*Proceedings of the Royal Society of Edinburgh*, **145A**, 885–891, *2015*
DOI:10.1017/S030821051500013X

# On a measure-theoretic area formula

**Valentino Magnani***
Dipartimento di Matematica, Università di Pisa,
Largo Pontecorvo 5, 56127 Pisa, Italy
(magnani@dm.unipi.it)

(MS received 23 January 2014; accepted 14 April 2014)

We review some classical differentiation theorems for measures, showing how they can be turned into an integral representation of a Borel measure with respect to a fixed Carathéodory measure. We focus our attention on the case when this measure is the spherical Hausdorff measure, giving a metric measure area formula. Our aim is to use certain covering derivatives as 'generalized densities'. Some consequences for the sub-Riemannian Heisenberg group are also pointed out.

*Keywords:* differentiation of measures; Hausdorff measure; area formula

2010 *Mathematics subject classification:* Primary 28A15
Secondary 28A78

It is well known that computing the Hausdorff measure of a set by an integral formula is usually related to rectifiability properties; namely, the set must be close to a linear subspace at any small scale. The classical area formula provides this relationship by an integral representation of the Hausdorff measure. Whenever a rectifiable set is thought of as a countable union of Lipschitz images of subsets in a Euclidean space, the area formula holds in metric spaces [4].

Over the last decade, the development of geometric measure theory in a non-Euclidean framework has raised new theoretical questions on rectifiability and area-type formulae. The main problem in this setting stems from the gap between the Hausdorff dimension of the target and that of the source space of the parametrization. In fact, in general this dimension might be strictly greater than the topological dimension of the set. As a result, the parametrization from a subset of the Euclidean space cannot be Lipschitz continuous with respect to the Euclidean distance of the source space.

One example of this difficulty is that the above-mentioned area formula does not work for a large class of Heisenberg group-valued Lipschitz mappings [1]. For this reason, theorems on differentiation of measures constitute an important tool to overcome this problem. In this connection, we show how Federer's theorems [2, §§ 2.10.17 and 2.10.18] are able to disclose a purely metric area formula. The surprising aspect of this formula is that an 'upper covering limit' actually can be seen as a *generalized density* of a fixed Borel measure.

*Dedicated to the memory of Herbert Federer (1920–2010) with deep admiration.

To define these densities, we first introduce covering relations: if $X$ is any set, a *covering relation* is a subset $\mathcal{C}$ of $\{(x,S)\colon x \in S \in \mathcal{P}(X)\}$. In the following, the set $X$ is always assumed to be equipped with a distance $d$. Defining for $A \subset X$ the corresponding class $\mathcal{C}(A) = \{S\colon x \in A,\ (x,S) \in \mathcal{C}\}$, we say that $\mathcal{C}$ is *fine at* $x$ if for every $\varepsilon > 0$ there exists $S \in \mathcal{C}(\{x\})$ such that $\operatorname{diam} S < \varepsilon$. According to [2, § 2.8.16], the notion of a covering relation yields the following notion of a 'covering limit'.

Definition 1 (covering limit). If $\mathcal{C}$ is a covering relation that is fine at $x \in X$, $\mathcal{C}(\{x\}) \subset \mathcal{D} \subset \mathcal{C}(X)$ and $f\colon \mathcal{D} \to \bar{\mathbb{R}}$, then we define the *covering limits*

$$(\mathcal{C}) \limsup_{S\to x} f = \inf_{\varepsilon>0} \sup\{f(S)\colon S \in \mathcal{C}(\{x\}),\ \operatorname{diam} S < \varepsilon\}, \tag{1}$$

$$(\mathcal{C}) \liminf_{S\to x} f = \sup_{\varepsilon>0} \inf\{f(S)\colon S \in \mathcal{C}(\{x\}),\ \operatorname{diam} S < \varepsilon\}. \tag{2}$$

The covering relations made by closed balls clearly play an important role in the study of the area formula for the spherical Hausdorff measure.

Definition 2. The closed ball and the open ball of centre $x \in X$ and radius $r > 0$ are denoted by

$$\mathbb{B}(x,r) = \{y \in X\colon d(x,y) \leqslant r\} \quad \text{and} \quad B(x,r) = \{y \in X : d(x,y) < r\},$$

respectively. We denote by $\mathcal{F}_{\mathrm{b}}$ the family of all closed balls in $X$.

The next definition introduces the Carathéodory construction (see [2, § 2.10.1]).

Definition 3. Let $\mathcal{S} \subset \mathcal{P}(X)$ and let $\zeta\colon \mathcal{S} \to [0,+\infty]$ represent the *size function*. If $\delta > 0$ and $R \subset X$, then we define

$$\phi_\delta(R) = \inf\left\{\sum_{j=0}^{\infty} \zeta(E_j)\colon E_j \in \mathcal{F},\ \operatorname{diam}(E_j) \leqslant \delta \text{ for all } j \in \mathbb{N},\ R \subset \bigcup_{j\in\mathbb{N}} E_j\right\}.$$

The $\zeta$-*approximating measure* denoted by $\psi_\zeta$ is defined as $\psi_\zeta = \sup_{\delta>0} \phi_\delta$. Denoting by $\mathcal{F}$ the family of closed sets of $X$, for $\alpha, c_\alpha > 0$ we define $\zeta_\alpha\colon \mathcal{F} \to [0,+\infty]$ by

$$\zeta_\alpha(S) = c_\alpha \operatorname{diam}(S)^\alpha.$$

Then the $\alpha$-*dimensional Hausdorff measure* is $\mathcal{H}^\alpha = \psi_{\zeta_\alpha}$. If $\zeta_{\mathrm{b},\alpha}$ is the restriction of $\zeta_\alpha$ to $\mathcal{F}_{\mathrm{b}}$, then $\mathcal{S}^\alpha = \psi_{\zeta_{\mathrm{b},\alpha}}$ is the $\alpha$-*dimensional spherical Hausdorff measure*.

These special limits of definition 1 naturally arise in the differentiation theorems for measures and allow us to introduce a special 'density' associated with a measure.

Definition 4 (Federer density). Following the terminology of [2, § 2.1.2], we fix a measure $\mu$ over $X$. Let $\mathcal{S} \subset \mathcal{P}(X)$ and let $\zeta\colon \mathcal{S} \to [0,+\infty]$. Then we set

$$\mathcal{S}_{\mu,\zeta} = \mathcal{S} \setminus \{S \in \mathcal{S}\colon \zeta(S) = \mu(S) = 0 \text{ or } \mu(S) = \zeta(S) = +\infty\},$$

along with the covering relation $\mathcal{C}_{\mu,\zeta} = \{(x,S)\colon x \in S \in \mathcal{S}_{\mu,\zeta}\}$. We choose $x \in X$ and assume that $\mathcal{C}_{\mu,\zeta}$ is fine at $x$. We define the quotient function as follows

$$Q_{\mu,\zeta}\colon \mathcal{S}_{\mu,\zeta} \to [0,+\infty], \qquad Q_{\mu,\zeta}(S) = \begin{cases} +\infty & \text{if } \zeta(S) = 0, \\ \mu(S)/\zeta(S) & \text{if } 0 < \zeta(S) < +\infty, \\ 0 & \text{if } \zeta(S) = +\infty. \end{cases}$$

We are now in the position to define the *Federer density*, or *upper $\zeta$-density of $\mu$ at $x \in X$*, as follows:

$$F^{\zeta}(\mu, x) = (\mathcal{C}_{\mu,\zeta}) \limsup_{S \to x} Q_{\mu,\zeta}(S). \tag{3}$$

According to the following definition, we will use special notation when we consider Federer densities with respect to $\zeta_\alpha$ and $\zeta_{\mathrm{b},\alpha}$, respectively.

Definition 5. If $\mu$ is a measure over $X$, and $\mathcal{C}_{\mu,\zeta_{\mathrm{b},\alpha}}$ is fine at $x \in X$, then we set $\theta^\alpha(\mu, x) = F^{\zeta_{\mathrm{b},\alpha}}(\mu, x)$. If $\mathcal{C}_{\mu,\zeta_\alpha}$ is fine at $x$, then we set $\mathfrak{d}^\alpha(\mu, x) = F^{\zeta_\alpha}(\mu, x)$.

Remark 6. If $x \in X$ and there exists an infinitesimal sequence $(r_i)$ of positive radii such that all $\mathbb{B}(x, r_i)$ have positive diameter, then both $\mathcal{C}_{\mu,\zeta_{\mathrm{b},\alpha}}$ and $\mathcal{C}_{\mu,\zeta_\alpha}$ are fine at $x$. The same conclusion also holds if we assume that for each $\mathbb{B}(x, r_j)$ with vanishing diameter it holds that $\mu(\mathbb{B}(x, r_j)) > 0$.

We say that a family $\mathcal{F} \subset \mathcal{P}(X)$ covers $A \subset X$ *finely* if for each $a \in A$ and $\varepsilon > 0$ there exists $S \in \mathcal{F}$ with $a \in S \in \mathcal{F}$ with $\operatorname{diam}(S) < \varepsilon$ (see [2, § 2.8.1]). From the previous definitions, we can now state a revised version of 2.10.17(2) of [2].

Theorem 7. *Let $\mathcal{S} \subset \mathcal{P}(X)$, and let $\zeta\colon \mathcal{S} \to [0, +\infty]$ be a size function. If $\mu$ is a regular measure over $X$, $A \subset X$, $t > 0$, $\mathcal{S}_{\mu,\zeta}$ covers $A$ finely and for all $x \in A$ we have $F^\zeta(\mu, x) < t$, then $\mu(E) \leqslant t\psi_\zeta(E)$ for every $E \subset A$.*

Analogously, the next theorem is a revised version of 2.10.18(1) in [2].

Theorem 8. *Let $\mu$ be a measure over $X$, let $\mathcal{S}$ be a family of closed and $\mu$-measurable sets, let $\zeta\colon \mathcal{S} \to [0, +\infty)$, let $B \subset X$ and assume that $\mathcal{S}_{\mu,\zeta}$ covers $B$ finely. If there exist $c, \eta > 0$ such that for each $S \in \mathcal{S}$ there exists $\tilde{S} \in \mathcal{S}$ with the properties*

$$\hat{S} \subset \tilde{S}, \quad \operatorname{diam} \tilde{S} \leqslant c \operatorname{diam} S \quad \textit{and} \quad \zeta(\tilde{S}) \leqslant \eta \zeta(S), \tag{4}$$

*where $\hat{S} = \bigcup\{T \in \mathcal{S}\colon T \cap S \neq \emptyset,\ \operatorname{diam} T \leqslant 2 \operatorname{diam} S\}$, $V \subset X$ is an open set containing $B$ and for every $x \in B$ we have $F^\zeta(\mu, x) > t$, then $\mu(V) \geqslant t\psi_\zeta(B)$.*

These theorems provide both upper and lower estimates for a large class of measures, starting from upper and lower estimates of the Federer density. A slight restriction of the assumptions in the previous theorems together with some standard arguments leads us to a new metric area-type formula, where the integration of $F^\zeta(\mu, \cdot)$ recovers the original measure. This is precisely our first result.

Theorem 9 (measure-theoretic area formula). *Let $\mu$ be a Borel regular measure over $X$ such that there exists a countable open covering of $X$ whose elements have $\mu$ finite measure. Let $\mathcal{S}$ be a family of closed sets, let $\zeta\colon \mathcal{S} \to [0, +\infty)$ and assume that for some constants $c, \eta > 0$ and for every $S \in \mathcal{S}$ there exists $\tilde{S} \in \mathcal{S}$ such that*

$$\hat{S} \subset \tilde{S}, \quad \operatorname{diam} \tilde{S} \leqslant c \operatorname{diam} S \quad \textit{and} \quad \zeta(\tilde{S}) \leqslant \eta \zeta(S), \tag{5}$$

*where $\hat{S} = \bigcup\{T \in \mathcal{S}\colon T \cap S \neq \emptyset,\ \operatorname{diam} T \leqslant 2 \operatorname{diam} S\}$. If $A \subset X$ is Borel, $\mathcal{S}_{\mu,\zeta}$ covers $A$ finely and $F^\zeta(\mu, \cdot)$ is a Borel function on $A$ with*

$$\psi_\zeta(\{x \in A\colon F^\zeta(\mu, x) = 0\}) < +\infty \quad \textit{and} \quad \mu(\{x \in A\colon F^\zeta(\mu, x) = +\infty\}) = 0, \tag{6}$$

*then for every Borel set $B \subset A$ we have*

$$\mu(B) = \int_B F^\zeta(\mu, x)\, \mathrm{d}\psi_\zeta(x). \tag{7}$$

The second condition of (6) corresponds precisely to the absolute continuity of $\mu \llcorner A$ with respect to $\psi_\zeta \llcorner A$. This measure-theoretic area formula may recall a precise differentiation theorem, where the third condition of (5) indeed represents a kind of 'doubling condition' for the size function $\zeta$. In fact, the doubling condition for a measure allows us to obtain a similar formula, where the density is computed by taking the limit of the ratio between the measures of closed balls with the same centre and radius (see, for example, [2, §§ 2.9.8 and 2.8.17]).

On the one hand, the Federer density $F^\zeta(\mu, x)$ may be hard to compute, depending on the space $X$. On the other, (7) requires neither special geometric properties for $X$, such as those of the Besicovitch covering theorem (see the general condition 2.8.9 of [2]), nor an 'infinitesimal' doubling condition for $\psi_\zeta \llcorner A$, as in [2, theorem 2.8.17]. Moreover, there are no constraints that prevent $X$ from being infinite dimensional.

The absence of specific geometric conditions on $X$ is important, especially in relation to applications of theorem 9 to sub-Riemannian geometry, in particular for the class of the so-called Carnot groups, where the classical Besicovitch covering theorem may not hold. In these groups we have no general theorem to 'differentiate' an arbitrary Radon measure. Therefore, new differentiation theorems are important.

We provide two direct consequences of theorem 9, which correspond to the cases where $\psi_\zeta$ is the Hausdorff measure and the spherical Hausdorff measure, respectively.

Theorem 10 (differentiation with respect to the Hausdorff measure). *Let $\mu$ be a Borel regular measure over $X$ such that there exists a countable open covering of $X$ whose elements have $\mu$ finite measure. If $A \subset X$ is Borel, $\alpha > 0$ and $\mathcal{S}_{\mu,\zeta_\alpha}$ covers $A$ finely, then $\mathfrak{d}^\alpha(\mu, \cdot)$ is Borel. Moreover, if $\mathcal{H}^\alpha(A) < +\infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{H}^\alpha \llcorner A$, then for every Borel set $B \subset A$, we have*

$$\mu(B) = \int_B \mathfrak{d}^\alpha(\mu, x)\, \mathrm{d}\mathcal{H}^\alpha(x).$$

This theorem essentially assigns a formula to the density of $\mu$ with respect to $\mathcal{H}^\alpha$. Let us recall the formula for this density:

$$\mathfrak{d}^\alpha(\mu, x) = \inf_{\varepsilon > 0} \{\sup\{Q_{\mu,\zeta_\alpha}(S) \colon x \in S \in \mathcal{S}_{\mu,\zeta_\alpha},\ \operatorname{diam} S < \varepsilon\}\}.$$

Working under the general assumption that all open balls have positive diameter, we have $Q_{\mu,\zeta_\alpha}(S) = \mu(S)/\zeta_\alpha(S)$. More manageable formulae for $\mathfrak{d}^\alpha(\mu, \cdot)$ turn out to be very hard to find and this difficulty is related to the geometric properties of the single metric space. On the other hand, if we restrict our attention to the spherical Hausdorff measure, then the corresponding density $\theta^\alpha(\mu, \cdot)$ can be computed explicitly in several contexts, where it can also be precisely interpreted geometrically.

In this case, we assume that $X$ is *diametrically regular*; namely, for all $x \in X$ and $R > 0$ there exists $\delta_{x,R} > 0$ such that $(0, \delta_{x,R}) \ni t \to \operatorname{diam}(B(y, t))$ is continuous for every $y \in \mathbb{B}(x, R)$. This ensures that $\theta^\alpha(\mu, \cdot)$ in the next theorem is Borel. We are now in a position to state the measure-theoretic area-type formula for the spherical Hausdorff measure.

THEOREM 11 (differentiation with respect to the spherical Hausdorff measure). *Let $X$ be a diametrically regular metric space, let $\alpha > 0$ and let $\mu$ be a Borel regular measure over $X$ such that there exists a countable open covering of $X$ whose elements have $\mu$ finite measure. If $B \subset A \subset X$ are Borel sets and $\mathcal{S}_{\mu,\zeta_{\mathrm{b},\alpha}}$ covers $A$ finely, then $\theta^\alpha(\mu, \cdot)$ is Borel on $A$. In addition, if $\mathcal{S}^\alpha(A) < +\infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}^\alpha \llcorner A$, then we have*

$$\mu(B) = \int_B \theta^\alpha(\mu, x)\, \mathrm{d}\mathcal{S}^\alpha(x). \tag{8}$$

In the sub-Riemannian framework, for distances with special symmetries and the proper choice of the Riemannian surface measure $\mu$, the density $\theta^\alpha(\mu, \cdot)$ is a function that can be computed with a precise geometric interpretation. These ideas are detailed for intrinsic surface measures in [5]. However, we expect (8) to have further applications to computing the spherical Hausdorff measure of sets. This is the motivation for the present work.

Here we are mainly concerned with the purely metric area formula; hence, we provide only a glimpse of applications to the Heisenberg group, leaving details and further developments for subsequent work.

Let $\Sigma$ be a $C^1$ smooth curve in $\mathbb{H}$ equipped with the sub-Riemannian distance $\rho$. This distance is also called the Carnot–Carathéodory distance (see [3] for the relevant definitions). Whenever a left-invariant Riemannian metric $g$ is fixed on $\mathbb{H}$, we can associate $\Sigma$ with its intrinsic measure $\mu_{\mathrm{SR}}$ (see [6] for more details). We will assume that $\Sigma$ has at least one *non-horizontal point* $x \in \Sigma$; namely, $T_x\Sigma$ is not contained in the horizontal subspace $H_x\mathbb{H}$ that is spanned by the horizontal vector fields evaluated at $x$ [3]. Since $\Sigma$ is smooth, this implies that all of these points constitute an open subset of $\Sigma$.

If we fix the size function $\zeta_{\mathrm{b},2}(S) = \operatorname{diam}(S)^2/4$ on closed balls and $x$ is non-horizontal, then it is possible to compute $\theta^2(\mu_{\mathrm{SR}}, x)$ explicitly, to obtain

$$\theta^2(\mu_{\mathrm{SR}}, x) = \alpha(\rho, g), \tag{9}$$

where $\alpha(\rho, g)$ is precisely the maximum of the lengths of all intersections of vertical lines passing through the sub-Riemannian unit ball, centred at the origin. The length of intersections is computed with respect to the scalar product given by the fixed Riemannian metric $g$ at the origin. We observe that (9) does not depend on the transversality of $T_x\Sigma$ with respect to $H_x\mathbb{H}$, since this factor is included in the definition of $\mu_{\mathrm{SR}}$ [6]. As an application of theorem 11, we obtain

$$\mu_{\mathrm{SR}} = \alpha(\rho, g)\mathcal{S}^2 \llcorner \Sigma,$$

where $\mathcal{S}^2$ is the spherical Hausdorff measure induced by $\zeta_{\mathrm{b},2}$. The appearance of the geometric constant $\alpha(\rho, g)$ is a new phenomenon, due to the use of Federer's density. The non-convex shape of the sub-Riemannian unit ball centred at the origin allows $\alpha(\rho, g)$ to be strictly larger than the length $\beta(\rho, g)$ of the intersection of the same ball with the vertical line passing through the origin. The special non-convex shape of the sub-Riemannian unit ball shows that $\theta^2(\mu_{\mathrm{SR}}, x)$ and the upper spherical density

$$\Theta^{*2}(\mu_{\mathrm{SR}}, x) = \limsup_{r\to 0^+} \frac{\mu_{\mathrm{SR}}(\mathbb{B}(x,r))}{r^2}$$

are not equal. In fact, setting $t \in (\alpha(\rho, g), \beta(\rho, g))$, we get

$$\Theta^{*2}(\mu_{\mathrm{SR}}, x) = \beta(\rho, g) < t < \alpha(\rho, g) = \theta^2(\mu_{\mathrm{SR}}, x) \quad \text{for all } x \in \mathcal{N},$$

where $\mathcal{N} = \{x \in \Sigma \colon T_x\Sigma \text{ is not horizontal}\}$ and we also have

$$\mu_{\mathrm{SR}}(\mathcal{N}) = \alpha(\rho, g)\mathcal{S}^2(\mathcal{N}) > t\mathcal{S}^2(\mathcal{N}). \tag{10}$$

As a consequence of (10), in the inequality (1) of [2, § 2.10.19], with $\mu = \mu_{\mathrm{SR}}$ and $A = \mathcal{N}$, the constant $2^m$ with $m = 2$ cannot be replaced by 1. Moreover, even if we weaken this inequality, replacing the Hausdorff measure with the spherical Hausdorff measure, (10) still shows that $2^m$ with $m = 2$ cannot be replaced by 1. In the case when $m = 1$, it is possible to show, by an involved construction of a purely $(\mathcal{H}^1, 1)$ unrectifiable set of the Euclidean plane, that $2^m$ is even sharp (see the example in [2, § 3.3.19]). Somehow, our curve with non-horizontal points has played the role of a more manageable unrectifiable set. Incidentally, the set $\mathcal{N}$ is purely $(\mathcal{H}^2, 2)$ unrectifiable with respect to $\rho$ (see [1]).

The connection between rectifiability and densities was pointed out by Preiss and Tišer [7], who improve in a general metric space $X$ the upper estimate for $\sigma_1(X)$, related to the so-called Besicovitch $\frac{1}{2}$-problem. According to [7], $\sigma_k(X)$ for some positive integer $k$ is the infimum of all positive numbers $t$ having the property that, for each $E \subset X$ with $\mathcal{H}^k(E) < +\infty$ and such that

$$\liminf_{r\to 0^+} \frac{\mathcal{H}^k(E \cap B(x,r))}{c_k 2^k r^k} > t$$

for $\mathcal{H}^k$-almost every $x \in E$, implies that $E$ is countably $k$-rectifiable, where $\mathcal{H}^k$ arises from the Carathéodory construction of size function $\zeta(S) = c_k \operatorname{diam}(S)^k$.

If we equip $\mathbb{H}$ with the so-called Korányi distance $d$, then a different application of theorem 11 gives a lower estimate for $\sigma_2(\mathbb{H}, d)$. In fact, we can choose $\Sigma_0$ to be a bounded open interval of the vertical line of $\mathbb{H}$ passing through the origin. This set is purely $(\mathcal{H}^2, 2)$ unrectifiable. We define $\zeta^d_{b,2}(S) = \operatorname{diam}_d(S)^2/4$ on closed balls, where the diameter $\operatorname{diam}_d(S)$ refers to the distance $d$, and consider the intrinsic measure $\mu_{\mathrm{SR}}$ of $\Sigma_0$. By the convexity of the $d$-unit ball centred at the origin, the corresponding Federer density $\theta^2_d(\mu_{\mathrm{SR}}, x)$ at a non-horizontal point $x$ satisfies

$$\theta^2_d(\mu_{\mathrm{SR}}, x) = \alpha(d, g),$$

where $\alpha(d, g)$ is the length of the intersection of the Korányi unit ball centred at the origin with the vertical line passing through the origin. Using the previous notation, by theorem 11 we get

$$\mu_{\mathrm{SR}} = \alpha(d, g)\mathcal{S}^2_d \llcorner \Sigma_0,$$

where $\mathcal{S}^2_d$ is the spherical Hausdorff measure induced by $\zeta^d_{b,2}$. Since we have

$$\lim_{r\to 0^+} \frac{\mathcal{S}^2_d \llcorner \Sigma_0(B(x,r))}{r^2} = \lim_{r\to 0^+} \frac{\mu_{\mathrm{SR}}(B(x,r))}{\alpha(d,g) r^2} = \lim_{r\to 0^+} \frac{\mu_{\mathrm{SR}}(\mathbb{B}(x,r))}{\alpha(d,g) r^2} = 1,$$

and an easy observation shows that $\mathcal{S}^2_d \llcorner \Sigma_0 \leqslant 2\mathcal{H}^2_2 \llcorner \Sigma_0$, it follows that

$$\begin{aligned}\frac{1}{2} = \lim_{r\to 0} \frac{\mathcal{S}^2_d \llcorner \Sigma_0(B(x,r))}{2r^2} &\leqslant \liminf_{r\to 0^+} \frac{\mathcal{H}^2_d(\Sigma_0 \cap B(x,r))}{r^2} \\ &\leqslant \lim_{r\to 0^+} \frac{\mathcal{S}^2_d \llcorner \Sigma_0(B(x,r))}{r^2} \\ &= 1.\end{aligned}$$

This implies that $\sigma_2(\mathbb{H}, d) \geqslant \frac{1}{2}$. Up to this point, we have seen how the geometry of the sub-Riemannian unit ball affects the geometric constants in estimates between measures. However, the opposite direction is also possible. In fact, considering the previous subset $\mathcal{N}$ and taking into account (1) of [2, theorem 2.10.19] with $m = 2$, we get

$$\mu_{\mathrm{SR}}(\mathcal{N}) \leqslant 4\beta(\rho, g)\mathcal{S}^2(\mathcal{N});$$

hence, the equality in (10) leads us to the estimate

$$1 < \frac{\alpha(\rho, g)}{\beta(\rho, g)} \leqslant 4.$$

It turns out to be rather striking that abstract differentiation theorems for measures can provide information on the geometric structure of the sub-Riemannian unit ball. Precisely, we cannot find any left-invariant sub-Riemannian distance $\tilde{\rho}$ in the Heisenberg group such that the geometric ratio $\alpha(\tilde{\rho}, g)/\beta(\tilde{\rho}, g)$ is greater than 4. There is still a number of related questions, so this paper may be seen as a starting point for establishing deeper relationships between the results of sub-Riemannian geometry and measure-theoretic results.

In particular, further motivation to study sub-Riemannian metric spaces may also arise from abstract questions of geometric measure theory. Clearly, more investigations are needed in order to understand and carry out this demanding programme.

## Acknowledgements

The author acknowledges the support of the European Project ERC AdG GeMeThNES 'Geometric Measure Theory in Non-Euclidean Spaces'.

## References

1 L. Ambrosio and B. Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Annalen* **318** (2000), 527–555.

2 H. Federer. *Geometric measure theory* (Springer, 1969).

3 M. Gromov. Carnot–Carathéodory spaces seen from within. In *SubRiemannian geometry* (ed. A. Bellaiche and J. Risler), Progress in Mathematics, vol. 144 (Birkhäuser, 1996).

4 B. Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Am. Math. Soc.* **121** (1994), 113–123.

5 V. Magnani. A new differentiation, shape of the unit ball and perimeter measure. Preprint, 2014. (Available at http://arxiv.org/abs/1408.4726.)

6 V. Magnani and D. Vittone. An intrinsic measure for submanifolds in stratified groups. *J. Reine Angew. Math.* **619** (2008), 203–232.

7 D. Preiss and J. Tišer. On Besicovitch's $\frac{1}{2}$-problem. *J. Lond. Math. Soc. (2)* **45** (1992), 279–287.