} u_{i_s})\right)\,,$$

$$B^t_s = \Delta_{1/t}\left(f^{v_{i_s}}(x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}} u_{i_s})^{-1} f(x\delta_t u^t_{i_1} \cdots u^t_{i_s})\right)\,,$$

$$G^t = \Delta_{1/t}\left(f(x\delta_t u^t_{i_1} \cdots u^t_{i_\gamma})^{-1} f(x\delta_t u_{i_1} \cdots u_{i_\gamma})\right)\,.$$

We observe that $x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}} \in A_\varepsilon$ for $s = 1, \dots, \gamma$, so $D^t_s \to \partial f^{v_i}(u_{i_s})$ as $t \to 0$ and uniformly when $a \in M$. It remains to be seen that B^t_s, $s = 1, \dots, \gamma$, and G^t go to the unit element as $t \to 0$, uniformly as $a \in U$. Denote $y^t_s = x\delta_t u^t_{i_1} \cdots u^t_{i_{s-1}} \in$

A_ε and $\omega_{i_s} = \lg u_{i_s}$; in view of Corollary 3.8 we see that $y^t_s \exp(\beta(y^t_s, t, w_{i_s})w_{i_s}) \in A$, then we can further decompose $B^t_s = F^t_s\, N^t_s$, where

$$F^t_s = \Delta_{1/t}\left(f^{v_{i_s}}(x\delta_t u^t_{i_1}\cdots u^t_{i_{s-1}}u_{i_s})^{-1} f\left(x(\delta_t u^t_{i_1}\cdots u^t_{i_{s-1}})\exp(\beta(y^t_s,t,w_{i_s})w_{i_s})\right)\right),$$

$$N^t_s = \Delta_{1/t}\left(f\left(x(\delta_t u^t_{i_1}\cdots u^t_{i_{s-1}})\exp(\beta(y^t_s,t,w_{i_s})w_{i_s})\right)^{-1} f(x\delta_t u^t_{i_1}\cdots u^t_{i_s})\right).$$

We have seen that $\beta(y,\zeta,v_{i_s})/\zeta \to 1$ as $\zeta \to 0$, uniformly in $y \in A_\varepsilon$, then $\beta(y^t_s, a_s t, v_{i_s})/a_s t \to 1$ when a varies in M. Moreover

$$a_s\,\beta(y,t,w_{i_s}) = \beta(y, a_s t, v_{i_s}), \qquad s \in \{1,\dots,\gamma\}$$

so the following estimates hold

$$\begin{aligned}
\rho(F^t_s) &\le Lp(f)\,\frac{\mathrm{d}\left(\delta_t u_{i_s}, \exp(\beta(y^t_s,t,w_{i_s})w_{i_s})\right)}{t}\\
&= Lp(f)\,\mathrm{d}\left(\exp(w_{i_s}), \exp\left((\beta(y^t_s,t,w_{i_s})/t)w_{i_s}\right)\right)\\
&= Lp(f)\,a_s\,\mathrm{d}\Big(\exp(v_{i_s}), \exp\left((\beta(y_{sl},a_s t_l,v_{i_s})/(a_s t_l))v_{i_s}\right)\Big)
\end{aligned}$$

$$\le Lp(f)\,\left(\sup_{a\in U}|a|\right)\,\mathrm{d}\Big(\exp(v_{i_s}), \exp\left((\beta(y_{sl},a_s t_l,v_{i_s})/(a_s t_l))v_{i_s}\right)\Big), \tag{6}$$

$$\begin{aligned}
\rho(N^t_s) &\le Lp(f)\,\frac{\mathrm{d}\left(\delta_t u^t_{i_s}, \exp(\beta(y^t_s,t,w_{i_s})w_{i_s})\right)}{t}\\
&= Lp(f)\,\mathrm{d}\left(u^t_{i_s}, \exp\left((\beta(y^t_s,t,w_{i_s})/t)w_{i_s}\right)\right)
\end{aligned}$$

$$= Lp(f)\,\mathrm{d}\left(u^t_{i_s}, \exp\left((\beta(y^t_s,a_s t,v_{i_s})/(a_s t))w_{i_s}\right)\right). \tag{7}$$

The first of these two estimates follows by Lemma 3.5, whereas the second is due to the fact that the points $x(\delta_t u^t_{i_1}\cdots u^t_{i_{s-1}})\exp(\beta(y^t_s,t,w_{i_s})w_{i_s})$ and $x\delta_t u^t_{i_1}\cdots u^t_{i_s}$ are in A, where f is Lipschitz. Both last right terms of equations (6), (7) go to zero uniformly as $a \in M$. The same reasoning yields

$$\rho(G^t) \le Lp(f)\,\mathrm{d}(u^t_{i_1}\cdots u^t_{i_\gamma}, u_{i_1}\cdots u_{i_\gamma}) \longrightarrow 0, \tag{8}$$

where we have used the uniform convergence of any $u^t_{i_s}$ for $s = 1,\dots,\gamma$. Now we remember that $x \in \mathcal{I}(A_\varepsilon)$ and ε is arbitrary, so there exists a null set $N \subset A$ such that for any $x \in A\setminus N$ the equation (3) holds uniformly with respect to $a \in U$.

Step 2, (G-linearity and construction of differential)

One finds easily that partial derivatives are 1-homogeneous under dilations. We want to prove the homomorphism property of partial derivatives, that is $\partial_y f(u\omega) = \partial_y f(u)\partial_y f(\omega)$. To get this equality we use step 1, but we need at least of an infinitesimal sequence $(t_l) \subset \mathbb{R}\setminus\{0\}$, which connects the three directions in the following sense : for $\mathcal{H}^Q_d$-a.e. $y \in A$ we have $y\delta_{t_l}(u\omega)$, $y\delta_{t_l}u$, $y\delta_{t_l}\omega \in A$. In fact, equation (3) is not trivial when we have directions $z \in E$ such that $x\delta_{t_j}z \in A$ and $t_j \to 0$. To obtain the sequence (t_l) it is enough to consider the three arbitrary directions $u\omega$, u, $\omega \in \mathbb{G}$ and iterate Corollary 3.2 for any direction, extracting further subsequnces. In this situation, with $u = \prod_{s=1}^{\gamma_1} \exp(b_s v_{i_s})$ and $\omega = \prod_{s=1}^{\gamma_2} \exp(c_s v_{i_s})$, applying twice step 1 it follows

$$\lim_{x\delta_t z \in A,\, t \to 0} \Delta_{1/t}\Big(f(x)^{-1} f(x\delta_t(u\omega))\Big)$$

$$= \prod_{s=1}^{\gamma_1} \Delta_{b_s}\Big(\partial_x f\left(\exp(v_{i_s})\right)\Big) \prod_{s=1}^{\gamma_2} \Delta_{c_s}\Big(\partial_x f\left(\exp(v_{i_s})\right)\Big),$$

$$\partial_x f(u\omega) = \lim_{x\delta_t z \in A,\, t \to 0} \Delta_{1/t}\Big(f(x)^{-1} f(x\delta_t(u\omega))\Big) = \partial_x f(u)\partial_x f(\omega) \tag{9}$$

and directly from equation (3) we infer

$$\lim_{x\delta_t z \in A,\, t \to 0} \Delta_{1/t}\Big(f(x)^{-1} f(x\delta_t(u^{-1}))\Big) = \left(\partial_x f(u)\right)^{-1}. \tag{10}$$

Now we want to define the differential map $d_y f$ globally on $\mathbb{G}$ for $\mathcal{H}^Q_d$-a.e. $y \in A$. Consider the countable dense subset $D_0 = \{\prod_{s=1}^{\gamma} \exp(b_s v_{i_s}) \mid (b_s) \in \mathbb{Q}^\gamma\} \subset \mathcal{G}$. Define the countable set given by $D = \{\omega_1 \cdots \omega_j \mid j \in \mathbb{N},\ \omega_i \in D_0,\ i = 1, \dots, j\}$. For any $\omega \in D$, in view of Corollaty 3.2 we get a sequence (depending on ω) which allows to apply step 1, defining the partial derivative of f on direction ω for any $y \in A \setminus N_\omega$, where $\mathcal{H}^Q_d(N_\omega) = 0$. It follows that equation (3) is not trivial for all $\omega \in W$ and for all $y \in A \setminus \bigcup_{\omega \in D} N_\omega$. Thus, for $\mathcal{H}^Q_d$-a.e. $y \in A$ and $\omega \in D$, it is well defined the partial derivative

$$L_y(\omega) = \lim_{t \to 0,\, A \ni x\delta_t \omega} \Delta_{1/t}\left(f(y)^{-1} f(y\delta_t \omega)\right).$$

By density we extend L_y to all of $\mathbb{G}$, setting $L_y(z) = \lim_{l \to \infty} L_y(\omega_l)$ whenever $(\omega_l) \subset W$ and $\omega_l \to z$. In view of equations (9) and (10) the sequence $L_y(\omega_l)$ is convergent and the extension is well defined, so choosing another sequence $(z_l) \subset W$ which converges to z we obtain

$$\rho\left(L_y(\omega_l)^{-1} L_y(z_l)\right) = \rho\left(L_y(\omega_l^{-1}) L_y(z_l)\right) = \rho\left(L_y(\omega_l^{-1} z_l)\right) \le Lp(f)\mathrm{d}(\omega_l^{-1} z_l) \to 0$$

as $l \to \infty$, because $\omega_l z_l \in W$. The latter inequality also proves that $L_y(\omega_l)$ is a Cauchy sequence whenever (ω_l) is convergent. We have defined $L_y : \mathbb{G} \longrightarrow \mathbb{P}$ for $\mathcal{H}^Q_d$-a.e. $y \in A$. By definition of L_y and equations (9), (10) the G-linearity of L_y follows.

Step 3, (Differentiability)
In step 1 we have proved that for $\mathcal{H}_d^Q$-a.e. $y \in A$ it follows
(11)
$$\rho\Big(\Delta_{1/t}\left(f(y)^{-1}f(y\delta_t z)\right), \prod_{s=1}^{\gamma}\Delta_{a_s}\partial_y f^{v_{i_s}}\left(\exp(v_{i_s})\right)\Big) \longrightarrow 0 \qquad \text{as} \qquad t \to 0\,,$$
uniformly when $z = \prod_{s=1}^{\gamma_1}\delta_{a_s}v_{i_s}$, $a \in M$, and $y\delta_t z \in A$.

We want to prove that the uniform limit (11) implies the differentiability. Assume by contradiction the existence of $\sigma > 0$ and $(z_l) \subset \mathbb{G}$ such that $z_l \to 0$ and
$$\rho\left(f(y)^{-1}f(yz_l), L_y(z_l)\right) \geq \sigma \mathrm{d}(z_l)\,,$$
define $z_l = \delta_{t_l}w_l$, with $t_l = \mathrm{d}(z_l)$, obtaining
$$\rho\left(\Delta_{1/t_l}\left(f(y)^{-1}f(y\delta_{t_l}w_l)\right), L_y(w_l)\right) \geq \sigma\,. \tag{12}$$
Represent $w_l = \prod_{s=1}^{\gamma}\exp(b_s^l v_{i_s})$, $(\mathrm{d}(w_l) = 1)$, and consider rational vectors $(b_s^{lj}) \in \mathbb{Q}^\gamma \cap M$ such that $\omega_{lj} = \prod_{s=1}^{\gamma}\exp(b_s^{lj}v_{i_s}) \in D_0$ and $\omega_{lj} \to \omega_l$ as $j \to \infty$. The explicit definition of L_y implies $L_y(\omega_{lj}) = \prod_{s=1}^{\gamma}\Delta_{b_s^{lj}}\left(\partial_x f^{v_{i_s}}(\exp(v_{i_s}))\right)$. As we will see in Subsection 3.1, any G-linear map is continuous in the topologies induced by the metrics, so that
$$L_y(\omega_l) = \lim_{j\to\infty} L_y(\omega_{lj}) = \lim_{j\to\infty}\prod_{s=1}^{\gamma}\Delta_{b_s^{lj}}\Big(\partial_x f^{v_{i_s}}\Big(\exp(v_{i_s})\Big)\Big) = \prod_{s=1}^{\gamma}\Delta_{b_s^l}\Big(\partial_x f^{v_{i_s}}(v_{i_s})\Big)\,.$$
Replacing $L_y(\omega_l)$ in equation (12) we have
$$\rho\left(\Delta_{1/t_l}\left(f(y)^{-1}f(y\delta_{t_l}w_l)\right), \prod_{s=1}^{\gamma}\Delta_{b_s^l}\Big(\partial_x f^{v_{i_s}}\Big(\exp(v_{i_s})\Big)\Big)\right) \geq \sigma\,,$$
so from uniform convergence of equation (11) it follows
$$\rho\left(\Delta_{1/t_l}\left(f(y)^{-1}f(y\delta_{t_l}w_l)\right), \prod_{s=1}^{\gamma}\Delta_{b_s^l}\Big(\partial_x f^{v_{i_s}}\Big(\exp(v_{i_s})\Big)\Big)\right) \longrightarrow 0\,,$$
which is a contradiction. This concludes the proof of differentiability. □

Remark. It should be noted that the differential does not depend on the explicit construction we have done in Theorem 3.9, where were involved the generating basis (v_i) and the extensions f^{v_i}. This fact follows from the uniqueness (Proposition 2.2). The choice of the generating basis can be interpreted as a fixed sistem of coordinates to represent the differential.

In the following example we consider the Heisenberg group $\mathbb{H}^3$, linearly isomorphic to $\mathbb{R}^3$, with horizontal vector fields $X = \partial_x - \frac{y}{2}\partial_z$, $Y = \partial_y + \frac{x}{2}\partial_z$ of $\mathbb{R}^3$, see for instance [17]. Note that the power 3 indicates the topological dimension of the Heisemberg group, this notation is not frequent in the literature.

Example. An application of the differentiability theorem is given in [3], where it is used to prove that the Heisenberg group $(\mathbb{H}^3, \mathrm{d})$ is purely $\mathcal{H}^k_{\mathrm{d}}$-unrectifiable for $k = 2$, 3, 4, that is any countably $\mathcal{H}^k_{\mathrm{d}}$-rectifiable set $S \subset \mathbb{H}^3$ is $\mathcal{H}^k_{\mathrm{d}}$-negligible. In this case the Lipschitz maps which parametrize the set have domains in arbitrary subsets of $\mathbb{R}^k$ and codomain in $\mathbb{H}^3$. This unrectifiability result is not strange because the "model space" considered in the definition of rectifiability is euclidean, so it is not bi-Lipschitz equivalent to any stratified non abelian group.

Remark. Notice that the differentiability theorem can be used to get some classical properties of suitable defined rectifiable sets in stratified groups. In fact, we can replace the subsets of an euclidean domain with that of a *stratified* subgroup, in the classic definition of rectifiability. So, in the special case when the subgroup is indeed a stratified group, the differentiability theorem gives us tangent spaces $\mathcal{H}^Q_d$-a.e. on the rectifiable set (if Q is the Hausdorff dimension of the subgroup) and one could go on as in [3], [20]. Moreover the Area formula gives a way to compute the intrinsic measure of rectifiable sets (see Example 4). This approach is followed in [29]. But actually, all of the above considerations are not possible for any stratified group. It is enough to consider the Heisenberg group $\mathbb{H}^3$ (which is the smallest non abelian stratified group), whose subgroups of topological dimension 2 are not stratified. The problem of differentiability of Lipschitz maps with domain in one of these proper subgroups is actually open. However this is not the unique way to define rectifiable sets on non abelian stratified groups, indeed in the theory of sets of finite perimeter in the Heisenberg group of [15] a different definition is used.

3.1. **G-linear maps.** This subsection of the work essentially makes it more self contained. We will show some elementary properties of G-linear maps which will be useful in the next subsection and in the proof of the Area formula. In fact, a G-linear map $L : \mathbb{G} \longrightarrow \mathbb{P}$ is linear when it is read between the corresponding Lie algebras through the exponential map. This fact follows from well known results on homomorphism of Lie groups, if one has in principle only the continuity of the map (see [36]). Moreover, a G-linear map has the *contact property* of sending horizontal vectors of the domain in horizontal vectors of the codomain. Other properties concern standard norm estimates of composition and product of G-linear maps.

Proposition 3.10. *Any continuous homomorphism* $L : \mathbb{G} \longrightarrow \mathbb{P}$ *read between the Lie algebras* $\ln \circ L \circ \exp : \mathcal{G} \longrightarrow \mathcal{P}$ *is linear.*

PROOF. The map $\varphi = \exp \circ L \circ \ln$ is clearly continuous. Theorem 3.39 of [36] implies that φ is C^∞ and applying Theorem 3.32 of [36] we find that $d\varphi = L : \mathcal{G} \longrightarrow \mathcal{P}$, so the proof is complete. □

Definition 8. We define $HG(\mathbb{G}, \mathbb{P})$ as the set of all G-linear maps between $\mathbb{G}$ and $\mathbb{P}$. Moreover given $T, L \in HG(\mathbb{G}, \mathbb{P})$ and $t \in \mathbb{R}$ we define the new functions $\Delta_t T,\ T \cdot L,\ -T : \mathbb{G} \longrightarrow \mathbb{P}$ as $\Delta_t T(u) = \Delta_t(T(u)),\ T \cdot L(u) = T(u)L(u),\ -T(u) = (T(u))^{-1}$ for any $u \in \mathbb{G}$. We define $HG(\mathcal{G}, \mathcal{P})$ as the set of all maps $L : \mathcal{G} \longrightarrow \mathcal{P}$ such that $\exp \circ L \circ \ln \in HG(\mathbb{G}, \mathbb{P})$.

Remark. It turns out any map of $HG(\mathbb{G}, \mathbb{P})$ induces uniquely a map of $HG(\mathcal{G}, \mathcal{P})$ and viceversa. We will prove that any map $T \in HG(\mathcal{G}, \mathcal{P})$ is linear, preserves the bracket operation and $L(V_1) \subset W_1$. Moreover there is a natural group of dilations, $T \longrightarrow \Delta_\lambda T$, $\lambda > 0$, both on $HG(\mathbb{G}, \mathbb{P})$ and $HG(\mathcal{G}, \mathcal{P})$.

Definition 9. Given $T, L \in HG(\mathbb{G}, \mathbb{P})$ we define $\rho(T, L) = \sup_{\mathrm{d}(u) \leq 1} \rho\left(T(u), L(u)\right)$ as the distance between T and L. If $L(u)$ is the unit element of $\mathbb{P}$ for any $u \in \mathbb{G}$ (the null map), we denote with $\rho(T)$ the distance between T and L. An analogous definition holds for maps of $HG(\mathcal{G}, \mathcal{P})$.

Proposition 3.11. *Any function* $T \in HG(\mathbb{G}, \mathbb{P})$ *is continuous and the distance of Definition 9 is a finite number, making* $HG(\mathbb{G}, \mathbb{P})$ *a complete metric space. Moreover, for any* $u \in \mathbb{G}$ *we have the estimate* $\rho(T(u)) \leq \rho(T)\, \mathrm{d}(u)$*.*

PROOF. Fix a generating basis $\{v_i \mid i = 1, \ldots, k\}$ such that

$$E = \left\{ \prod_{s=1}^{\gamma} \exp(a_s v_{i_s}) \mid (a_s) \subset U \right\} \supset \{u \in \mathbb{G} \mid \mathrm{d}(u) \leq 1\},$$

where $U \subset \mathbb{R}^\gamma$. By triangle inequality of the distance in $\mathbb{P}$ we get the estimate

$$\rho(T) \leq \left(\sup_{a \in U} |a|\right) \sum_{i=1}^{\gamma} \rho\left(T(v_{i_s})\right) < \infty .$$

The homogeneity of ρ implies the inequality $\rho\left(T(u)\right) \leq \rho(T)\, \mathrm{d}(u)$ for every $u \in \mathbb{G}$. Considering the map $T^{-1} \cdot L \in HG(\mathbb{G}, \mathbb{P})$ we have proved that the distance between T and L is finite. Of course $\rho(T) = 0$ implies that T is the null map, the triangle inequality and symmetry property of the distance follow directly from that of the metric ρ in $\mathbb{P}$. The homogenteity of ρ on $\mathbb{G}$ gives the homogeneity of the distance in $HG(\mathbb{G},\mathbb{P})$. Even the continuity is straightforward from the same inequality. The completeness of $HG(\mathbb{G},\mathbb{P})$ easily follows by the completeness of $\mathbb{P}$. □

Corollary 3.12. *Let $L:\mathbb{G}\longrightarrow\mathbb{P}$ be an injective G-linear map and $L(\mathbb{G})=S$. Then S is a stratified subgroup and $L^{-1}:S\longrightarrow\mathbb{G}$ is G-linear with*

$$d\left(L^{-1}(y)\right)\le d(L^{-1})\,\rho(y),\quad d(L^{-1})<\infty\,. \tag{13}$$

Proof. Clearly S is a subgroup of $\mathbb{P}$ and the contact property of L implies the stratification. In fact,

$$[L(V_i),L(V_1)]=L([V_i,V_1])=L(V_{i+1})\,,$$

so S is a stratified subgroup and $L^{-1}:S\longrightarrow\mathbb{G}$ is G-linear. Proposition 3.11 leads us to the conclusion. □

Corollary 3.13. *Consider $L,T\in HG(\mathbb{G},\mathbb{P})$ and $S\in HG(\mathbb{P},\mathbb{T})$, where $(\mathbb{G},\mathrm{d})$, $(\mathbb{P},\rho)$, $(\mathbb{T},\nu)$, are stratified Lie algebras. Then $S\circ L\in HG(\mathbb{G},\mathbb{T})$, $L\cdot T\in HG(\mathbb{G},\mathbb{P})$ and*

$$\nu(S\circ L)\le\ \nu(S)\,\rho(L)\qquad \rho(L\cdot T)\le\rho(L)+\rho(T) \tag{14}$$

Proof. It is an easy computation, using Proposition 3.11 and the triangle inequality. □

Corollary 3.14. *Any map $L\in HG(\mathcal{G},\mathcal{P})$ is linear.*

Proof. Proposition 3.11 implies the continuity and Proposition 3.10 gives the linearity. □

Corollary 3.15. *Any map $L\in HG(\mathcal{G},\mathcal{P})$ has the contact property $L(V_1)\subset W_1$.*

Proof. This property follows straightforward from Theorem 3.9 and Proposition 3.6, because Proposition 3.11 implies the Lipschitz property for L. But one could also observe that, given $v\in V_1$, the curve $\Delta_t(Lv)=L(\delta_t v)$ is Lipschitz in the variable t if and only if $Lv\in W_1$. □

Collecting all we have seen we get the following characterization.

Proposition 3.16. *Any homomorphism $L:\mathbb{G}\longrightarrow\mathbb{P}$ is Lipschitz if and only it is G-linear.*

Proof. Proposition 3.11 implies the Lipschitz property of L if it is G-linear. Viceversa, consider a Lipschitz homomorphism $L:\mathbb{G}\longrightarrow\mathbb{P}$. From the differentiability Theorem 3.9 one gets the a.e. differentiability. Then, passing to the corresponding Lie algebras, taking a differentiability point x with differential $\varphi \in HG(\mathbb{G},\mathbb{P})$ and writing the Definition 7 for $y = x\delta_t z$ it follows

$$\lim_{t\to 0} \frac{\rho\left(L(\delta_t z), \varphi(\delta_t z)\right)}{t} = \lim_{t\to 0} \rho\left(\Delta_{1/t}(L(\delta_t z)), \varphi(z)\right) = 0\,. \tag{15}$$

For the horizontal direction $z = v \in V_1$ we have $\delta_t v = tv$, the continuity of L and Proposition 3.10 imply the linearity, so if $Lv = \sum_{i=1} w_i$, with $w_i \in W_i$, the limit (15) gives

$$\Delta_{1/t}L(tv) = \Delta_{1/t}\left(tL(v)\right) = \sum_{i=1}^{m} t^{1-i} w_i \longrightarrow \varphi(v) \quad \text{as} \quad t \to 0\,.$$

The latter limit forces $w_i = 0$ for any $i > 1$, then $L(v) = \varphi(v) \in W_1$. The homomorphism property gives $L(V_i) \subset W_i$, so if $z = \sum_{i=1}^m z_i$, with $z_i \in V_i$, we have

$$L(\delta_t z) = \sum_{i=1}^{m} t^i L(z_i) = \Delta_t\left(L(z)\right)\,,$$

because $L(z_i) \in W_i$. This proves the homogeneity of L, so $L \in HG(\mathcal{G},\mathcal{P})$. □

3.2. **Jacobians.** In this subsection we give the definition of jacobian and we study its properties. In particular we emphasize its relation with G-linear maps. In fact, we get an explicit factorization formula for the jacobian, which involves the geometry of the G-linear map we consider. Precisely in the factorization there appears the algebraic jacobian of the map, where by algebraic jacobian we mean the classical jacobian for linear maps between Hilbert spaces of finite dimension. As explained in Section 2, we have a left invariant Riemannian metric on $\mathbb{G}$ and any tangent space is endowed with a scalar product. Thus, given a q dimensional subspace $\tilde{P}$ of $\mathcal{G}$, we will denote with $\mathcal{H}^q \llcorner \tilde{P}$ the q-Hausdorff measure on the subspace $\tilde{P}$ with the metric induced by the scalar product.

Inspired by the work [3], we first define the jacobian of a G-linear map in a tautological way. In other words, our definition of jacobian trivially satisfies the Area formula in principle for any G-linear map.

Definition 10 (Jacobian)**.** Let $L : \mathbb{G} \longrightarrow \mathbb{P}$ be a G-linear map. The *jacobian* $J_Q(L)$ of L is defined by

$$J_Q(L) = \frac{\mathcal{H}^Q_\rho\left(L(B_1)\right)}{\mathcal{H}^Q_d(B_1)}\,.$$

A covering argument together with the homogeneity and the homomorphism property of L shows that the above definition is independent of the set we consider, that is we can replace the set B_1 with any measurable set with finite and positive measure.

The next proposition shows that the jacobian is zero for singular G-linear maps, and provides a formula for the Hausdorff dimension of their image.

Proposition 3.17. *Let $L : \mathbb{G} \longrightarrow \mathbb{P}$ be a G-linear map, let $P = L(\mathbb{G})$ and let q_0 be the topological dimension of P. Then, the Hausdorff dimension of P in the metric ρ is $Q_0 = \sum_{j=1}^n j \dim\left(\tilde{L}(V_j)\right)$ and*

$$\mathcal{H}_\rho^{Q_0} \llcorner P = \alpha_P \, \mathcal{H}^{q_0} \llcorner \tilde{P} \tag{16}$$

where $\alpha_P = \mathcal{H}_\rho^{Q_0} \llcorner P(B_1^\rho) / \mathcal{H}^{q_0} \llcorner \tilde{P}(B_1^\rho)$ and $\tilde{P} = \ln P$, $(\ln = \exp^{-1})$.

Proof. We define $\tilde{L} = \ln \circ L \circ \exp$, observing that $\tilde{P} = \tilde{L}(\mathcal{G})$. The contact property $\tilde{L}(V_1) \subset W_1$ implies that $\tilde{L}(V_i) \subset W_i$ for any $i \leq n$ ($W_i = 0$ for $i > m$). So the set $\tilde{P} = \tilde{L}(\mathcal{G})$ is a subspace, a graded subalgebra and a subgroup of $\mathcal{P}$. We denote with q_0 the dimension of $\tilde{P}$ as a vector space. Consider the restriction of the dilation Δ_r on $\tilde{P}$, $\tilde{\Delta}_r : \tilde{P} \longrightarrow \tilde{P}$, and fix an orthonormal basis $(z_i) \subset \tilde{P}$. Then define the isometry $\phi : \mathbb{R}^{q_0} \longrightarrow \tilde{P}$, $\phi(x) = \sum_{i=1}^k x_i z_i$, observing that $D_r = \phi^{-1} \circ \tilde{\Delta}_r \circ \phi(x) = \sum_{i=1}^k r^{d_i} x_i$, where d_i is the degree of $z_i \in W_{d_i}$ (uniquely defined). It follows that the jacobian of D_r is r^{Q_0}, being $Q_0 = \sum_{i=1}^{q_0} d_i$. The homogeneity of dilations gives $B_r^\rho \cap \tilde{P} = \tilde{\Delta}_r(B_1^\rho \cap \tilde{P})$ and by the invariance of Hausdorff measure under isometries we have

$$\mathcal{H}^{q_0}(B_r^\rho \cap \tilde{P}) = \mathcal{H}^{q_0}\left(\phi^{-1} \circ \tilde{\Delta}_r(B_1^\rho \cap \tilde{P})\right) = \mathcal{L}^{q_0}\left(\phi^{-1} \circ \tilde{\Delta}_r(B_1^\rho \cap \tilde{P})\right) .$$

Thus, by the Euclidean change of variable formula it follows

$$\mathcal{H}^{q_0}(B_r^\rho \cap \tilde{P}) = \mathcal{L}^{q_0}\left(D_r\left(\phi^{-1}(B_1^\rho \cap \tilde{P})\right)\right)$$
$$= r^{Q_0}\, \mathcal{L}^{q_0}\left(\phi^{-1}(B_1^\rho \cap \tilde{P})\right) = r^{Q_0} \mathcal{H}^{q_0}(B_1^\rho \cap \tilde{P}) .$$

From the latter inequality we deduce that $\mathcal{H}_\rho^{Q_0} \llcorner \tilde{P}$ is a locally finite measure on $\tilde{P}$. Making the same reasoning we find that $\mathcal{H}_\rho^{Q_0} \llcorner \tilde{P}$ is left invariant under translations. In fact, fixed $\xi \in \tilde{P}$ and denoting with $T_\xi : \tilde{P} \longrightarrow \tilde{P}$ the left translation of the group $T_\xi(\eta) = \xi\eta$, the map $\phi^{-1} \circ T_\xi \circ \phi$ has jacobian one and it is an isometry. So, both measures $\mathcal{H}_\rho^{Q_0} \llcorner \tilde{P}$ and $\mathcal{H}^{q_0} \llcorner \tilde{P}$ are locally finite and left invariant, hence they are proportional. Moreover, taking into account that the exponential map is an isometry, we have $\mathcal{H}_\rho^{Q_0} \llcorner \tilde{P}(B_1^\rho) = \mathcal{H}_\rho^{Q_0} \llcorner P(\overline{B}_1^\rho)$. To get a non-ambiguous equality we have overlined the ball contained in $\mathbb{G}$ (despite the convention made in Definition 4). □

Proposition 3.18. *Let $L : \mathbb{G} \longrightarrow \mathbb{P}$ be an injective G-linear map, with $P = L(\mathcal{G})$. Then the jacobian of the map is given by the formula*

$$J_Q(L) = \alpha_P \, \beta_Q \, J^a(\tilde{L}) \, , \tag{17}$$

where $\alpha_P = \mathcal{H}^Q_\rho \llcorner P(B^\rho_1)/\mathcal{H}^q \llcorner \tilde{P}(B^\rho_1)$, $\beta_Q = \mathcal{L}^q(B_1)/\mathcal{H}^Q_d(B_1)$, $\tilde{L} = \ln \circ L \circ \exp : \mathcal{G} \longrightarrow \mathcal{P}$ and $J^a(L)$ denotes the algebraic jacobian of the linear map $\tilde{L}$.

Proof. From Proposition 3.17 and the injectivity of $\tilde{L}$ the space $\tilde{P} = \tilde{L}(\mathcal{G})$ has topological dimension q and Hausdorff dimension Q, moreover we have

$$\mathcal{H}^Q_\rho \left(L(B_1) \right) = \mathcal{H}^Q_\rho (\tilde{L}(B_1)) = \alpha_P \, \mathcal{H}^q(\tilde{L}(B_1)) \, .$$

The euclidean Area formula gives

$$\mathcal{H}^q(\tilde{L}(B_1)) = J^a(\tilde{L}) \, \mathcal{L}^q(B_1) \, ,$$

so the proof is complete. □

Remark. The coefficient

$$\alpha_P \, \beta_Q = \frac{\mathcal{H}^Q_\rho \llcorner P(B^\rho_1) \, \mathcal{L}^q(B_1)}{\mathcal{H}^q \llcorner \tilde{P}(B^\rho_1) \, \mathcal{H}^Q_d(B_1)}$$

represents a sort of *distortion factor*, which depends on both the measures $\mathcal{H}^Q_d$, $\mathcal{H}^Q_\rho$ and on the subspace P we consider. Notice that when $\mathbb{G} = \mathbb{P}$, so d $= \rho$ and $\tilde{P} = \mathcal{G}$, we get $\mathcal{H}^Q_\rho \llcorner P(B^\rho_1) = \mathcal{H}^Q_d(B_1)$, $\mathcal{H}^q \llcorner \tilde{P}(B^\rho_1) = \mathcal{L}^q(B_1)$ and the distortion factor reduces to 1.

4. The Area formula

In this section we prove the Area formula in the geometry of stratified groups.

Proposition 4.1 (Linearization). *Let $f : A \subset \mathbb{G} \longrightarrow \mathbb{P}$ be a measurable function, $\lambda > 1$, and*

$$E = \{x \in \mathcal{I}(A) \mid \text{there exists } d_x f : \mathbb{G} \longrightarrow \mathbb{P} \text{ and is injective}\}.$$

Then E has a measurable countable partition $\mathcal{F}$, such that for any $T \in \mathcal{F}$ there is an injective G-linear map $\varphi : \mathbb{G} \longrightarrow \mathbb{P}$ with the properties

$$\lambda^{-1}\rho(\varphi(z)) \le \rho(d_x f(z)) \le \lambda\,\rho(\varphi(z)) \quad \text{for any } z \in \mathbb{G} \text{ and any } x \in T \tag{18}$$

$$Lp\left(f_{|T} \circ (\varphi_{|T})^{-1}\right) \le \lambda \ \text{ and } \ Lp\left(\varphi_{|T} \circ (f_{|T})^{-1}\right)\right) \le \lambda. \tag{19}$$

Proof. By linearity of G-linear maps when represented between Lie algebras (see Corollary 3.14) we get a countable dense subset $\tilde{M}$ of all G-linear maps from $\mathcal{G}$ to $\mathcal{P}$. The set $\tilde{M}$ has the isometric correspondent $M = \{\varphi \mid \varphi = \exp \circ \tilde{\varphi} \circ \ln : \mathbb{G} \longrightarrow \mathbb{P},\ \tilde{\varphi} \in \tilde{M}\}$. Choose $\varepsilon > 0$ such that $\lambda^{-1} + \varepsilon < 1 < \lambda - \varepsilon$ and define the measurable set $S(\varphi, k) = \big\{y \in E \mid (\star) \text{ holds}\big\}$ with $\varphi \in M$ and $k \in \mathbb{N}$, where

$$(\star) \qquad \begin{cases} (\lambda^{-1} + \varepsilon)\,\rho\,(\varphi(z)) \le \rho\,(d_y f(z)) \le (\lambda - \varepsilon)\,\rho\,(\varphi(z)) & \forall z \in \mathbb{G} \\ \rho\,\big(f(z), f(y)\, d_y f(y^{-1}z)\big) \le \varepsilon\,\rho(\varphi(y^{-1}z)) & \forall z \in B_{y,1/k}. \end{cases}$$

Now we prove that any $y \in E$ is contained in some $S(\varphi, k)$. Define $\tilde{L} = \ln \circ d_y f \circ \exp$ and choose a positive $\varepsilon_1 < \min_{|w|=1} |\tilde{L}|$, where $|\cdot|$ is the norm of the fixed scalar product on the Lie algebras. By the density we find $\tilde{\varphi} \in \tilde{M}$ such that $\|\tilde{L} - \tilde{\varphi}\| \le \varepsilon_1$ as linear maps, so $\tilde{\varphi}$ has to be injective on $\mathcal{G}$. The maps $\tilde{\varphi} : \mathcal{G} \longrightarrow \mathcal{P}$ and $\tilde{L} : \mathcal{G} \longrightarrow \mathcal{P}$ are injective, so by Corollary 3.12 the maps $\tilde{\varphi}^{-1}$ and $\tilde{L}^{-1}$ are G-linear, besides inequality (13) will be used in the subsequent estimates. We will make our calculations for $\tilde{\varphi}$ because there is a correspondence of $\varphi = \exp \circ \tilde{\varphi} \circ \ln \in M$ with $\rho(\tilde{\varphi}(\ln z)) = \rho(\varphi(z))$, for any $z \in \mathbb{G}$. By inequality (1) we get

$$\rho(\tilde{L}, \tilde{\varphi}) \le C\,\|\tilde{L} - \tilde{\varphi}\|^{1/m} \le C\,\varepsilon_1^{1/m},$$

where m is the degree of nilpotency of $\mathcal{P}$. The estimates (14) imply

$$\rho(\tilde{L} \circ \tilde{\varphi}^{-1}) = \rho\left((\tilde{\varphi} \cdot (-\tilde{\varphi}) \cdot \tilde{L}) \circ \tilde{\varphi}^{-1})\right) \le 1 + \rho(\tilde{\varphi}, \tilde{L})\, d(\tilde{\varphi}^{-1}),$$

$$d(\tilde{\varphi}^{-1}) = \rho(\tilde{L}^{-1} \circ \tilde{L} \circ \tilde{\varphi}^{-1}) \le d(\tilde{L}^{-1})\rho(\tilde{L} \circ \tilde{\varphi}^{-1}),$$

hence, choosing ε_1 small enough, depending on $\tilde{L}, C, \varepsilon$ and λ, we have

$$\rho(\tilde{L} \circ \tilde{\varphi}^{-1}) \le \frac{1}{1 - \rho(\tilde{\varphi}, \tilde{L}) d(\tilde{L}^{-1})} \le \frac{1}{1 - C\varepsilon_1^{1/m} d(\tilde{L}^{-1})} < \lambda - \varepsilon,$$

$$\rho(\tilde{\varphi}\circ\tilde{L}^{-1}) = \rho\left((\tilde{L}\cdot(-\tilde{L})\cdot\tilde{\varphi})\circ\tilde{L}^{-1})\right) \leq 1+\rho(\tilde{L},\tilde{\varphi})\, d(\tilde{L}^{-1})$$
$$\leq 1 + C\varepsilon_1^{1/m} d(\tilde{L}^{-1}) < (\lambda^{-1}+\varepsilon)^{-1}\,.$$

The last two equations prove the first equation of $(\star)$, taking into account the equality $\rho(\tilde{L}(\ln z)) = \rho(d_y f(z))$, for any $z \in \mathbb{G}$. The definition of differentiability and the Lipschitz property of φ^{-1} leads to the second equation of $(\star)$ for k large depending on φ and ε. From the σ-compactness of $\mathbb{G}$ the set $S(\varphi,k)$ has a countable partition of measurable sets $T\subset S(\varphi,k)$ with $\mathrm{diam}(T)\leq 1/k$, so if we prove properties (18) and (19) for any T, we have finished the proof. Consider two points $u,y\in T\subset S(\varphi,k)$, by the definition of $S(\varphi,k)$, the first equation of $(\star)$ leads to (18). The second equation of $(\star)$ relative to y gives

$$\rho(f(u),f(y)) \leq \rho(d_y f(y^{-1}u)) + \varepsilon\,\rho(\varphi(y^{-1}u))\,, \tag{20}$$

$$\rho(f(u),f(y)) \geq \rho(d_y f(y^{-1}u)) - \varepsilon\,\rho(\varphi(y^{-1}u))\,, \tag{21}$$

adding the first one of $(\star)$, with $z=y^{-1}u$, to both equations (20) and (21) we get (19). □

An important tool for Proposition 4.3 is the following, see for instance [9].

Lemma 4.2. *Let (X,d,μ) be an Ahlfors regular space of dimension Q. Then, any ball B of radius R can be covered by at most $C\,(R/r)^Q$ balls of radius r, with C depending only on the regularity constants for X.*

The next proposition is an extension of the Sard Theorem in stratified groups when the dimension of the target is grater than that of the domain.

Proposition 4.3. *Let $f: A \longrightarrow \mathbb{P}$ be a Lipschitz map and $A\subset\mathbb{G}$ a measurable set. If the differential of f is non-injective at $\mathcal{H}^Q_d$-a.e. point of A, then $\mathcal{H}^Q_\rho(f(A))=0$.*

PROOF. Clearly it is not restrictive to assume that A contains only the points where f is differentiable and the differential is singular. So, let consider a point $x\in A$ where $d_x f$ is not injective and let $P_x = d_x f(\mathbb{G})$ be the corrisponding subgroup of $\mathcal{P}$. From Proposition 3.17 it follows in particular that P_x is an Ahlfors regular space of dimension Q_x. The singularity of $d_x f$ implies $Q_x\leq Q-1$. Denote with C_x the constant of Lemma 4.2 applied to $X=P_x$ and define the family of sets

$$E_j = \left\{x\in A \mid C_x\leq j\,\right\}\cap B_j \qquad \text{with}\, j\in\mathbb{N}$$

Consider $x \in E_j$ and $\varepsilon > 0$; denote with $I^\rho_r(E)$ the open set of points with distance from E less than r in the metric ρ. By differentiablility we obtain

$$f(B_{x,r}) \subset f(x) I^\rho_{\varepsilon r}\big(d_x f(B_r)\big) \tag{22}$$

for any $r \le r_{x,\varepsilon}$. Observe that $d_x f(B_r) \subset B^\rho_{cr} \cap P_x$, where $c = 2Lp(f)$, then using Lemma 4.2 we find $N \le C_x \varepsilon^{-Q_x}$ balls $B^l_\varepsilon \subset P_x$ of radius $c\varepsilon r$ which cover $B^\rho_{cr} \cap P_x$. Defining $\omega_Q = \mathcal{H}^Q_\rho(B^\rho_1)$ we see that the inclusion

$$I^\rho_{\varepsilon r}(B^\rho_{cr} \cap P_x) \subset \bigcup_{l=1}^N I^\rho_{\varepsilon r}(B^l_\varepsilon)$$

implies

$$\mathcal{H}^Q_{\rho,\infty}\left(I^\rho_{\varepsilon r}(B^\rho_{cr} \cap P_x)\right) \le j\varepsilon^{-Q_x}\omega_Q(c+1)^Q(\varepsilon r)^Q \le j\varepsilon\omega_Q(c+1)^Q r^Q = j\varepsilon C_Q \mathcal{H}^Q_d(B_r)$$

then for any $r \le r_{x,\varepsilon}$ and $x \in E_j$ it follows

$$\mathcal{H}^Q_\rho\big(f(B_{x,r})\big) \le j\varepsilon C_Q \mathcal{H}^Q_d(B_r). \tag{23}$$

Now we fix $j \in \mathbb{N}$ and consider the covering $\{B_{x,r} \mid x \in E_j \text{ and (22) holds for some } r \le r_{x,\varepsilon}/5 \le 1\}$. By a Vitali procedure we can extract a disjoint family of balls B_{x_l,r_l} contained in $I^{\mathrm{d}}_1(E_j)$ and such that $E_j \subset \bigcup_{l=1}^\infty B_{x_l,5r_l}$ (see [9]). The estimate (23) proves

$$\mathcal{H}^Q_\rho\big(f(E_j)\big) \le j\varepsilon C_Q \mathcal{H}^Q_d\left(I^{\mathrm{d}}_1(E_j)\right)$$

The free choice and the independence of ε and j lead us to the conclusion. $\square$

Definition 11. For any function $f : A \subset \mathbb{G} \longrightarrow \mathbb{P}$ and $B \subset A$, we define the *multiplicity function* relative to B as $N(f,B,y) = \sharp(\{f^{-1}(y) \cap B\}) \in \mathbb{N} \cup \{+\infty\}$, where $\sharp$ indicates the cardinality of the set.

Remark. It is worth to observe that even in the case when the Hausdorff dimension of $\mathbb{G}$ is less than the Hausdorff dimension of the target $\mathbb{P}$, it can happen that there does not exist a Lipschitz map $f : \mathbb{G} \longrightarrow \mathbb{P}$ with injective differential at some differentiability point. In fact, recalling notation of Section 2, $\mathcal{G} = V_1 + V_2 + \cdots + V_n$ and $\mathcal{P} = W_1 + W_2 + \cdots + W_m$, if $\dim(V_{j_0}) > \dim(W_{j_0})$ for some $j_0 \le \min\{m,n\}$ and $Q = \sum_{i=1}^n i\dim(V_i) \le \sum_{j=1}^m j\dim(W_j)$ the contact property of any G-linear map L implies $L(V_{j_0}) \subset W_{j_0}$, so L is non-injective. In this case the Area formula is a straightforward consequence only of Proposition 4.3. This remark points out the typical rigidity of the stratified geometry. In other words the conditions we have assumed on the stratification prevent any Lipschitz embedding of $\mathbb{G}$ into $\mathbb{P}$.

Theorem 4.4 (Area formula). *Given a measurable set $A \subset \mathbb{G}$ and a Lipschitz map $f : A \longrightarrow \mathbb{P}$, then the following formula holds*

$$\int_A J_Q(d_x f)\ d\mathcal{H}^Q_d(x) = \int_{\mathbb{P}} N(f, A, y)\ d\mathcal{H}^Q_\rho(y)\,. \tag{24}$$

PROOF. We start observing that (24) holds when A is negligible, because Lipschitz map have the Lusin property. Thus, in view of Theorem 3.9, we can exclude from the beginning the null subset of A where the function is not differentiable, assuming the differentiability at any point of A. We define the set $A' = \{x \in A \mid d_x f \text{ is injective}\}$ and $Z = A \setminus A'$. The set additivity of $N(f, \cdot, y)$ gives

$$\int_{\mathbb{P}} N(f, A', y)\, d\mathcal{H}^Q_\rho(y) + \int_{\mathbb{P}} N(f, Z, y)\, d\mathcal{H}^Q_\rho(y) = \int_{\mathbb{P}} N(f, A, y)\, d\mathcal{H}^Q_\rho(y)\,,$$

so the proof is achieved if we show the following equalities

$$\int_{\mathbb{P}} N(f, A', y)\, d\mathcal{H}^Q_\rho(y) = \int_A J_Q(d_x f)\, d\mathcal{H}^Q_d(x)\,, \tag{25}$$

$$\int_{\mathbb{P}} N(f, Z, y)\, d\mathcal{H}^Q_\rho(y) = 0\,. \tag{26}$$

We start from (25), applying Proposition 4.1 we get a measurable countable partition $\mathcal{F}$ of A' where we have an approximation of f controlled by a paramenter $\lambda > 1$. Consider an element $T \in \mathcal{F}$ contained in some $S(\varphi, k)$; the equation (18) implies

$$\lambda^{-Q}\mathcal{H}^Q_\rho(\varphi(T)) \le \mathcal{H}^Q_\rho\big((d_x f \circ \varphi^{-1} \circ \varphi)(T)\big) \le \lambda^Q \mathcal{H}^Q_\rho(\varphi(T)) \quad \text{for any } x \in T$$

By definition of jacobian, taking the average on T of the above inequality we find

$$\lambda^{-Q}\mathcal{H}^Q_\rho(\varphi(T)) \le \int_T J_Q(d_x f)\, d\mathcal{H}^Q_d(x) \le \lambda^Q \mathcal{H}^Q_\rho(\varphi(T))$$

using (19)

$$\lambda^{-2Q}\mathcal{H}^Q_\rho(f(T)) \le \int_T J_Q(d_x f) d\mathcal{H}^Q_d(x) \le \lambda^{2Q}\mathcal{H}^Q_\rho(f(T)). \tag{27}$$

The map f is injective on T, so adding (27) on all these sets it follows

$$\lambda^{-2Q}\int_{\mathbb{P}} N(f, A', y) d\mathcal{H}^Q_\rho(y) \le \int_{A'} J_Q(d_x f)\, d\mathcal{H}^Q_d(x) \le \lambda^{2Q}\int_{\mathbb{P}} N(f, A', y) d\mathcal{H}^Q_\rho(y).$$

Letting $\lambda \to 1^+$ we have (25). The equation (26) follows directly from Proposition 4.3. □

Corollary 4.5. *Given a Lipschitz map $f : A \subset \mathbb{G} \longrightarrow \mathbb{P}$ and a summable function $u : A \subset \mathbb{G} \longrightarrow \mathbb{R}$ we have*

$$\int_A u(x)\, J_Q(d_x f)\; d\mathcal{H}^Q_d(x) = \int_{\mathbb{P}} \sum_{x \in f^{-1}(y)} u(x)\; \mathcal{H}^Q_\rho(y)\,.$$

PROOF. We use the standard argument of approximating u with finite linear combinations of characteristic functions, see for example [10]. □

Example. We consider the Heisenberg group $\mathbb{H}^5$, with horizontal vector fields $X_i = \partial_{x^i} - \frac{y^i}{2}\partial_z$ and $Y_i = \partial_{y^i} + \frac{x^i}{2}\partial_z$, for $i = 1, 2$. We have $[X_i, Y_i] = Z = \partial_z$ for $i = 1, 2$, getting a basis of $\mathbb{R}^5$, which can be identified with the Lie algebra of $\mathbb{H}^5$. Thus, an element of $\mathbb{H}^5$ can be written as $\exp\left(\sum_{i=1}^2 (x^i\, X_i + y^i\, Y_i) + z\, Z\right)$, where $\exp : \mathbb{R}^5 \longrightarrow \mathbb{H}^5$. Then, we represent an element of $\mathbb{H}^5$ as $(x, y, z) \in \mathbb{R}^5$, with $x = (x^1, x^2)$ and $y = (y^1, y^2)$. The Baker-Campbell-Hausdorff formula gives the explicit group operation (denoted with $\circledcirc$) in our coordinates

$$(x, y, z) \circledcirc (\xi, \eta, \zeta) = \left(x + \xi, y + \eta, z + \zeta + \frac{(x^1\eta^1 + x^2\eta^2 - y^1\xi^1 - y^2\xi^2)}{2}\right)\,.$$

The restriction of the operation to the subset $\mathbb{G} = \{(x, y, z) \in \mathbb{H}^5 \mid x^2 = 0\}$ gives

$$(x^1, y, z) \circledcirc (\xi^1, \eta, \zeta) = \left(x^1 + \xi^1, y + \eta, z + \zeta + \frac{(x^1\eta^1 - y^1\xi^1)}{2}\right)\,,$$

so $\mathbb{G}$ is a subgroup of $\mathbb{H}^5$. Moreover $\mathbb{G}$ is a stratified group. In fact, the horizontal space $V_1 = \mathrm{span}(X_1, Y_1, \partial_{y^2})$ is left invariant under the translations of the subgroup and $[X_1, Y_1] = Z$, so the generating condition is achieved with $V_2 = \mathrm{span}(Z)$.

Consider an injective Lipschitz map $f : A \subset \mathbb{G} \longrightarrow \mathbb{H}^5$ and fix $S = f(A)$. The set $S \subset \mathbb{H}^5$ can be seen as a hypersurface of $\mathbb{H}^5$ with Hausdorff dimension 5 ($\mathbb{H}^5$ has Hausdorff dimension 6). In view of Remark 3, there exists a tangent hyperplane to S in $\mathcal{H}^5_{\mathrm{d}}$-a.e. $y \in S$, $T_y(S) = d_x f(\mathbb{G})$, with $y = f(x)$ and the Area formula gives

$$\mathcal{H}^5_{\mathrm{d}}(S) = \int_A J_5(d_x f)\, d\mathcal{H}^5_{\mathrm{d}}(x)\,.$$

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