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# THE QUOTIENT OF A COMPLETE SYMMETRIC VARIETY

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*Dedicated to Ernest Vinberg on the occasion of his 70th birthday*

Abstract. We study the quotient of a completion of a symmetric variety $G/H$ under the action of $H$. We prove that this is isomorphic to the closure of the image of an isotropic torus under the action of the restricted Weyl group. In the case the completion is smooth and toroidal we describe the set of semistable points.

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## 1. Introduction

Let $G$ be a semisimple simply connected algebraic group over an algebraically closed field of characteristic different from 2. Given an involution $\sigma$ of $G$ with fixed subgroup $G^\sigma$, we fix a subgroup $G^\sigma \subset H \subset N_G(G^\sigma)$.

Our goal in this paper is the study of the action of $H$ on certain completions of $G/H$ with the methods of geometric invariant theory.

The study of such problems starts with the famous paper [12] of Kostant and Rallis in which the $H$ action on the quotient $\operatorname{Lie} G/\operatorname{Lie} H$ is studied. This can be considered as an infinitesimal version of our study. Results similar to those in [12] have been later obtained by Richardson in [14] in the case of the quotient $G/H$.

In particular Richardson has proved, among other things, that if we take the image $S_H$ in $G/H$ of an anisotropic maximal torus $S$ in $G$ and consider the action of the restricted Weyl group $\widetilde{W}$ on $S_H$ (see below for the definitions), the GIT quotient $H \,\backslash\!\backslash\, G/H$ is isomorphic to $\widetilde{W}\backslash S_H$. Furthermore, he shows that the closed $H$ orbits in $G/H$ are precisely the orbits of elements in $S_H$.

In this paper we generalize these two results to the case of a completion $Y$ of $G/H$. In particular we reprove the results of Richardson mentioned above.

To state our result take a smooth toroidal projective $G$-equivariant completion $Y$ of $G/H$. In $Y$ consider the closure $Y_S$ of $S_H$. The $\widetilde{W}$ action on $S_H$ extends to an action on $Y_S$. Fix an ample line bundle $\mathcal{L}$ on $Y$. Our first result is that the GIT

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quotient $H \mathbin{/\mkern-6mu/}_{\mathcal{L}} Y$ relative to $\mathcal{L}$ is isomorphic to $\widetilde{W}\backslash Y_S$. In particular this quotient does not depend on the choice of $\mathcal{L}$. (Theorem 4.1).

We then pass to the study of the set $Y^{ss} \subset Y$ of semistable points in $Y$ with respect to $\mathcal{L}$. Also in this case we show that $Y^{ss}$ does not depend on the choice of $\mathcal{L}$ (Remark 5.6) and we describe rather precisely the intersection of $Y^{ss}$ with any $G$ orbit. In particular we show that given two $H$ orbits $O_1 \subset \bar{O}_2$ in $Y^{ss}$ then they both lie in the same $G$ orbit (Proposition 6.1) and that a $H$ orbit $O$ in $Y^{ss}$ is closed if and only if it meets $Y_S$ (Theorem 6.4). These last facts allow us to give a version of our results in the case of any $G$ stable open subset in $Y$.

The proofs of our results are rather straightforward in characteristic zero and are based on the careful analysis of sections of line bundles on $Y$ given in [1] and [2]. However to carry out our proofs in positive characteristic we have to deal with a number of rather technical results which often do not appear in the literature and which, in view of this, we have decided to explain here.

## 2. Preliminaries

In this section we introduce notations, recall some simple properties and describe the spherical weights relative to a given involution.

Let us choose an algebraically closed field $\Bbbk$ whose characteristic is not equal to 2. Usually all algebraic group schemes in this paper are going to be affine and defined over $\Bbbk$ but, occasionally we are going to consider group schemes defined over the ring $\boldsymbol{A} := \mathbb{Z}[1/2]$ and flat over Spec $\boldsymbol{A}$. Gothic letters are going to denote Lie algebras.

Let $G$ be a semisimple and simply connected algebraic group. Let $\bar{G}$ be the adjoint quotient of $G$ and $Z$ the kernel of the projection of the isogeny $G \to \bar{G}$. This is a possibly not reduced subgroup of $G$ whose associated reduced subvariety is given by the center of $G$.

Let $\sigma$ be an involution of $G$ and let $H^\circ = G^\sigma$ be the subgroup of elements fixed by $\sigma$. We consider also the inverse image $\bar{H}$ under the isogeny $G \to \bar{G}$ of the subgroup of $\bar{G}$ of elements fixed by the involution of $\bar{G}$ induced by $\sigma$. We recall that $H^\circ$ is connected and reductive and that $\bar{H}$ is a possibly not reduced subgroup of $G$ whose associated reduced subgroup is the normalizer $N_G(H^\circ)$ of $H^\circ$. It is known that the connected component of the identity of $\bar{H}$ with reduced structure is equal to $H^\circ$ (see [4]).

Let now $H^\circ \subset H \subset \bar{H}$ be a possibly not reduced subgroup of $G$. The quotient $G/H = \operatorname{Spec} \Bbbk[G]^H$ is called a *symmetric variety*.

We fix an *anisotropic maximal torus* $S$ of $G$, that is a torus of $G$ such that $\sigma(s) = s^{-1}$ for all $s \in S$, and having maximal dimension among the tori with this property. The dimension $\ell$ of $S$ is called the *rank* of the symmetric variety. We choose also a $\sigma$ stable maximal torus $T$ of $G$ containing $S$ and a Borel subgroup containing $T$ with the property that the intersection $B \cap \sigma(B)$ has minimal possible dimension. Occasionally we will also need to consider *isotropic tori*, that is tori contained in $H$.

**2.1. Ring of definition.** It will be important for us that the classification of involutions is independent of the characteristic (see [16]). So we can use Kac classification or Satake classification to construct the involutions. If we use Kac classification, we see that we can assume that $G$, $\sigma$ and hence $H^\circ$ are all defined over $\boldsymbol{A}$ and that there is a maximally isotropic maximal torus (this means a maximal torus of $G$ containing a maximal torus of $H^\circ$) defined over $\boldsymbol{A}$ and a $\sigma$ stable Borel subgroup containing this torus also defined over $\boldsymbol{A}$. On the other hand if we use Satake classification, we see that we can assume that $G$, $\sigma$, $H^\circ$, the maximal torus $T$, the torus $S$ and the Borel subgroup $B$ are all defined over $\boldsymbol{A}$. However, occasionally, we will need to work with an $\boldsymbol{A}$ form of $G$, where both maximally isotropic and maximally anisotropic maximal tori are defined and split over $\boldsymbol{A}$.

We start with a flat $\boldsymbol{A}$ form $\boldsymbol{G}$ of $G$, and with a $\sigma$ defined over $\boldsymbol{A}$ constructed using Kac diagrams. So $H^\circ$ is defined over $\boldsymbol{A}$ and there exists an $\boldsymbol{A}$ split maximal torus $N$ of $H^\circ$ defined over $\boldsymbol{A}$ and an $\boldsymbol{A}$ split maximal torus $M$ of $G$ defined over $\boldsymbol{A}$ containing $N$. The characters of $M$, $N$ are defined over $\boldsymbol{A}$ and the root decomposition of the Lie algebra of $G$ is also defined over $\boldsymbol{A}$. In particular all Borel subgroups containing $M$ are defined over $\boldsymbol{A}$. Let $B_M$ be a $\sigma$ stable Borel subgroups of $G$ containing $M$. Let $\Psi$ and $\Psi^+ \subset \Psi$ be the corresponding sets of roots and positive roots of the Lie algebra of $G$ with respect to $B_M$. Finally notice that also $\bar{H}$, hence $H$, can be assumed to be defined over $\boldsymbol{A}$.

We want to show that in $G$ there is a maximally anisotropic maximal torus defined and split over $\boldsymbol{A}$. This slightly strength a result in [6].

**Lemma 2.1.** *There is a torus $S$ in $G$ defined and split over $\boldsymbol{A}$ such that $\sigma(s) = s^{-1}$ for all $s \in S$ and $S$ has maximal dimension among the tori with this property. Moreover there is a maximal torus $T$ of $G$ containing $S$ defined and split over $\boldsymbol{A}$. Finally the root decomposition of $\mathfrak{g}$ with respect to the action of the torus $T$ is defined over $\boldsymbol{A}$ and there is a Borel $B$ subgroup containing $T$ and defined and flat over $\boldsymbol{A}$ such that the dimension of $\sigma(B) \cap B$ is the minimal possible. The two Borel subgroups $B$ and $B_M$ are conjugated by an element of $\boldsymbol{G}(\boldsymbol{A})$.*

*Proof.* For each root $\beta \in \Psi$ denote by $u_\beta(t)$ the corresponding one parameter subgroup. This subgroup can be defined over $\boldsymbol{A}$. We construct (see [11, Section VI.7]) the torus $T$ as follows. Let $\mathcal{B} \subset \Psi^+$ be a set of roots maximal among the subsets with the following properties:

(i) $\beta \in \mathcal{B}$ implies $\sigma(\beta) = \beta$ and $\sigma(u_\beta(t)) = u_\beta(-t)$;
(ii) $\beta, \beta' \in \mathcal{B}$ implies $\beta + \beta', \beta - \beta' \notin \Psi$.

For each $\beta \in \mathcal{B}$ set

$$g_\beta = u_\beta(1)u_{-\beta}(-1/2) \quad \text{and} \quad g_{\mathcal{B}} = \prod_{\beta \in \mathcal{B}} g_\beta.$$

Notice that since by ii) the roots in $\mathcal{B}$ are orthogonal to each other, the elements $g_\beta$ as $\beta$ runs in $\mathcal{B}$ commute and $g_{\mathcal{B}}$ is well defined and lies in $\boldsymbol{G}(\boldsymbol{A})$.

We then set $T = g_{\mathcal{B}} M g_{\mathcal{B}}^{-1}$ and $S = g_{\mathcal{B}} M_{\mathcal{B}} g_{\mathcal{B}}^{-1}$, where $M_{\mathcal{B}}$ is the subtorus of $T$ corresponding to the coroots in $\mathcal{B}$. By [11, Section VI.7] $T$ and $S$ have all the required properties.

Our claims about the root decomposition now follows from the analogous properties for the torus $M$, and under this hypothesis it is clear that each Borel containing $T$ is defined and flat over $\boldsymbol{A}$. Also notice that $g_{\mathcal{B}} B_M g_{\mathcal{B}}^{-1}$ is a Borel containing $T$

so it must be conjugated to $B$ by an element in $N_G(T)$. Now by Lemma 2.7 in [6] every element of the Weyl group has a representative in $\boldsymbol{G}(\boldsymbol{A})$ proving the last claim. □

We finish this section with a simple Lemma regarding invariants. Since we are going to deal with not necessarily reduced algebraic groups, let us recall that if $L$ is a not necessarily reduced algebraic group and $V$ is a representation of $L$, a $L$ invariant vector $v \in V$ is a vector whose image under the coaction $V \to V \otimes \Bbbk[L]$ is $v \otimes 1$.

**Lemma 2.2.** *Let* $\mathbf{L}$ *be an algebraic group scheme defined and flat over* $\boldsymbol{A}$ (*we do not assume that* $\mathbf{L}$ *is either connected or reduced in general*) *and let* $V$ *be a finite dimensional representation of* $\mathbf{L}$ *defined and flat over* $\boldsymbol{A}$. *Assume that* $V(\mathbb{C})^{\mathbf{L}(\mathbb{C})} \neq 0$. *Then there is an* $\mathbf{L}$ *invariant vector defined over* $\boldsymbol{A}$ *whose reduction modulo* $p$ *is different from* $0$ *for all odd primes* $p$. *In particular* $V(\Bbbk)^{\mathbf{L}(\Bbbk)} \neq 0$.

*Proof.* Let $V_{\boldsymbol{A}}$ be an $\boldsymbol{A}$ lattice compatible with the action. The action of $\mathbf{L}$ on $V$ is given by the coaction map $a^\sharp \colon V_{\boldsymbol{A}} \to \boldsymbol{A}[\mathbf{L}] \otimes_{\boldsymbol{A}} V_{\boldsymbol{A}}$. If $B$ is an $\boldsymbol{A}$ algebra, set $a^\sharp_B = \mathrm{id}_B \otimes_{\boldsymbol{A}} a^\sharp$. Thus an element $v$ in $V(B) := B \otimes_{\boldsymbol{A}} V_{\boldsymbol{A}}$ is fixed by $\mathbf{L}$ if $a^\sharp_B(v) = 1 \otimes v$. Let now $F \colon V_{\boldsymbol{A}} \to \boldsymbol{A}[\mathbf{L}] \otimes_{\boldsymbol{A}} V_{\boldsymbol{A}}$ be given by $F(v) = a^\sharp(v) - 1 \otimes v$ and $F_B = \mathrm{id}_B \otimes F$. $V(B)^{\mathbf{L}} = \ker F_B$. In particular notice that when $B$ is a field of characteristic zero we have, since $V_{\boldsymbol{A}}$ is a free $\boldsymbol{A}$-module, that $\ker F_B = B \otimes_{\boldsymbol{A}} \ker F$. In particular $V(\mathbb{C})^{\mathbf{L}(\mathbb{C})} = \mathbb{C} \otimes_{\boldsymbol{A}} \ker F$ is defined over $\boldsymbol{A}$. Moreover since also $\boldsymbol{A}[\mathbf{L}] \otimes_{\boldsymbol{A}} V_{\boldsymbol{A}}$ has no torsion, we have that if $n \in \mathbb{Z} \smallsetminus \{0\}$, $v \in V_{\boldsymbol{A}}$ and $nv \in \ker F$ then $v \in \ker F$. So $\ker F$ is a direct summand of $V_{\boldsymbol{A}}$.

It follows that $\Bbbk \otimes_{\boldsymbol{A}} \ker F$ injects into a non-zero subspace of $V(\Bbbk)^{\mathbf{L}(\Bbbk)}$ proving our claim. □

**2.2. Spherical weights and the restricted root system.** We want to describe now the Weyl modules of $G$ which have a non-zero $H$ invariant vector.

If $A$ is a torus, we denote with $\Lambda_A$ its character lattice $\mathrm{Hom}(A, \Bbbk^*)$. Given a surjective homomorphism $A \to B$ between tori, we are going to consider $\Lambda_B$ as a sublattice of $\Lambda_A$.

Let $\Lambda = \Lambda_T$ and let $r \colon \Lambda \to \Lambda_S$ be the surjective homomorphism induced by the inclusion $S \subset T$. Let also $\Phi$ be the root system of $\mathfrak{g}$ with respect to $T$, $\Phi^+$ (resp. $\Delta$) be the choice of positive roots (resp. the simple roots) corresponding to the Borel $B$ and $\Lambda^+$ be the dominant weights with respect to $B$.

Every character $\lambda$ of $T$ extends uniquely to a one dimensional character of $B$ and we define $\mathcal{L}_\lambda$ as the line bundle $G \times_B \Bbbk_{-\lambda}$ on $G/B$. Every line bundle on $G/B$ is isomorphic to a line bundle of this form. For $\lambda \in \Lambda^+$ the *Weyl module* $V_\lambda$ is defined as the dual of the space of sections $\Gamma(G/B, \mathcal{L}_\lambda)$. With the choices of the previous section, all these objects are defined over $\boldsymbol{A}$. Furthermore, it is well known [10] that $\Gamma(G/B, \mathcal{L}_\lambda)$ and hence $V_\lambda$ is flat over $\boldsymbol{A}$. Occasionally we will have to consider also line bundles on a partial flag variety $G/P$, where $P$ is a parabolic subgroup containing $B$. The natural projection $G/B \to G/P$ induces an inclusion of $\mathrm{Pic}(G/P)$ in $\mathrm{Pic}(G/B) = \Lambda$ and allows us to identify $\mathrm{Pic}(G/P)$ with the sublattice $\Lambda_P$ of $\Lambda$ consisting of those characters $\lambda$ of $B$ which extend to $P$. For

$\lambda \in \mathrm{Pic}(G/P)$ we are going, by abuse of notation, to denote by $\mathcal{L}_\lambda$ the line bundle $G \times_P \Bbbk_{-\lambda}$ on $G/P$.

We define the monoid of dominant $H$ *spherical weights* as

$$\Omega_H^+ = \left\{\lambda \in \Lambda^+ : \Gamma(G/B, \mathcal{L}_\lambda)^H \neq 0\right\}$$

and the lattice of spherical weights $\Omega_H$ as the lattice generated by $\Omega_H^+$. We set also $\Omega = \Omega_{H^\circ}$ and $\Omega^+ = \Omega_{H^\circ}^+$.

Recall that, since $H$ has an open orbit in $G/B$, if $\lambda \in \Lambda^+$ then the space of $H$ invariant sections $\Gamma(G/B, \mathcal{L}_\lambda)^H$ is at most one dimensional. A non-zero vector in $\Gamma(G/B, \mathcal{L}_\lambda)^H$ will be called a *spherical vector*.

Let us now give a description of $\Omega$. In characteristic zero $\Omega$ has been described by Helgason [9] using analytic methods. An algebraic proof of these results has been given by Vust [19]. The Theorem of Vust is stated in characteristic zero but its proof can be used verbatim in any characteristic different from 2 once we replace $V_\lambda$ with $V_\lambda^*$. Moreover Vust's proof can also be easily adapted to describe the lattice $\Omega_H$. Let $S_H = S/S \cap H$ then we have the following Theorem.

**Theorem 2.3** (Vust [19, Théorème 3])**.** *Let $\lambda \in \Lambda^+$ then $\lambda \in \Omega_H^+$ if and only if $\sigma(\lambda) = -\lambda$ and $r(\lambda) \in \Lambda_{S_H}$.*

We will need also to study quasi invariants under the action of $\bar{H}$, so we define a dominant weight $\lambda$ to be *quasi spherical* if the representation $\Gamma(G/B, \mathcal{L}_\lambda)$ has a line fixed by $\bar{H}$. We denote the monoid of quasi spherical dominant weights by $\Pi^+$ and we set $\Pi$ equal to the sublattice spanned by $\Pi^+$ and call it the lattice of quasi spherical weights.

Quasi spherical weights have been described in terms of spherical weights and exceptional roots by De Concini and Springer in [6].

Let $\Phi_0$ (resp. $\Delta_0$) be the set of roots (resp. simple roots) fixed by $\sigma$ and let $\Phi_1$ (resp. $\Delta_1$) be the complement of $\Phi_0$ in $\Phi$ (resp. of $\Delta_0$ in $\Delta$). With our choices of the Borel subgroup $B$ we have $\sigma(\alpha) \in \Phi^-$ for all $\alpha \in \Phi_1^+ = \Phi_1 \cap \Phi^+$ (see [4]). Moreover the involution $\sigma$ induces an involution $\bar{\sigma}$ of $\Delta_1$, where $\bar{\sigma}(\alpha)$ is the unique simple root such that $\sigma(\alpha) + \bar{\sigma}(\alpha)$ lies in the span of $\Delta_0$. A simple root $\alpha \in \Delta_1$ is said to be *exceptional* if $\bar{\sigma}(\alpha) \neq \alpha$ and $\kappa(\sigma(\alpha), \alpha) \neq 0$, $\kappa$ being a nondegenerate bilinear form on $\Lambda$ invariant under the action of the Weyl group. We denote by $\{\omega_\alpha\}_{\alpha \in \Delta}$, the fundamental weights with respect to the simple basis $\Delta$. We have,

**Theorem 2.4** (De Concini and Springer [6, Lemma 4.6 and Theorem 4.8])**.**

(i) *For each $\lambda \in \Pi^+$ the line fixed by $\bar{H}$ is unique.*
(ii) *$\Pi^+$ is generated as a monoid by $\Omega^+$ and the fundamental weights $\omega_\alpha$ corresponding to the exceptional roots.*

The set of spherical weights is related to the *restricted root system* as follows. Let us quickly recall how restricted roots are defined. If $\alpha \in \Phi$ is not fixed by $\sigma$, we define the *restricted root* $\tilde{\alpha}$ as $\alpha - \sigma(\alpha)$ and the restricted root system $\tilde{\Phi} \subset \Lambda$ as the set of all restricted roots. This is a (not necessarily reduced) root system (see [14]) of rank $\ell$ and the subset $\tilde{\Phi}^+$ (resp. $\tilde{\Delta}$) of restricted roots $\tilde{\alpha}$ with $\alpha$ positive (resp. $\alpha$ simple) is a choice of positive roots (resp. a simple basis) for $\tilde{\Phi}$.

We collect in the following lemma some known and easy consequences of the previous theorems.

**Lemma 2.5.**
(i) $\Pi \cap \Lambda^+ = \Pi^+$ *and* $\Omega_H \cap \Lambda^+ = \Omega_H^+$;
(ii) *In the adjoint case we have* $\Omega_{\bar{H}} = \mathbb{Z}[\tilde{\Phi}]$;
(iii) *In the simply connected case we have*

$$\Omega = \{\lambda \in \Lambda \colon \sigma(\lambda) = -\lambda \text{ and } \tfrac{2\kappa(\lambda,\tilde{\alpha})}{\kappa(\tilde{\alpha},\tilde{\alpha})} \in \mathbb{Z} \text{ for all } \tilde{\alpha} \in \tilde{\Phi}\};$$

(iv) *If* $\lambda \in \Lambda$ *and* $n\lambda \in \Omega$ *for some positive natural number* $n$, *then* $\sigma(\lambda) = -\lambda$;
(v) *The restriction of* $r$ *to* $\Omega_H$ *is injective and* $r(\Omega_H) = \Lambda_{S_H}$.

In particular, by (iii), $\Omega^+$ is the set of dominant weights of the root system $\tilde{\Phi}$, so it is a free monoid of rank $\ell$ and a basis of it is given by fundamental weights $\tilde{\omega}_{\tilde{\alpha}}$ with respect to $\tilde{\Delta}$. Notice that if $\alpha$ is exceptional also $\beta = \bar{\sigma}(\alpha)$ is exceptional. If this is the case, we shall call $\tilde{\alpha} \in \tilde{\Delta}$ an exceptional restricted simple root and we recall that $\tilde{\omega}_{\tilde{\alpha}} = \omega_\alpha + \omega_\beta$.

Finally we apply Lemma 2.2 to our situation.

**Corollary 2.6.** *If* $\lambda \in \Omega_H^+$, *then* $V_\lambda$ *has a nonzero vector fixed by* $H$ *and if* $\lambda \in \Pi^+$ *then* $V_\lambda$ *has a line fixed by* $\bar{H}$. *More precisely there is a vector of* $V_\lambda$ *defined over* $\boldsymbol{A}$ *whose reduction modulo any odd prime is different from* 0 *and fixed by* $H$ (*respectively spans a line fixed by* $\bar{H}$).

*Proof.* Let $G$, $\sigma$, $V_\lambda$ be defined over $\boldsymbol{A}$ as explained above. Let $M$, $N$, $B_M$ be as in Section 2.1. In particular any character of the group $H^\circ$ is a character of $N$ hence it is defined over $\boldsymbol{A}$.

Let now $\lambda \in \Omega_H^+$. Since $V_\lambda(\mathbb{C})$ contains a non-zero vector fixed by $H$, the claim follows from Lemma 2.2.

In general notice that since $H^\circ$ is a spherical subgroup (it has an open orbit in $G/B$) it acts on two different lines in $V_\lambda(\mathbb{C})$ stabilized by $H^\circ$ with different characters. In particular any line in $V_\lambda(\mathbb{C})$ which is stabilized by $H^\circ(\mathbb{C})$ must be defined over $\boldsymbol{A}$: indeed let $R$ be such a line and consider the character $\chi$ of $H^\circ$ given by the action of $H^\circ$ on $R$. Recall that with our choices all characters of $H^\circ$ are defined over $\boldsymbol{A}$. Applying Lemma 2.2 to $V_\lambda \otimes \chi^{-1}$ we see that the line $R$ must be defined over $\boldsymbol{A}$. In particular the line stabilized by $\bar{H}(\mathbb{C})$ in $V_\lambda(\mathbb{C})$ is stabilized by $H^\circ$ so it is defined over $\boldsymbol{A}$ and it is $\bar{H}$ stable. □

**2.3. Line bundles on $G/H$.** In this section we want to study some properties of the line bundles on $G/H$. We begin with a remark on $\bar{H}/H$.

**Lemma 2.7.** *The coordinate ring of* $\bar{H}/H$ *is isomorphic to the group algebra* $\Bbbk[\Omega_H/\Omega_{\bar{H}}]$.

*Proof.* Let $H \cap S = H \times_G S$ be the scheme theoretic intersection of $H$ and $S$. By Proposition 7 in [19] we have $H = H^\circ \cdot (H \cap S)$. Thus,

$$\bar{H}/H \simeq \bar{H} \cap S / H \cap S \simeq \ker\{S_H \twoheadrightarrow S_{\bar{H}}\},$$

where the kernel has to be considered scheme theoretically. Now by Lemma 2.5 v) we have $S_{\bar{H}} \simeq \operatorname{Spec} \Bbbk[\Omega_{\bar{H}}]$ and $S_H \simeq \operatorname{Spec} \Bbbk[\Omega_H]$.

It follows that, if we denote by $e^\chi$ the function on $S_H$ corresponding to $\chi \in \Omega_H$, the coordinate ring of the kernel is then given by $\Bbbk[S_H]/\langle e^\chi - 1 \colon \chi \in \Omega_{\bar H}\rangle \simeq \Bbbk[\Omega_H/\Omega_{\bar H}]$, proving the claim. □

We denote by $x_H$ the point of $G/H$ corresponding to the coset $eH$ and by $q_H \colon G/H \to G/\bar H$ the projection induced by inclusion $H \subset \bar H$.

The line bundles on $G/H$ are parametrized by the set of one dimensional characters $\Lambda_H$ of $H$ by associating to a line bundle $\mathcal{L}$ the character by which $H$ acts on the fiber of $\mathcal{L}$ over $x_H$.

If $\lambda \in \Pi^+$ by Theorem 2.4, the line fixed by $\bar H$ in $V_\lambda^*$ is unique and we can consider the character $-\chi_H(\lambda)$ given by the action of $H$ on this line. The map $\chi_H \colon \Pi^+ \to \Lambda_H$ extends to a group homomorphism $\chi_H \colon \Pi \to \Lambda_H$ and by Lemma 2.5(i) the kernel of this homomorphism is given by $\Omega_H$. In particular for any $\xi \in \Pi/\Omega_{\bar H}$ we can consider a line bundle $\mathcal{L}_\xi$ on $G/\bar H$ whose associated isomorphism class is given by $\chi_{\bar H}(\xi)$.

**Proposition 2.8.** *The vector bundles $(q_H)_*(\mathcal{O}_{G/H})$ and $\bigoplus_{\xi\in\Omega_H/\Omega_{\bar H}} \mathcal{L}_\xi$ on $G/\bar H$ are $G$-equivariantly isomorphic.*

*Proof.* Set $\Xi_H = \Omega_H/\Omega_{\bar H}$. Notice first that by Lemma 2.7 the map $q_H$ is a covering of degree equal to the cardinality of $\Xi_H$. So the two vector bundles $(q_H)_*(\mathcal{O}_{G/H})$ and $\bigoplus_{\xi\in\Xi_H} \mathcal{L}_\xi$ have the same rank.

If $\xi \in \Xi_H$, then $q_H^*(\mathcal{L}_\xi)$ is trivial for all $\xi$ as a $G$-linearized line bundle. So by adjunction we have a $G$-equivariant monomorphism of sheaves $\mathcal{L}_\xi \to (q_H)_*(\mathcal{O}_{G/H})$. Thus, for any subset $R \subset \Xi_H$ there exists a $G$-equivariant map $\gamma_R \colon \bigoplus_{\xi\in R} \mathcal{L}_\xi \to (q_H)_*(\mathcal{O}_{G/H})$. Since $\gamma_R$ is equivariant, the induced map at the level of the total spaces of vector bundles has constant rank.

We claim that $\gamma_R$ is of rank $|R|$. If $|R| = 1$, this is clear by the above considerations. We proceed by induction. Write $R = R' \cup \{\xi\}$. $\gamma'_R$ is of rank $|R|-1$. Assume $\gamma_R$ is not of maximal rank. We clearly get an inclusion $j \colon \mathcal{L}_\xi \to \bigoplus_{\xi'\in R'} \mathcal{L}_{\xi'}$. In particular there exists $\xi' \in R'$ such that the composition of $j$ with the projection onto $\mathcal{L}_{\xi'}$ is a non-zero $G$-equivariant morphism and thus an isomorphism of line bundles. Since $\xi \neq \xi'$, this is a contradiction.

If we apply this to $R = \Xi_H$ and use the fact that $(q_H)_*(\mathcal{O}_{G/H})$ and $\bigoplus_{\xi\in\Xi_H} \mathcal{L}_\xi$ have the same rank, we get that $\gamma_{\Xi_H}$ is a isomorphism as desired. □

## 3. Completions of Symmetric Varieties

An *embedding* of a symmetric variety $G/H$ is a normal connected $G$-variety $Y$ together with an open $G$-equivariant inclusion $\jmath_Y \colon G/H \subset Y$. We set $y_0$ equal to the image of $x_H$ under this embedding and call it the basepoint of $Y$. We are also going to consider the finite covering $\pi_Y \colon G/H^\circ \to Y$ of $\jmath_Y$ given by $\pi_Y(gH^\circ) = g{\cdot}y_0$. We denote by $Y_0$ the image of $\jmath_Y$ and set $\partial Y = Y \smallsetminus Y_0$ and $\Delta_Y$ equal to the set of irreducible components of $\partial Y$ of codimension 1 in $Y$.

A line bundle $\mathcal{L}$ on $Y$ is said to be *spherical* if $\pi_Y^*(\mathcal{L})$ is isomorphic to the trivial line bundle on $G/H^\circ$. We denote $\mathrm{SPic}(Y)$ the subgroup of the Picard group $\mathrm{Pic}(Y)$ of $Y$ of spherical line bundles. We also say that a line bundle is *strictly spherical* if restricted to the open orbit $G/H$ it is isomorphic to the trivial line bundle and we denote by $\mathrm{SPic}_0(Y)$ the subgroup of $\mathrm{Pic}(Y)$ of classes of strictly spherical line bundles.

Many of the properties of $Y$ can be deduced from corresponding properties of the associated toric variety $Y_S$. This is defined as the closure of the orbit $S \cdot y_0$ in $Y$. Notice that since $S \cdot y_0$ is isomorphic to $S_H$, $Y_S$ is a toric variety for the torus $S_H$. The normalizer $N_{H^\circ}(S)$ of $S$ in $H^\circ$ acts on $Y_S$ and the action of the centralizer $Z_{H^\circ}(S)$ of $S$ is trivial. It follows that we have an action of the *restricted Weyl group* $\widetilde{W} = N_{H^\circ}(S)/Z_{H^\circ}(S)$ on $Y_S$.

We are now going to describe an open subvariety $Y_S^+$ of $Y_S$ with the property the $\widetilde{W}$ translates of $Y_S^+$ cover $Y_S$. Let $\Lambda_S^\vee$ be the lattice of one parameter subgroups of $S$. If $\eta \in \Lambda_S^\vee$ and there exists the limit $\lim_{t\to\infty} \eta(t) \cdot y_0$, we denote this limit by $y_\eta$. We say that $\eta \in \Lambda_S^\vee$ is positive if $\tilde{\alpha}(\eta(t))$ is a nonnegative power of $t$ for all $\tilde{\alpha} \in \tilde{\Delta}$ and let $Y_S^+$ the union of the $S$ orbits of the elements $\{y_\eta \colon \eta \in \Lambda_S^\vee$ is positive$\}$. It is then immediate to verify that $Y_S = \widetilde{W} Y_S^+$. Indeed if $y \in Y_S$ there is a $\eta \in \Lambda_S^\vee$ and a $s \in S$ such that $y = s(\lim_{t\to\infty} \eta(t) \cdot y_0)$. Since $\eta$ is $\widetilde{W}$ conjugate to a positive one parameter subgroup, we deduce $y$ is $\widetilde{W}$ conjugate to an element in $Y_S^+$.

**3.1. The wonderful compactification of a symmetric variety.** The so called wonderful compactification $X$ of the symmetric variety $G/\bar{H}$ has been introduced in characteristic zero in [4] and in arbitrary characteristic in [6]. We want to very briefly recall some of the basic properties of $X$ and introduce some notations.

Recall that by Lemma 2.5 and Theorem 2.3 a basis of the character lattice $\Lambda_{S_{\bar{H}}}$ is given by the set $\tilde{\Delta} = \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_\ell\}$ of simple restricted roots (with an arbitrarily chosen numbering). Thus we get an action, defined over $\boldsymbol{A}$, of $S_{\bar{H}}$ on the affine space $\mathbb{A}^\ell$ given by $s(a_1, \ldots, a_\ell) = (\tilde{\alpha}_1(s)a_1, \ldots, \tilde{\alpha}_\ell(s)a_\ell)$. The following theorem (Theorem 3.1 in [4], Proposition 3.10, Theorem 3.10 and Theorem 3.13 in [6]) can be taken implicitly as the defnition of the wonderful compactification.

**Theorem 3.1.** *The wonderful compactification $X$ of $G/\bar{H}$ is the unique $G/\bar{H}$ embedding such that*

(i) *$X$ is a smooth projective $G$-variety and the closure of every $G$ orbit in $X$ is smooth*;
(ii) *$\partial X$ is a divisor with normal crossing and smooth irreducible components*;
(iii) *given a $G$ orbit $\mathcal{O} \subset X$, $\bar{\mathcal{O}}$ is the transversal intersection of the irreducible divisors in $\Delta_X$ containing it*;
(iv) *The intersection of any number of divisors in $\Delta_X$ is a $G$ orbit closure. In particular the intersection of all divisors in $\Delta_X$ is the unique closed $G$ orbit in $X$*;
(v) *There exists a scheme $\boldsymbol{X}$ defined and flat over $\boldsymbol{A}$ whose specialization to $\Bbbk$ is isomorphic to $X$. Moreover the point $x_0 = \jmath_X(x_{\bar{H}})$ is defined over $\boldsymbol{A}$*;
(vi) *Let $\boldsymbol{G}$ be as in Section 2.1. There is an action of $\boldsymbol{G}$ on $\boldsymbol{X}$ that specializes over $\Bbbk$ to the action of $G$ on $X$*;
(vii) *We have an isomorphism $X_S^+ \simeq \mathbb{A}^\ell$ as $S_{H^\circ}$ toric varieties defined over $\boldsymbol{A}$.h*

Since any projective $G$-variety is isomorphic to a variety $G/P$ with $P$ a parabolic subgroup containing $B$, we have already remarked that its Picard group can be identified with a sublattice of $\Lambda$. Thus composing with the homomorphism induced by the inclusion of the unique closed orbit, we get a homomorphism $j\colon \operatorname{Pic}(X) \to \Lambda$. One has the following result (Theorems 4.2 and 4.8 in [6]).

**Theorem 3.2.**

(i) *The homomorphism $j$ is injective and its image is the sublattice $\Pi$ of $\Lambda$.*
(ii) *The map $D \to j(\mathcal{O}(D))$ is a bijection between $\Delta_X$ and $\tilde{\Delta}$.*

Notice that combining these two results we easily see that we get a bijection between the subsets $\Gamma \subset \tilde{\Delta}$ and the set of $G$ orbit closures defined by associating to $\Gamma$ the intersection

$$X_\Gamma := \bigcap_{\{D\colon j(\mathcal{O}(D))\in\Gamma\}} D.$$

In particular $X_{\tilde{\Delta}}$ is the unique closed orbit while $X = X_\varnothing$. Let also $X_{\tilde{\alpha}} = X_{\{\tilde{\alpha}\}}$ for $\tilde{\alpha} \in \tilde{\Delta}$.

For each $\lambda \in \Pi$ we choose a line bundle $\mathcal{L}_\lambda$ on $X$ such that $j(\mathcal{L}_\lambda) = \lambda$ in the following way. First we choose a basis $\mathcal{B}$ of $\Pi$ and for each $\beta \in \mathcal{B}$ we take a line bundle with the required property. Now, for $\lambda = \sum_{\beta\in\mathcal{B}} c_\beta\, \beta \in \Pi$, $c_\beta \in \mathbb{Z}$, we set $\mathcal{L}_\lambda := \bigotimes_{\beta\in\mathcal{B}} \mathcal{L}_\beta^{\otimes c_\beta}$. We denote the restriction of these line bundles to $X_{\tilde{\Delta}}$ by the same symbol.

If $L \subset \Lambda$ is a sublattice of $\Lambda$, then our definition allows us to consider the graded rings

$$R_L(X) := \bigoplus_{\lambda\in L} \Gamma(X, \mathcal{L}_\lambda) \quad \text{and} \quad R_L(X_{\tilde{\Delta}}) := \bigoplus_{\lambda\in L} \Gamma(X_{\tilde{\Delta}}, \mathcal{L}_\lambda).$$

The ring $R(X) = R_\Pi(X)$ it is called the *Cox ring* of $X$ and it was studied in the case of the variety $X$ in [3], where it was called the ring of all sections. The fact that $G$ is simply connected implies that each line bundle on $X$ has a canonical $G$ linearization. It follows $G$ acts on $R_L(X)$ and $R_L(X_{\tilde{\Delta}})$.

The space $\Gamma(X, \mathcal{L}_\lambda)$ of sections of $\mathcal{L}_\lambda$ has been described as a $G$-module in [4] and [6]. Let us recall here this description.

Recall that a good filtration of a $G$-module $W$ is a filtration $W = W_0 \supset W_1 \supset \cdots \supset W_m = \{0\}$ by $G$ submodules such that for each $i = 1, \dots m$, $W_{i-1}/W_i$ is isomorphic to $\Gamma(G/B, \mathcal{L}_{\lambda_i})$ for a suitable dominant weight $\lambda_i$.

The result in [6] implies that $\Gamma(X, \mathcal{L}_\lambda)$ has a good filtration. To be more precise first of all one shows that for any $\lambda \in \Pi$ the map

$$\Gamma(X, \mathcal{L}_\lambda) \to \Gamma(X_{\tilde{\Delta}}, \mathcal{L}_\lambda)$$

is surjective.

Now for any $\lambda, \mu \in \Pi$ set $\mu \leqslant_\sigma \lambda$ if $\lambda - \mu \in \mathbb{N}[\tilde{\Delta}]$.

Notice that, for $\tilde{\alpha} \in \tilde{\Delta}$, there is a $G$ invariant section $s_{\tilde{\alpha}}$ of $\mathcal{L}_{\tilde{\alpha}}$, unique up to multiplication by a non-zero scalar, whose divisor is $X_{\tilde{\alpha}}$.

If $\nu = \sum_{\tilde{\alpha}} n_{\tilde{\alpha}}\tilde{\alpha} \geqslant_\sigma 0$, consider $s^\nu := \prod_{\tilde{\alpha}} s_{\tilde{\alpha}}^{n_{\tilde{\alpha}}}$. If $\lambda \geqslant_\sigma \mu$, the multiplication by $s^{\lambda-\mu}$ defines a $G$-equivariant injective map from $\Gamma(X, \mathcal{L}_\mu)$ to $\Gamma(X, \mathcal{L}_\lambda)$ whose image we denote by $s^{\lambda-\mu}\Gamma(X, \mathcal{L}_\mu)$.

For any $\nu \geqslant_\sigma 0$ we now set

$$F_{\lambda,\nu} = \sum_{\mu \leqslant_\sigma \lambda-\nu} s^{\lambda-\mu}\Gamma(X, \mathcal{L}_\mu).$$

The $F_{\lambda,\nu}$ form a decreasing filtration of $\Gamma(X, \mathcal{L}_\lambda)$ by $G$ submodules. In [4], [6] the associated graded is computed and we have that the division by $s^\nu$ and restriction of sections to $X_{\tilde{\Delta}}$ gives an isomorphism $F_{\lambda,\nu}/(\sum_{\nu' >_\sigma \nu} F_{\lambda,\nu'}) \simeq V^*_{\lambda-\nu}$ so that

$$\mathrm{Gr}_F\, \Gamma(X, \mathcal{L}_\lambda) = \bigoplus_{\mu\in\Pi^+,\, \mu\leqslant_\sigma \lambda} s^{\lambda-\mu} V^*_\mu. \tag{1}$$

Clearly the filtration $F_{*,*}$ respects multiplication. This implies that the associated graded

$$\mathrm{Gr}_F\, R(X) := \bigoplus_{\lambda\in\Pi} \mathrm{Gr}_F\, \Gamma(X, \mathcal{L}_\lambda)$$

of $R(X)$ has a ring structure. Furthermore, (1) gives a ring isomorphism

$$\mathrm{Gr}_F\, R(X) \simeq R_\Pi(X_{\tilde{\Delta}})[s_{\tilde{\alpha}_1}, \dots, s_{\tilde{\alpha}_\ell}]. \tag{2}$$

In the previous section we have studied spherical weights. We want to prove now that $\lambda$ is spherical precisely when $\mathcal{L}_\lambda$ is spherical.

The homomorphism $\pi^*_X\colon \mathrm{Pic}(X) \to \mathrm{Pic}(G/H^\circ)$ can be identified with the homomorphism $\chi\colon \Pi \to \Lambda_{H^\circ}$ associating to $\lambda \in \Pi$, the character $\chi(\lambda)$ by which $H^\circ$ acts on the fiber of $\mathcal{L}_\lambda$ on the point $x_0$.

We claim that $\chi(\lambda) = \chi_{H^\circ}(\lambda)$ is the dual of the character by which $H^\circ$ acts on the line fixed by $\bar{H}$ in $V^*_\lambda$ introduced in the previous section. To see this we may assume $\lambda \in \Pi^+$.

We fix $\lambda \in \Pi^+$ and $\mathcal{L} = \mathcal{L}_\lambda$. In this case $\mathcal{L}$ has no base points over $X_{\tilde{\Delta}}$, so, since $X_{\tilde{\Delta}}$ is the unique closed orbit in $X$, by Theorem 3.1(iv) it also has no base points over $X$. Thus by the reductivity of $\bar{H}$, there is a positive integer $m$ and a line $L \subset \Gamma(X, \mathcal{L}^m)$ stable under the action of $\bar{H}$ and such that if $\sigma \in L - \{0\}$, $\sigma$ does not vanish on $x_0$. It follows that $H^\circ$ acts on $L$ by the character $-m\chi(\lambda)$.

Take the filtration $\{F_{m\lambda,\nu}\}$ of $\Gamma(X, \mathcal{L}^m)$. There is a unique submodule $F_{m\lambda,\nu}$ such that $L \subset F_{m\lambda,\nu} - \sum_{\nu' >_\sigma \nu} F_{m\lambda,\nu'}$. So $L$ has non-zero image in $V^*_{m\lambda-\nu}$ and thus coincides with the unique $\bar{H}$ stable line in $V^*_{m\lambda-\nu}$. We deduce that $m\chi(\lambda) = \chi_{H^\circ}(m\lambda - \nu)$. Since $\nu$ lies in $\mathbb{Z}[\tilde{\Phi}]$, we have $\chi_{H^\circ}(m\lambda - \nu) = \chi_{H^\circ}(m\lambda)$, whence $m\chi_{H^\circ}(\lambda) = m\chi(\lambda)$. Finally since $H^\circ$ is connected its character group has no torsion and we get that $\chi_{H^\circ}(\lambda) = \chi(\lambda)$ as desired. We deduce the following lemma.

**Lemma 3.3.** *Let $\lambda \in \Pi$ then $\pi^*_X(\mathcal{L}_\lambda)$ is trivial if and only if $\lambda \in \Omega$. Moreover if $\pi_H\colon G/H \to X$ is defined by $\pi_H(gH) = g\cdot x_0$ then $\pi^*_H(\mathcal{L}_\lambda)$ is trivial if and only if $\lambda \in \Omega_H$.*

*Proof.* The first claim has just been proved. As for the second it follows since by Theorem 2.3 a character $\lambda$ lies in $\Omega_H \cap \Pi^+$ if and only if the line in $V^*_\lambda$ stable under $\bar{H}$ is pointwise invariant under $H$. □

**3.2. Toroidal compactifications and ring of definition.** An embedding $Y$ of $G/H$ is called *toroidal* if there exists a basepoint preserving $G$-equivariant map $\phi\colon Y \to X$.

Presently we are going to explain their construction and show that they are defined and flat over $\boldsymbol{A}$.

Let $L_{\mathbb{R}} = \mathrm{Hom}(\Lambda_S, \mathbb{R})$ and $L_{\mathbb{R}}^{\vee} = \Lambda_S \otimes_{\mathbb{Z}} \mathbb{R}$ bet its dual. The $S$, or $S_H$, toric varieties are described by fans in $L_{\mathbb{R}}$. In particular take the cochamber $C \subset L_{\mathbb{R}}$ of dominant elements with respect to $\tilde{\Delta}$ and let $\mathcal{T}_H$ be the $S_H$ toric variety associated to $C$. $\mathcal{T}_H$ has a natural $\boldsymbol{A}$ form $\boldsymbol{\mathcal{T}}_H$. In particular in the adjoint case $\boldsymbol{\mathcal{T}}_{\bar{H}} \simeq \mathbb{A}^{\ell}_{\boldsymbol{A}}$.

Choose an $\boldsymbol{A}$ form of $G$ as in Section 2.1. Consider for any $H$ the finite field extension $\mathbb{Q}(G/\bar{H}) \subset \mathbb{Q}(G/H)$. $\mathbb{Q}(G/\bar{H})$ is the field of rational functions on $\boldsymbol{X}$ and we take $\boldsymbol{X}_H$ equal to the normalization of $\boldsymbol{X}$ in $\mathbb{Q}(G/H)$. Let $\phi_H\colon \boldsymbol{X}_H \to \boldsymbol{X}$ denote the normalization map and let $X_H = \boldsymbol{X}_H(\Bbbk)$.

**Lemma 3.4.** *$\boldsymbol{X}_H$ is a projective normal and Cohen–Macaulay embedding of $G/H$. $\phi_H$ is a finite flat morphism. In particular $\boldsymbol{X}_H$ is proper and flat over $\boldsymbol{A}$.*

*Proof.* The projectivity and normality of $\boldsymbol{X}_H$ are clear from the definitions. Let us show that $\boldsymbol{X}_H$ is Cohen–Macaulay.

To see this, let us recall $X$ is covered by the $G$ translates of an open set $\mathcal{U}$ of the form $X_S^+ \times U$, where $U$ is the unipotent radical of the parabolic $P \subset B$ such that $X_{\tilde{\Delta}} \simeq G/P$. By Theorem 3.1 we have that $X_S^+$ and $U$ are defined over $\boldsymbol{A}$ and so is the isomorphism $X_S^+ = \mathcal{T}_{\bar{H}} \simeq \mathbb{A}^{\ell}$. In particular the open set $\mathcal{U}$ is defined over $\boldsymbol{A}$ and we denote by $\boldsymbol{\mathcal{U}}$ the associated subscheme of $\boldsymbol{X}$ and $\mathbf{U}$ the subgroup scheme of $\boldsymbol{G}$ defining $U$.

It easily follows that $\boldsymbol{X}_H$ is covered by the the $\boldsymbol{G}(\boldsymbol{A})$ translates of the preimage $\boldsymbol{\mathcal{U}}_H$ of $\boldsymbol{\mathcal{U}}$ and that $\boldsymbol{\mathcal{U}} \simeq \boldsymbol{\mathcal{T}}_H \times \mathbf{U}$. Since $\boldsymbol{\mathcal{T}}_H$ is Cohen–Macaulay, also $\boldsymbol{\mathcal{U}}$ is Cohen–Macaulay and everything follows.

Since any finite morphism between a Cohen–Macaulay scheme and a smooth scheme is flat, we deduce that $\phi_H$ is flat and all the other claims are clear. $\square$

We are now going to follow the method of [5] to build all toroidal compactifications. For each $\tilde{\alpha} \in \tilde{\Delta}$ we have already chosen a line bundle $\mathcal{L}_{\tilde{\alpha}}$ on $X$ together with a $G$ invariant section $s_{\tilde{\alpha}} \in \Gamma(X, \mathcal{L}_{\tilde{\alpha}})$. We can then consider the vector bundle $\mathcal{V} := \bigoplus_{\tilde{\alpha} \in \tilde{\Delta}} \mathcal{L}_{\tilde{\alpha}}$ and the $G$ invariant section $s := \bigoplus_{\tilde{\alpha} \in \tilde{\Delta}} s_{\tilde{\alpha}}$ of $\mathcal{V}$. Set $\mathcal{V}^* = \{v = (v_{\tilde{\alpha}}) \in \mathcal{V}\colon v_{\tilde{\alpha}} \neq 0\ \forall \tilde{\alpha} \in \tilde{\Delta}\}$. By our previous identifications $\mathcal{V}^*$ is a principal $S_{\bar{H}}$ bundle. If $Z$ is an $S_{\bar{H}}$-variety, we can take the associate bundle $\mathcal{V}^* \times_{S_{\bar{H}}} Z$ on $X$ with fiber $Z$. In particular $\mathcal{V} = \mathcal{V}^* \times_{S_{\bar{H}}} \mathbb{A}^{\ell}$, where $S_{\bar{H}}$ acts on $\mathbb{A}^{\ell}$ via the characters $\tilde{\alpha} \in \tilde{\Delta}$.

Now take $Z$ to be a $S_{\bar{H}}$ embedding over $\mathbb{A}^{\ell}$. The corresponding fan $F_Z$ is a (partial) decomposition of the fundamental Weyl cochamber $C$. The map $Z \to \mathbb{A}^{\ell}$ induces a map $\mathcal{V}^* \times_{S_{\bar{H}}} Z \to \mathcal{V}$ and we define $X_Z$ as the fiber product

$$\begin{array}{ccc} X_Z & \xrightarrow{s_Z} & \mathcal{V}^* \times_{S_{\bar{H}}} Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{s} & \mathcal{V}. \end{array}$$

The $G$ action on $\mathcal{V}$ preserves $\mathcal{V}^*$ and commutes with the $S_{\bar{H}}$ action. So $G$ also acts on $\mathcal{V}^* \times_{S_{\bar{H}}} Z$, the map $\mathcal{V}^* \times_{S_{\bar{H}}} Z \to \mathcal{V}$ is $G$-equivariant and $X_Z$ is a $G$-variety.

In the case of a general $G/H$ we set $X_{H,Z}$ equal to the normalization of $X_Z$ in the field of rational function on $G/H$. We clearly have the cartesian diagram

$$\begin{array}{ccc} X_{H,Z} & \xrightarrow{\mu_Z} & X_Z \\ \downarrow & & \downarrow \\ X_H & \xrightarrow{\mu} & X. \end{array}$$

In particular the morphisms $\mu_Z$ and $s_{H,Z} := s_Z \mu_Z$ are flat. One has the following result (see [5]).

**Theorem 3.5.**

(i) *Every toroidal embedding of $G/H$ is of the form $X_{H,Z}$ for some $S_{\bar{H}}$ embedding $Z$ over $\mathbb{A}^\ell$. In particular it is defined and flat over $\boldsymbol{A}$.*

(ii) *$X_{H,Z}$ is complete* (*resp. projective*) *if and only if the projection $Z \to \mathbb{A}^\ell$ is proper* (*resp. projective*)*.*

(iii) *Every $G$ orbit in $X_{H,Z}$ is of the form $\mathcal{O}_{\mathcal{K}} := (s_{H,Z})^{-1}(\mathcal{V}^* \times_{S_H} \mathcal{K})$ for a unique $S_{\bar{H}}$ orbit $\mathcal{K}$ in $Z$.*

(iv) *Let $\mathcal{F}_Z$ be the fan in $L_{\mathbb{R}}$ whose cones are the $\widetilde{W}$ translates of the cones in $F_Z$. Then $\mathcal{F}_Z$ is the fan corresponding to $S_H$ embedding $Z_H := (X_{H,Z})_{S_H}$. Furthermore, each $G$ orbit in $X_{H,Z}$ intersects $Z_H$ in a unique $N_{H^\circ}(S)$ orbit* (*notice that in accord with* (iii) *these orbits are in canonical bijection with $S_{\bar{H}}$ orbits in $Z$*)*.*

(v) *The divisors in $\Delta_{X_{H,Z}}$ are defined over $\boldsymbol{A}$.*

*Proof.* All these statements are proved in [5] in the case of an embedding of $G/\bar{H}$.

To see (i) in the general case take a toroidal embedding $Y$ of $G/H$. Let us take the quotient by the finite group scheme $\bar{H}/H$. We get an embedding of $G/\bar{H}$ which is obviously toroidal and hence of the form $X_Z$ for a suitable $S_{\bar{H}}$ embedding $Z$. If we now consider $X_{H,Z}$, we get a morphism $Y \to X_{H,Z}$ which is $G$-equivariant birational and finite. Since both $Y$ and $X_{H,Z}$ are normal, it follows that the above morphism is a $G$-equivariant isomorphism.

The proof of the remaining statements is now easy and we leave it to the reader. □

**Remark 3.6.** (1) Let us point out that our result in particular implies that the $G$ orbits in $X_{H,Z}$ are exactly the preimages of $G$ orbits in $X_Z$.

(2) It is not hard to see that $X_{H,Z}$ is smooth if and only if $Z_H$ is smooth. Equivalently if and only if the $S_H$ embedding whose fan is $F_Z$ is smooth. This depends very much on the lattice $\mathrm{Hom}(\Lambda_{S_H}, \mathbb{Z}) \subset L_{\mathbb{R}}$.

(3) There exists an open affine covering $\{U_i = \mathrm{Spec}\, R_i\}$ of the $\boldsymbol{A}$ form of $X_{H,Z}$ such that $R_i$ are free $\boldsymbol{A}$-modules and $U_i \cap U_j = \mathrm{Spec}\, R_{ij}$, where $R_{ij}$ is also a free $\boldsymbol{A}$-module.

**3.3. Line bundles on a toroidal embedding.** In this section we assume $Y$ to be a smooth toroidal compactification of $G/H$ with the $\boldsymbol{A}$ structure described in the previous section.

We have the following lemma about the structure of the Picard group of $Y$.

**Lemma 3.7.** *Let $Y$ be a equivariant smooth toroidal compactification of $G/H$ then*

(i) *We have the following sequence describing the Picard group of $Y$:*

$$0 \to \bigoplus_{D\in\Delta_Y} \mathbb{Z}\mathcal{O}(D) \to \mathrm{Pic}(Y) \xrightarrow{\jmath_Y^*} \Lambda_H \to 0.$$

(ii) $\mathrm{SPic}_0(Y) = \bigoplus_{D\in\Delta_Y} \mathbb{Z}\mathcal{O}(D)$.

(iii) *For each closed $G$ orbit $O$ of $Y$ consider the restriction $\imath_O^*\colon \mathrm{SPic}_0(Y) \to \mathrm{Pic}(O)$ of line bundles to $O$. Then the product of these restriction maps $\imath^*\colon \mathrm{SPic}_0(Y) \to \prod \mathrm{Pic}(O)$ is injective.*

(iv) *All line bundles on $Y$ are defined and flat over $\boldsymbol{A}$.*

*Proof.* The only thing we need to show to prove (i) is the injectivity of the map from $\bigoplus_{D\in\Delta_Y} \mathbb{Z}\mathcal{O}(D)$ to $\mathrm{Pic}(Y)$. Notice that, since $G$ is semisimple and simply connected and $\mathrm{Pic}(Y)$ discrete, every line bundle has a unique $G$ linearization. Thus $\mathrm{Pic}(Y) \simeq \mathrm{Pic}_G(Y)$. It follows that it is enough to prove the injectivity of the map from $\bigoplus_{D\in\Delta_Y} \mathbb{Z}\mathcal{O}(D)$ to $\mathrm{Pic}_G(Y)$. Consider the restriction map $\mathrm{Pic}_G(Y) \to \mathrm{Pic}_{S_H}(Y_S^+)$. $\mathrm{Pic}_{S_H}(Y_S^+)$ is isomorphic to $\bigoplus \mathbb{Z}\mathcal{O}(D')$, where the sum is take over all $S_H$-equivariant divisors $D'$. So the claim follows from Theorem 3.5. Since $\mathrm{SPic}_0$ is the kernel of $\jmath_Y^*$, this also proves (ii).

(iii) follows from the previous considerations and the fact that, up to isomorphism a $S_H$-equivariant line bundle on $Y_S^+$ is completely determined by its restriction to the closed orbits, that is the $S_H$ fixpoints in $Y_S^+$.

Finally let $D \in \Delta_Y$. By Theorem 3.5(v) it is defined over $\boldsymbol{A}$. We know that $\Lambda_H = \mathrm{Pic}(G/H)$ is generated by the codimension 1 irreducible $B$ orbits in $G/H$ and that these orbits are defined over $\boldsymbol{A}$ by Lemma 2.7 in [6] and Lemma 2.1. Thus (iv) follows from (i). ☐

Let $F_Y$ be the fan associated to the toric variety $Y_S^+$ and let $F_Y(i)$ be the set of faces of $F_Y$ of dimension $i$. In particular the closed orbits of $Y$ are parametrized by $F_Y(\ell)$ while $F_Y(1)$ can be identified with $\Delta_Y$ the set of $G$ invariant divisors. For each $\rho \in F_Y$ we set $y_\rho := y_\eta$ for $\eta$ a generic element in $\rho$ and denote by $O_\rho = G\,y_\rho$ the associated $G$ orbit.

By Theorem 3.5 and the description of the equivariant Picard group of a toric variety we have the following description of the strictly spherical line bundles on $Y$:

$$\mathrm{SPic}_0(Y) = \{\underline{\lambda} = (\lambda_\tau) \in \prod_{\tau\in F_Y(\ell)} \Omega_H \colon \lambda_\tau = \lambda_{\tau'} \text{ on } \tau\cap\tau'\}. \tag{3}$$

We can think of $\underline{\lambda}$ as a real valued function on the Weyl cochamber $C$ which coincides with the linear form $\lambda_\tau$ on the face $\tau$. We denote by $\mathcal{L}_{\underline{\lambda}}$ a line bundle whose class is given by $\underline{\lambda}$. In particular we can describe in this way the line bundles $\mathcal{O}(D)$ for each divisor $D \in \Delta_Y$. Indeed let $v_D \in \Lambda_{S_H}^\vee$ be a not divisible element of

$\Lambda_{S_H}$ in the 1-dimensional face of $F_Y$ associated to $D$. For each $\tau \in F_Y(\ell)$ notice that, since $Y$ is smooth, the set $\{v_{D'} \colon D' \in \Delta_Y\} \cap \tau$ is a basis of $\Lambda^\vee_{S_H}$.

So we can define $\alpha_{D,\tau} \in \Lambda_{S_H}$ to be the weight which is equal to zero if $v_D \notin \tau$ while if $v_D \in \tau$ it is 1 on $v_D$ and zero on each $v_{D'} \in \tau$ with $D' \neq D$. It is then easy to see that $\underline{\alpha}_D = (\alpha_{D,\tau})_{\tau \in F_Y(\ell)}$ is the class of $\mathcal{O}(D)$ in $\mathrm{SPic}_0(Y)$.

Now we want to describe the sections of a strictly spherical line bundle on $Y$ in the case of characteristic 0. The proofs are very similar to the one given in [1]. A description of the section of a line bundle on a general spherical variety is given in [2] and we could have used that result as well. However the description we are going to give is more suited to our purpose.

For $D \in \Delta_Y$ let $s_D$ be a $G$ invariant section of $\Gamma(Y, \mathcal{O}(D))$ vanishing on $D$. If $\underline{\lambda} = \sum_D a_D \underline{\alpha}_D$, we set $s^{\underline{\lambda}} = \prod_D s_D^{a_D}$. Also for a given $\mu \in \Omega_H^+$ consider the line bundle $\phi^*(\mathcal{L}_\mu)$, where $\phi \colon Y \to X$ is the $G$-equivariant projection from $Y$ to $X$. This line bundle corresponds to the element $\underline{\mu} \in \mathrm{SPic}_0(Y)$ with $\mu_\tau = \mu$ for all $\tau \in F_Y(\ell)$ under the identification of $\Omega_H$ with $\Lambda_{S_H}$ given by Lemma 2.5(v). In particular $V_\mu^*$ is a submodule of $\Gamma(Y, \phi^*(\mathcal{L}_\mu))$.

For $\underline{\lambda} \in \mathrm{SPic}_0(Y)$ set

$$\begin{aligned}\mathcal{A}(\underline{\lambda}) &= \{\mu \in \Omega_H^+ \colon \forall \tau \in F_Y(\ell),\ \ \lambda_\tau - \mu = \sum_{v_D \in \tau} a_D \underline{\alpha}_D \text{ with } a_D \geqslant 0\} \\ &= \{\mu \in \Omega_H^+ \colon \mu \leqslant \underline{\lambda} \text{ on } C\}.\end{aligned}$$

We then have the following theorem whose proof is completely analogous to the one given in [1].

**Theorem 3.8.** *Assume $Y$ to be a smooth toroidal compactification of $G/H$ and assume the field to be of characteristic zero and let $\underline{\lambda} \in \mathrm{SPic}_0(Y)$ then*

$$\Gamma(Y, \mathcal{L}_{\underline{\lambda}}) = \bigoplus_{\mu \in \mathcal{A}(\underline{\lambda})} s^{\underline{\lambda} - \underline{\mu}} V_\mu^*.$$

From the above result we can deduce, as in [1, Section 4.2] and [15], the following corollary. Let $\Omega_H^{++}$ be the set of elements of $\Omega_H$ that are in the interior of the Weyl cochamber $C$.

**Corollary 3.9.** *Let $Y$ be a smooth toroidal compactification of $G/H$, let $\underline{\lambda} \in \mathrm{SPic}_0(Y)$ and assume the field to be of characteristic zero. Then*

(i) *For every $\mu \in \Lambda^+$ $V_\mu^*$ is an irreducible summand of $\Gamma(Y, \mathcal{L})$ if and only if $\Gamma(Y_S, \mathcal{L}|_{Y_S})$ has a section of $S_H$ weight equal to $\mu$;*

(ii) *For every line bundle $\mathcal{L}$ generated by global sections, the restriction map*

$$\Gamma(Y, \mathcal{L}) \to \Gamma(Y_S, \mathcal{L}|_{Y_S})$$

*is surjective;*

(iii) *$\mathcal{L}_{\underline{\lambda}}$ is an ample line bundle if and only if it is very ample;*

(iv) *$\mathcal{L}_{\underline{\lambda}}$ is an ample line bundle if and only if $\lambda_\tau \in \Omega_H^{++}$ and $\lambda_\tau < \lambda_{\tau'}$ on $\tau' \smallsetminus \tau$ for all faces $\tau$ and $\tau'$ of $F_Y$ of maximal dimension.*

**Remark 3.10.** We have limited our discussion to strictly spherical line bundle and to characteristic 0. Using Frobenius splitting methods it is easy to generalize the previous results as in [6]. However the stated result are enough for our purpose here.

## 4. The Quotient of a Symmetric Variety

Let $Y$ be an embedding of $G/H$ and let $K$ be a subgroup such that $H^\circ \subset K \subset H$. Any line bundle on $Y$ has a $G$ linearization, so in particular it has a $K$ linearization. Recall that if $\mathcal{L}$ is an ample line bundle on $Y$ a point $y$ on $Y$ is said to be $\mathcal{L}$ semistable (with respect to the action of $K$) if for some $n > 0$ there exists $f \in \Gamma(Y, \mathcal{L}^n)^K$ such that $f(y) \neq 0$. We denote by $Y^{ss}(\mathcal{L})$ the set of $\mathcal{L}$ semistable points, or in case $\mathcal{L}$ is chosen just semistable points. $Y^{ss}(\mathcal{L})$ is a possibly empty open subset of $Y$. By [13] Theorem 3.21 there exists a good quotient of the set of $\mathcal{L}$ semistable points which we shall denote by $K \mathbin{\backslash\!\backslash}_{\mathcal{L}} Y$.

In this section we are going to prove the following theorem.

**Theorem 4.1.** *Let $Y$ be an equivariant projective embedding of $G/H$ and let $\mathcal{L}$ be an ample and spherical line bundle on $Y$ then the inclusion $Y_S \subset Y$ induces an isomorphism of algebraic varieties between $\widetilde{W}\backslash Y_S$ and $K \mathbin{\backslash\!\backslash}_{\mathcal{L}} Y$.*

We will prove this theorem by computing the invariant sections. We will analyze first the case of the wonderful compactification and the case of the quotient of the open affine part $G/H$.

**4.1. Invariants and semiinvariants of the Cox ring of the wonderful compactification.** In this section we compute the $H$ invariants of the ring $R(X)$. We use the notations introduced in Section 3.1.

**Lemma 4.2.** *Let $\lambda \in \Pi$. Then $(\mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H = \mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)^H)$. In particular the dimension of the space of invariants $\Gamma(X, \mathcal{L}_\lambda)^H$ equals the cardinality of the set $K_\lambda := \{\mu \in \Omega^+ \colon \mu \leqslant_\sigma \lambda\}$ if $\lambda \in \Omega_H$ and is zero otherwise.*

*Proof.* In characteristic zero the equality
$$(\mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H = \mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)^H)$$
is an immediate consequence of the linear reductivity of $H$.

Also (in arbitrary characteristic) by equation (1) we have that
$$(\mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H = \bigoplus_{\mu\in\Pi^+,\, \mu\leqslant_\sigma\lambda} s^{\lambda-\mu}(V_\mu^*)^H.$$
By Vust criterion (Theorem 2.3) $(V_\mu^*)^H$ is one dimensional if $\mu \in \Omega_H^+$ and it is zero otherwise. So, since by Lemma 2.5 $\mathbb{Z}[\tilde{\Phi}] \subset \Omega_H$ we have that $(\mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H$ has dimension equal to $|K_\lambda|$ if $\lambda \in \Omega_H$ and it is zero otherwise.

In general $(\mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H \supset \mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)^H)$ so
$$\dim \Gamma(X, \mathcal{L}_\lambda)^H \leqslant \dim(\mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))^H.$$

On the other hand by Theorem 3.1 and Lemma 3.7 the variety $X$ and the spaces $\Gamma(X, \mathcal{L}_\lambda)$ are all defined over $\boldsymbol{A}$. Lemma 2.2 then clearly implies that in positive characteristic $\dim \Gamma(X, \mathcal{L}_\lambda)^H$ can only increase. This together with the previous inequality implies our claim. □

We compute now the ring $R(X)^H$. By Lemma 4.2, for each $\tilde{\alpha} \in \tilde{\Delta}$ we can choose $p_{\tilde{\alpha}} \in \Gamma(X, \mathcal{L}_{\tilde{\omega}_{\tilde{\alpha}}})$ an $H^\circ$ invariant section which does not vanish on $X_{\tilde{\Delta}}$. So, if $\lambda = \sum a_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} \in \Omega^+$, we can define

$$p^\lambda = \prod_{\tilde{\alpha} \in \tilde{\Delta}} p_{\tilde{\alpha}}^{a_{\tilde{\alpha}}}. \tag{4}$$

**Proposition 4.3.** *The set $\{s^\mu p^\lambda : \mu \in \Pi,\, \mu \geqslant_\sigma 0,\, \lambda \in \Omega_H^+\}$ is a $\Bbbk$ basis of $R(X)^H$. In particular the ring $R(X)^{H^\circ}$ is a polynomial ring in the variables $s_{\tilde{\alpha}}, p_{\tilde{\alpha}}$ with $\tilde{\alpha} \in \tilde{\Delta}$.*

*Proof.* Notice first that by Lemma 4.2, if $\lambda \in \Omega_H$, $\Gamma(X, \mathcal{L}_\lambda)^H = \Gamma(X, \mathcal{L}_\lambda)^{H^\circ}$ so it is enough to prove the claim in the case of $H^\circ$.

The image of $p_{\tilde{\alpha}}$ in the graded ring $\mathrm{Gr}_F(R(X))$ defines an $H^\circ$ invariant element $\bar{p}_{\tilde{\alpha}}$ of $V^*_{\tilde{\omega}_{\tilde{\alpha}}}$. So by the description of the $H^\circ$ invariants of $\mathrm{Gr}_F(R(X))$ the image of the elements $s^\mu p^\lambda$ in the graded ring $\mathrm{Gr}_F(R(X))$ is a $\Bbbk$ basis of the space of $H^\circ$ invariants. This implies that the elements $s^\mu p^\lambda$ are linearly independent.

By construction, the elements $s^\mu p^\lambda$ are $H^\circ$ invariants. So, again by Lemma 4.2 they are a $\Bbbk$ basis of $R(X)^{H^\circ}$. □

The computation of semi invariants is similar. If $V$ is a representation of $\bar{H}$, we denote by $V_{si}^{\bar{H}}$ the subspace spanned by the set of semi invariant vectors, i. e., vectors spanning lines fixed by $\bar{H}$.

By Theorem 2.4, we know that there are semi invariants which are not $H^\circ$ invariants only if there exists an exceptional simple root. Set $\Delta_e = \{\alpha \in \Delta_1 : \alpha$ is exceptional$\}$ and $\tilde{\Delta}_{ne} = \{\tilde{\alpha} \in \tilde{\Delta} : \alpha$ is not exceptional$\}$. By Theorem 2.4 the set $\{\omega_\alpha : \alpha \in \Delta_e\} \cup \{\tilde{\omega}_{\tilde{\alpha}} : \tilde{\alpha} \in \tilde{\Delta}_{ne}\}$ is a basis of $\Pi$. Let $\bar{q}_\alpha \in V^*_{\omega_\alpha}$ be a non-zero $\bar{H}$ semi invariant. $\bar{q}_\alpha$ is unique up to multiplication by a non-zero scalar. So the ring $\big(R_\Pi(X_{\tilde{\Delta}})\big)_{si}^{\bar{H}}$ of semi invariants is a polynomial ring in the generators $\bar{q}_\alpha$ with $\alpha \in \Delta_e$ and $\bar{p}_{\tilde{\alpha}}$ (the restriction of $p_{\tilde{\alpha}}$ to $X_{\tilde{\Delta}}$) with $\tilde{\alpha} \in \tilde{\Delta}_{ne}$. Using Corollary 2.6 and arguing as in Lemma 4.2 we deduce that there exists $q_\alpha \in \Gamma(X, \mathcal{L}_{\omega_\alpha})_{si}^{\bar{H}}$ such that its restriction to $X_{\tilde{\Delta}}$ is equal to $\bar{q}_\alpha$. If $\lambda = \sum_{\alpha \in \Delta_e} c_\alpha \omega_\alpha + \sum_{\tilde{\alpha} \in \tilde{\Delta}_{ne}} c_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} \in \Pi^+$, we define $q^\lambda = \prod_{\alpha \in \Delta_e} q_\alpha^{c_\alpha} \cdot \prod_{\tilde{\alpha} \in \tilde{\Delta}_{ne}} p_{\tilde{\alpha}}^{c_{\tilde{\alpha}}}$. The arguments given in the case of invariants can now be easily adapted implying

**Proposition 4.4.** *Let $\lambda \in \Pi$. Then $(\mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)))_{si}^{\bar{H}} = \mathrm{Gr}_F(\Gamma(X, \mathcal{L}_\lambda)_{si}^{\bar{H}})$. Moreover the set $\{s^\mu q^\lambda : \mu \in \Pi,\, \mu \geqslant_\sigma 0,\, \lambda \in \Pi^+\}$ is a $\Bbbk$ basis of $R(X)_{si}^{\bar{H}}$. In particular the ring $R(X)_{si}^{\bar{H}}$ is a polynomial ring in the variables $s_{\tilde{\alpha}}$ with $\tilde{\alpha} \in \tilde{\Delta}$, $p_{\tilde{\alpha}}$ with $\tilde{\alpha} \in \tilde{\Delta}_{ne}$ and $q_\alpha$ with $\alpha \in \Delta_e$.*

**4.2. Filtration of the coordinate ring of $G/H$ and Richardson theorem.** We want now to use the wonderful variety $X$ to define a filtration of the coordinate ring of $G/H$. In the case in which $H$ is the diagonal subgroup in $G = H \times H$, these ideas already appear in [18]. This will be used to describe the $H$ invariants of this ring. This description of the invariants has already been given by Richardson [14] but a proof in our setting seems natural.

First we make explicit the relation between the coordinate ring of $G/H$ and the ring $R_{\Omega_H}(X)$.

For each $\tilde{\alpha} \in \tilde{\Delta}$ we choose a trivialization $\varphi_{\tilde{\omega}_{\tilde{\alpha}}} \colon \pi^*(\mathcal{L}_{\tilde{\omega}_{\tilde{\alpha}}}) \to \mathcal{O}_{G/H^\circ}$. Given $\lambda = \sum_{\tilde{\alpha}\in\tilde{\Delta}} c_{\tilde{\alpha}}\tilde{\omega}_{\tilde{\alpha}} \in \Omega$ we obtain the trivialization of $\pi^*(\mathcal{L}_\lambda)$ given by $\bigotimes_{\tilde{\alpha}\in\tilde{\Delta}} \varphi_{\tilde{\omega}_{\tilde{\alpha}}}^{\otimes c_{\tilde{\alpha}}}$. With these choices the pull back of sections defines a ring homomorphism:

$$\pi_H^* \colon R_{\Omega_H}(X) \to \Bbbk[G/H].$$

Notice also that since $s_{\tilde{\alpha}}$ is $G$ invariant the functions $\pi_H^*(s_{\tilde{\alpha}})$ are constant and we can normalize them to be equal to 1.

For each $\lambda \in \Omega_H$ we consider the $G$ submodule

$$\mathbb{F}_\lambda := \pi_H^*(\Gamma(X,\, \mathcal{L}_\lambda))$$

of $\Bbbk[G/H]$. Notice that since the $s_\alpha$ are all equal to 1, we clearly have that if $\mu <_\sigma \lambda$, $\mathbb{F}_\mu \subset \mathbb{F}_\lambda$. Also, since the image of $\pi_H$ is dense in $X$ we have that $\pi_H^*$ restricted to $\Gamma(X,\, \mathcal{L}_\lambda)$ is an isomorphism onto $\mathbb{F}_\lambda$. Furthermore, if we set $\mathbb{F}'_\lambda = \sum_{\mu <_\sigma \lambda} F_\mu$, we have $\mathbb{F}_\lambda/\mathbb{F}'_\lambda \simeq V_\lambda^*$.

**Proposition 4.5.** *The map $\pi_H^*$ induces an isomorphism of rings*

$$\varphi \colon \frac{R_{\Omega_H}(X)}{(s_{\tilde{\alpha}} - 1 \colon \tilde{\alpha} \in \tilde{\Delta})} \to \Bbbk[G/H].$$

*Proof.* The mapping $\varphi$ is clearly well defined and its surjectivity follows immediately from Proposition 2.8.

Let us now show that $\varphi$ is injective. As above set $\Xi_H = \Omega_H/\Omega_{\bar{H}}$ and for all cosets $\xi \in \Xi_H$ define $R_\xi = \bigoplus_{\lambda\in\xi} \Gamma(X,\, \mathcal{L}_\lambda)$ so that $R_{\Omega_H}(X) = \bigoplus_{\xi\in\Xi_H} R_\xi$ is a $\Xi_H$ grading of the ring $R_{\Omega_H}(X)$.

On the other hand by Proposition 2.8, the coordinate ring $\Bbbk[G/H]$ decomposes as the direct sum $\bigoplus_{\xi\in\Xi_H} \Gamma(G/\bar{H},\, \mathcal{L}_\xi)$ and the restriction of $\pi_H^*$ decomposes as the direct sum $\bigoplus_{\xi\in\Xi_H} \jmath_\xi^*$, where $\jmath_\xi^* \colon R_\xi \to \Gamma(G/\bar{H},\, \mathcal{L}_\xi)$ is induced by the inclusion $\jmath_X$ of $G/\bar{H}$ in $X$.

Also, since the elements $\{s_{\tilde{\alpha}} - 1 \colon \tilde{\alpha} \in \tilde{\Delta}\}$ lie in $R_0$ the ideal $I$ that they generate decomposes as the direct sum $I = \bigoplus_{\xi\in\Xi_H} I_\xi$ with $I_\xi = I \cap R_\xi$, for each $\xi \in \Xi_H$. Thus $\jmath_\xi^*$ induces a map

$$\varphi_\xi \colon R_\xi/I_\xi \to \Gamma(G/\bar{H},\, \mathcal{L}_\xi)$$

and it is enough to see that $\varphi_\xi$ is injective for each $\xi \in \Xi_q$.

Fix $\xi \in \Xi_H$. Let $g = \sum_{\lambda\in A} g_\lambda \in R_\xi$ with $g_\lambda \in \Gamma(X,\, \mathcal{L}_\lambda)$ and $A$ a finite subset of the coset $\xi$. Assume $\pi_H^*(g) = 0$. By assumption there exists $\mu \in \xi$ such that $\mu \geqslant_\sigma \lambda$ for all $\lambda \in A$. Set $g' = \sum_{\lambda\in A} s^{\mu-\lambda} g_\lambda$ and notice that $g' \equiv g \mod I_\xi$ and that $g' \in \Gamma(X,\, \mathcal{L}_\mu)$. We have $\pi_H^*(g') = \pi_H^*(g) = 0$ and since $\pi_H^*$ restricted to $\Gamma(X,\, \mathcal{L}_\mu)$ is injective, $g' = 0$ and $g \in I_\xi$ as desired. □

**Corollary 4.6.** *The $G$ submodules $\mathbb{F}_\lambda$, $\lambda \in \Omega_H$, induce a good (increasing) filtration of the coordinate ring $\Bbbk[G/H]$.*

We are now going to use this filtration to study the ring of invariants $\Bbbk[G/H]^K$. We first need a well known lemma.

**Lemma 4.7.** *Fix a dominant weight $\lambda \in \Omega^+$.*

(i) *Let $\phi \in V_\lambda^*$ be a nonzero $H^\circ$ invariant and let us consider the decomposition of $V_\lambda^*$ with respect to the action of $T$. Then the lowest weight component of $\phi$ is not zero.*

(ii) *Let $p^\lambda \in \Gamma(X, \mathcal{L}_\lambda)$ denote the $H^\circ$ invariant defined in formula (4) and consider the decomposition of $p^\lambda \in \Gamma(X, \mathcal{L}_\lambda)$ with respect to the action of $T$. Then the lowest weight component of $p^\lambda$ is not zero.*

*Proof.* In $V_\lambda^*$ there is a non-zero vector $v$ fixed by the maximal unipotent subgroup $U^-$ opposite to $B$. This vector is unique up to a non-zero scalar and has weight $-\lambda$ which is the lowest weight of $V_\lambda^*$. Write $\phi = w + av$ with $w$ lying in the unique $T$ stable complement $V'$ of the one dimensional space spanned by $v$ and $a \in \Bbbk$. $V'$ is $B$ stable. We need to prove $a \neq 0$.

Consider the $G$ submodule $W$ generated by $\phi$. Since $W$ must contain a non-zero vector fixed by $U^-$, it has to contain $v$.

On the other hand $B \cdot H^\circ$ is dense in $G$ so the subspace $W$ is equal to the space spanned by the vectors $b \cdot \phi$ with $b \in B$. If $a = 0$, then $W$ would be contained in $V'$ giving a contradiction. This proves (i).

To see (ii) it suffices to consider the image $\bar{p}^\lambda$ in $\Gamma(X, \mathcal{L}_\lambda)/F'_{\lambda,\lambda} \simeq V_\lambda^*$ which is non-zero by the very definition of $p^\lambda$. □

We can now prove Richardson Theorem (see [14, Corollary 11.5]). Notice that, since $N_{H^\circ}(S) \subset H^\circ \subset K$ the inclusion of $S_H$ in $G/H$ induces a map from $\widetilde{W}\backslash S_H$ to $K \backslash\!\backslash G/H$.

**Theorem 4.8.** *Let $H^\circ \subset K \subset H$ be a subgroup of $H$. Then the inclusion $S_H \subset G/H$ induces an isomorphism $\widetilde{W}\backslash S_H \simeq K \backslash\!\backslash G/H$.*

*Proof.* By the definition of $\widetilde{W}$, the restriction of functions from $G/H$ to $S_H$ induces a homomorphism

$$d\colon \Bbbk[G/H]^K \to \Bbbk[S_H]^{\widetilde{W}}.$$

We claim that $d$ is an isomorphism.

To see this we first make some remarks on the $K$ invariants of $\Bbbk[G/H]$. For $\lambda \in \Omega_H^+$ let $f^\lambda := \pi_H^*(p^\lambda)$. Arguing as in Proposition 4.3 it is easy to see that the elements $f^\lambda$ with $\lambda \in \Omega_H^+$ are a basis of $\Bbbk[G/H]^K$ as a vector space. In particular for each $\lambda \in \Omega_H^+$, $\mathbb{F}_\lambda^K/\mathbb{F}'^K_\lambda$ is one dimensional and spanned by the class of $f^\lambda$ (notice that $\Omega_H \subset \Omega_K$).

The computation of the $\widetilde{W}$ invariants of the ring $\Bbbk[S_H]$ is also very simple. Let $\Bbbk[S_H] = \bigoplus_{\lambda \in \Lambda_{S_H}} \Bbbk\varphi_\lambda$, where $\varphi_\lambda$ is a function of weight $\lambda$. We know by Theorem 2.3 and Lemma 2.5 that the restriction $r$ of character from $T$ to $S$ induces an isomorphism between $\Lambda_{S_H}$ and $\Omega_H$ so we identify the two lattices. Also a weight $\lambda \in \Omega_H$ is dominant with respect to $\tilde{\Delta}$ if and only if it is dominant with respect to

$\Delta$. If $\lambda \in \Lambda^+_{S_H}$, we set

$$\psi_\lambda = \sum_{\psi \in \widetilde{W}\cdot\varphi_{-\lambda}} \psi.$$

The elements $\psi_\lambda$ with $\lambda \in \Omega^+_H$ are clearly a basis of $\Bbbk[S_H]^{\widetilde{W}}$.

Given $\lambda \in \Omega^+_H$, let $U_\lambda$ denote the span of elements $\psi_\mu$ with $\mu \in \Omega^+_H$ and $\mu \leqslant_\sigma \lambda$.

Notice that $f^\lambda_S := d(f^\lambda)$ lies in $U_\lambda$. Indeed $f^\lambda_S$ is a $\widetilde{W}$ invariant and its weights are in a subset of the weights appearing in $V^*_\lambda$. Thus for each $\lambda \in \Omega^+_H$, $d(\mathbb{F}_\lambda)^K \subseteq U_\lambda$. We claim that $d$ maps isomorphically $\mathbb{F}^K_\lambda$ onto $U_\lambda$. This will imply our claim.

By an easy induction we need to show that $f^\lambda_S \notin \sum_{\mu <_\sigma \lambda,\, \mu\in\Omega^+_H} U_\mu$. Using Lemma 4.7 it suffices to prove that the restriction to $S_H$ of a lowest weight vector $h$ in $\mathbb{F}_\lambda$ is non-zero. The closure of $S_H$ in $X$ contains the unique point of the closed orbit $X_{\tilde{\Delta}}$ fixed by $B$. $h$ does not vanish at this point. Since $h$ is non-zero at a point in the closure of $S_H$, it cannot vanish on $S_H$ proving that $f^\lambda_S$ does not lie in $\sum_{\mu <_\sigma \lambda,\, \mu\in\Omega^+_H} U_\mu$. □

**Remark 4.9.** Notice that in particular $K \mathbin{\backslash\!\backslash} G/H$ does not depend on the choice of the subgroup $K$ between $H^\circ$ and $H$. However it is not true in general that the $K$ orbits in $G/H$ are the same of the $H$ orbits in $G/K$. To see this it is enough to take $G = \mathrm{SL}(2, \mathbb{C})$ And $\sigma$ the conjugation by $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. Then $H^\circ$ is the diagonal torus and it is easy to check that $H^\circ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \bar{H} \neq \bar{H} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \bar{H}$.

To complete our picture we show, with a different proof, another result of Richardson which tells us the orbits of the elements in $S_H$ are precisely the closed orbits in $G/H$.

**Proposition 4.10.** *Let $H^\circ \subset H$, $K \subset \bar{H}$. Let $s \in S_H$ then $Ks$ is closed in $G/H$.*

*Proof.* Since $H^\circ$ has finite index both in $H$ and in $K$, it is enough to study the case $H = K = H^\circ$.

Fix $V$ to be a finite dimensional faithful representation of $G$. Consider the map $\chi\colon G/H^\circ \to G \subset GL(V)$ given by $\chi(gH^\circ) = g\sigma(g)^{-1}$. By [17] Theorem 5.4.4 this is a closed immersion. Notice that $\chi(h\cdot x) = h\chi(x)h^{-1}$ for all $h \in H^\circ$ and $x \in G/H^\circ$.

For $s \in S_H$ set $x = \chi(s) = s^2$. The element $s$ is semisimple in $GL(V)$. We want to prove that the orbit $\{hxh^{-1}\colon h \in H^\circ\}$ is closed in $GL(V)$.

Let $p(t)$ be the minimal polynomial of $x$. Since $x$ is semisimple, $p(t)$ does not have multiple roots. For all $\lambda \in \Bbbk^*$ and $y \in G$ set $V_\lambda(y) = \{v \in \mathfrak{g}\colon \mathrm{Ad}_y(v) = \lambda v\}$. Notice that for $y \in T$

$$V_\lambda(y) = \begin{cases} \bigoplus_{\alpha\in\Phi\colon \alpha(y)=\lambda} \mathfrak{g}_\alpha & \text{if } \lambda \neq 1, \\ \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi\colon \alpha(y)=1} \mathfrak{g}_\alpha & \text{if } \lambda = 1. \end{cases}$$

Given $\lambda \in \Bbbk^*$ and $\mu = \frac{1}{2}(\lambda + \lambda^{-1})$ set

$$W_\mu(y) = \begin{cases} V_\lambda(y) \oplus V_{\lambda^{-1}}(y) & \text{if } \lambda \neq \pm 1, \\ V_\lambda(y) & \text{if } \lambda = \pm 1. \end{cases}$$

Notice that we have

$$\mathfrak{h} = \bigoplus_{\mu} (\mathfrak{h} \cap W_\mu(x)) \tag{5}$$

since $\mathfrak{h} = \mathfrak{t}^\sigma \oplus \bigoplus_{\alpha\in\Phi_0} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\Phi_1^+} \mathbb{k}(x_\alpha + \sigma(x_\alpha))$ and $\sigma(\alpha)(x) = \alpha(x)^{-1}$ so that $x_\alpha + \sigma(x_\alpha) \in W_{\alpha(x)+\alpha(x)^{-1}}$. Let $n_\mu = \dim(\mathfrak{h} \cap W_\mu(x))$. We define the closed subset $M \subset G$ as follows. An element $y \in G$ lies in $M$ if

(i) $p(y) = 0$;
(ii) the characteristic polynomial of $\mathrm{Ad}_y$ is equal to the characteristic polynomial of $\mathrm{Ad}_x$;
(iii) $\dim(\mathfrak{h} \cap W_\mu(y)) \geqslant n_\mu$ for all $\mu$.

By condition (iii) and relation (5) it follows that if $y \in M$, $\dim(\mathfrak{h} \cap W_\mu(y)) = n_\mu$ for all $\mu$. In particular $\dim \mathfrak{h} \cap V_1(y) = \dim \mathfrak{h} \cap V_1(x)$ so that $\dim \mathfrak{h} \cap Z_{\mathfrak{g}}(y) = \dim \mathfrak{h} \cap Z_{\mathfrak{g}}(x)$.

Now by condition (i) if $y \in M$, $y$ is semisimple and so by [17, Theorem 5.4.4] we have that $Z_{\mathfrak{g}}(y)$ is the Lie algebra of $Z_G(y)$. It follows that $\dim \mathfrak{h} \cap Z_{\mathfrak{g}}(y) = \dim H \cap Z_G(y) = n_1$ and finally $\dim H^\circ y = \dim H - n_1$ so that every $H^\circ$ orbit in $M$ has the same dimension.

In particular every $H^\circ$ orbit in $M$ is closed. Since $x \in M$, the proposition follows. □

**Remark 4.11.** Notice that our proof works also under the slightly more general assumption that $G$ is reductive. This will be useful later on.

**Remark 4.12.** We notice that if $H^\circ \subset K \subset H$ and $s \in S$ then $Ks = Hs$ in $G/H$. Indeed by Remark 4.9 we have that $K \mathbin{\backslash\!\!\backslash} G/H = H \mathbin{\backslash\!\!\backslash} G/H$ so the natural map $Kx \mapsto Hx$ from the set of $K$ orbits into the set of $H$ orbits is a bijection at the level of closed orbits. At this point everything follows from the fact that $Hs$ is a union of closed $K$ orbits.

**4.3. The quotient of a smooth projective toroidal compactification.** Let us now assume that $\mathcal{L}$ is a line bundle generated by global sections but not necessarily ample on a given smooth, projective toroidal compactification $Y$ of $G/H$. This is going to be useful in order to prove Theorem 4.1 in its full generality.

Since $\mathcal{L}$ is spherical, $\pi_Y^*(\mathcal{L})$ is trivial. Notice now that the $G$ linearization of $\mathcal{L}$ restricts to a $N_{H^\circ}(S)$ linearization of $\mathcal{L}|_{Y_S}$ and that $Z_{H^\circ}(S)$ acts trivially on $\mathcal{L}|_{Y_S}$. To see this first notice that $Z_{H^\circ}(S)$ acts trivially on the fiber of $\mathcal{L}$ over $y_0$. Indeed $\pi_Y^*(\mathcal{L})$ is trivial, hence $H^\circ$ acts trivially on the fiber of $\mathcal{L}$ over $y_0$ and $Z_{H^\circ}(S) \subset H^\circ$. Now, $Z_{H^\circ}(S)$ commutes with $S$ and $S \cdot y_0$ is dense in $Y_S$ so necessarily $Z_{H^\circ}(S)$ acts trivially on $\mathcal{L}|_{Y_S}$.

Notice that $Y^{ss}(\mathcal{L}) = Y^{ss}(\mathcal{L}^n)$ for any $n > 0$, so $K \mathbin{\backslash\!\!\backslash}_{\mathcal{L}^n} Y \simeq K \mathbin{\backslash\!\!\backslash}_{\mathcal{L}} Y$. Since $\Omega/\Omega_K$ is finite, up to taking a power of $\mathcal{L}$ we can always assume that $\pi_{Y,K}^*(\mathcal{L}) \simeq \mathcal{O}_{G/K}$, where $\pi_{Y,K} \colon G/K \to Y$ is defined by $\pi_{Y,K}(gK) = g \cdot y_0$. We call these line bundles $K$ spherical. We have

**Theorem 4.13.** *Let $\mathcal{L}$ be a $K$ spherical line bundle on $Y$. Then*

$$\Gamma(Y, \mathcal{L})^K \simeq \Gamma(Y_S, \mathcal{L}|_{Y_S})^{\widetilde{W}}.$$

We need first a well known general fact on flat schemes over $\boldsymbol{A}$ whose proof we give for completeness.

**Lemma 4.14.** *Let $U$ be a projective flat scheme over $\boldsymbol{A}$ such that there exists an open affine covering $\{U_1, \ldots, U_n\}$ of $U$ with $U_i = \operatorname{Spec} R_i$ and $U_i \cap U_j = \operatorname{Spec} R_{ij}$, where $R_i$ and $R_{ij}$ are free $\boldsymbol{A}$-modules. Let $\mathcal{L}$ be a locally free sheaf over $U$. For each ring extension $\boldsymbol{A} \to B$ let $U_B = U \times_{\operatorname{Spec}(\boldsymbol{A})} \operatorname{Spec}(B)$ and $\mathcal{L}_B$ the pull back of $\mathcal{L}$ to $U_B$. Then*:

(i) *$\Gamma(U, \mathcal{L})$ is a finitely generated free $\boldsymbol{A}$-module*;
(ii) *the map $B \otimes_{\boldsymbol{A}} \Gamma(U, \mathcal{L}) \to \Gamma(U_B, \mathcal{L}_B)$ is injective*;
(iii) *moreover if $B$ is a field of characteristic $0$ then we have an isomorphism $B \otimes_{\boldsymbol{A}} \Gamma(U, \mathcal{L}) \simeq \Gamma(U_B, \mathcal{L}_B)$.*

*Proof.* The fact that $\Gamma(U, \mathcal{L})$ is finitely generated is the content of Theorem III.5.2a) in [8].

We can refine the covering $U_i$ in such a way it has the same properties and moreover it is such that $\mathcal{L}|_{U_i}$ is defined by a free $R_i$-module of rank 1. We have the exact sequence:

$$0 \to \Gamma(U, \mathcal{L}) \xrightarrow{r_1} \prod \Gamma(U_i, \mathcal{L}) \xrightarrow{r_2} \prod \Gamma(U_i \cap U_j, \mathcal{L}),$$

where $r_1$ and $r_2$ are given by restriction of sections. In particular $\Gamma(U, \mathcal{L})$ is a submodule of $\prod \Gamma(U_i, \mathcal{L})$ which is a free $\boldsymbol{A}$-module, hence, since $\boldsymbol{A}$ is a PID, $\Gamma(U, \mathcal{L})$ is a free $\boldsymbol{A}$-module.

Let $M$ denote the image of $r_2$. $M$ is a submodule of $\prod \Gamma(U_i \cap U_j, \mathcal{L})$, so also $M$ is a free $\boldsymbol{A}$-module. Write $r_2 = \imath \circ r'$ with $r' \colon \prod \Gamma(U_i, \mathcal{L}) \to M$ and $\imath \colon M \to \prod \Gamma(U_i \cap U_j, \mathcal{L})$. For any $\boldsymbol{A}$ algebra $B$ we can tensor by $B$ and, since by definition $\Gamma(U_i \times_{\operatorname{Spec}(\boldsymbol{A})} \operatorname{Spec}(B), \mathcal{L}_B) = B \otimes_{\boldsymbol{A}} \Gamma(U_i, \mathcal{L})$, we get the exact sequence

$$0 \to \Gamma(U, \mathcal{L}) \otimes_{\boldsymbol{A}} B \xrightarrow{r_1 \otimes \mathrm{id}_B} \prod \Gamma(U_i \times_{\operatorname{Spec}(\boldsymbol{A})} \operatorname{Spec}(B), \mathcal{L}_B) \xrightarrow{r' \otimes \mathrm{id}_B} M \otimes_{\boldsymbol{A}} B \to 0 \tag{6}$$

from which deduce that the map from $\Gamma(U, \mathcal{L}) \otimes_{\boldsymbol{A}} B$ to $\Gamma(U_B, \mathcal{L}_B)$ is injective. This proves (i) and (ii).

Now assume that $B$ is a field of characteristic zero. $\imath$ is an inclusion between free $\boldsymbol{A}$-modules, so since $B$ has characteristic zero we have that $\imath \otimes \mathrm{id}_B \colon M \otimes_{\boldsymbol{A}} B \to \prod \Gamma(U_i \cap U_j, \mathcal{L}) \otimes_{\boldsymbol{A}} B = \prod \Gamma((U_i \cap U_j) \times_{\operatorname{Spec}(\boldsymbol{A})} \operatorname{Spec}(B), \mathcal{L}_B)$ is injective. It follows that $M \otimes_{\boldsymbol{A}} B$ is the image of $r_2 \otimes \mathrm{id}_B$ and by the exact sequence (6) we get that $\Gamma(U, \mathcal{L}) \otimes_{\boldsymbol{A}} B$ is equal to the space of sections of $\mathcal{L}_B$ on $U_B$. □

Notice that by Remark 3.6(3) we can apply this lemma to a toroidal compactification. We obtain

**Lemma 4.15.** *Let $Y$ be a smooth toroidal compactification of $G/H$ and let $\mathcal{L}$ be a $K$ spherical line bundle on $Y$ generated by global sections. Then the restriction of sections from $Y$ to $Y_S$ induces an isomorphism $\Gamma(X, \mathcal{L})^K \simeq \Gamma(Y_S, \mathcal{L}|_{Y_S})^{\widetilde{W}}$.*

*Proof.* We prove first that this map is injective. Consider the pull back $\pi^*_{Y,K}(\mathcal{L})$. By hypothesis this is isomorphic to the trivial line bundle on $G/K$. So a trivialization of it induces inclusions $\Gamma(Y,\, \mathcal{L}) \subset \Bbbk[G/K]$ and $\Gamma(Y_S,\, \mathcal{L}|_{Y_S}) \subset \Bbbk[S_K]$. We get a commutative diagram

$$\begin{array}{ccc} \Gamma(Y,\, \mathcal{L})^K & \longrightarrow & \Bbbk[G/K]^K \\ \downarrow & & \downarrow \\ \Gamma(Y_S,\, \mathcal{L}|_{Y_S})^{\widetilde{W}} & \longrightarrow & \Bbbk[S_K]^{\widetilde{W}}, \end{array}$$

where the horizontal maps are the induced by the pull back of sections and vertical maps are given by restriction of sections. Since the inclusions $G/H \subset Y$ and $S_H \subset Y_S$ are open, the two horizontal maps are injective and by Theorem 4.8 also the right vertical map is injective. It follows that also the vertical map on the left is injective.

In order to prove surjectivity it is enough to prove that we have enough invariants. First we prove this result in characteristic zero. Let $U$ be a $G$ submodule of $\mathbb{C}[G/K]$ and $U_S$ be its image in $\mathbb{C}[S_K]$. Observe that by Theorem 4.8 we have an isomorphism between $U^K$ and $U_S^{\widetilde{W}}$.

Set $U$ equal to the image of $\Gamma(Y,\, \mathcal{L})$ in $\Bbbk[G/K]$. By Corollary 3.9(ii), the restriction map $\Gamma(Y,\, \mathcal{L}) \to \Gamma(Y_S,\, \mathcal{L}|_{Y_S})$ is surjective for any spherical line bundle generated by global sections. Thus $U_S$ equals the image of $\Gamma(Y_S,\, \mathcal{L}|_{Y_S})$ in $\Bbbk[S_K]$ and this implies our claim.

Assume now that the base field $\Bbbk$ is of arbitrary characteristic. The description of $\Gamma(Y_S,\, \mathcal{L}|_{Y_S})$ as an $S$-module does not depend on the characteristic. It follows that there is a basis of $\Gamma(Y_S,\, \mathcal{L}|_{Y_S})$ on which $\widetilde{W}$ acts by permutations. Thus also the description of $\Gamma(Y_S,\, \mathcal{L}|_{Y_S})^{\widetilde{W}}$ and hence its dimension $d$ does not depend on the characteristic. On the other hand by Lemma 4.14(ii) we have that $d \leqslant \dim \Gamma(Y_{\Bbbk},\, \mathcal{L}_{\Bbbk})^K$. Since $\Gamma(Y_{\Bbbk},\, \mathcal{L}_{\Bbbk})^K$ injects into $\Gamma(Y_S,\, \mathcal{L}|_{Y_S})^{\widetilde{W}}$, everything follows. □

If we now set

$$A_{\mathcal{L}} := \bigoplus_n \Gamma(Y,\, \mathcal{L}^n) \quad \text{and} \quad B_{\mathcal{L}} := \bigoplus_n \Gamma(Y_S,\, \mathcal{L}^n|_{Y_S}),$$

we deduce from the above lemma that $\operatorname{Proj}(A_{\mathcal{L}}^K) = \operatorname{Proj}(B_{\mathcal{L}}^{\widetilde{W}})$ for any spherical line bundle $\mathcal{L}$ generated by global sections on a smooth toroidal projective embedding $Y$ of $G/H$. In particular Theorem 4.1 follows for such a compactification.

**4.4. Proof of Theorem 4.1.** We now prove Theorem 4.1 for any projective embedding $Y$ of $G/H$. Consider an equivariant resolution $\tilde{Y}$ of the closure of the image of $G/H$ in $Y \times X$. By construction this is a toroidal compactification and we have a $G$-equivariant birational projective morphism $\phi\colon \tilde{Y} \to Y$. Clearly $\phi(\tilde{Y}_S) = Y_S$.

As already noticed at the beginning of Section 4.3 we can assume $\mathcal{L}$ to be $K$ spherical. Let $\mathcal{M} = \phi^*(\mathcal{L})$ and notice that this is a $K$ spherical line bundle on $\tilde{Y}$ generated by global sections. Notice also that since $Y$ is normal we have $\Gamma(\tilde{Y},\, \mathcal{M}) = \Gamma(Y,\, \mathcal{L})$ and $A_{\mathcal{M}} = A_{\mathcal{L}}$. We have the following commutative diagram, where the horizontal map are given by pull back of sections, and the vertical maps are given by the restriction of sections:

$$\begin{array}{ccc} \Gamma(Y,\,\mathcal{L})^K & \xrightarrow{\ \simeq\ } & \Gamma(\tilde{Y},\,\mathcal{M})^K \\ \downarrow & & \downarrow \\ \Gamma(Y_S,\,\mathcal{L}|_{Y_S})^{\widetilde{W}} & \longrightarrow & \Gamma(\tilde{Y}_S,\,\mathcal{M}|_{\tilde{Y}_S})^{\widetilde{W}} \end{array}$$

Now the vertical map on the right is an isomorphism by the result obtained for a smooth toroidal compactification and the bottom map is injective, since $\tilde{Y}_S \to Y_S$ is surjective. So also the vertical map on the left is an isomorphism. So $B_{\mathcal{L}}^{\widetilde{W}} \simeq A_{\mathcal{L}}^K$ and

$$K \mathbin{\backslash\!\backslash}_{\mathcal{L}} Y = \operatorname{Proj}(A_{\mathcal{L}}^K) \simeq \operatorname{Proj}(B_{\mathcal{L}}^{\widetilde{W}}) = \widetilde{W} \backslash Y_S$$

as claimed. □

## 5. Semistable Points

In this section we want to give a more geometric description of the set of semistable points. We analyze first the case of flag varieties.

**5.1. Divisors of invariants in flag varieties.** We give first some definitions. If $I \subset \tilde{\Delta}$, we set $\Delta_I = \Delta_0 \cup \{\alpha \in \Delta_1 \colon \tilde{\alpha} \in I\}$ and define $\Phi_I$ to be the subroot system of $\Phi$ spanned by $\Delta_I$. We let $P_I$ denote the corresponding parabolic subgroup of $G$ and $\Lambda_I = \Lambda_{P_I}$ the set of characters of $P_I$. We also set $\Pi_I = \Pi \cap \Lambda_I$, $\Omega_{I,H} = \Omega_H \cap \Lambda_I$ and $\Omega_I = \Omega_{I,H^\circ}$. We have

$$\Omega_I = \bigoplus_{\tilde{\alpha} \in \tilde{\Delta} \smallsetminus I} \mathbb{Z}\, \tilde{\omega}_{\tilde{\alpha}} \qquad \Pi_I = \Omega_I + \bigoplus_{\alpha \in \Delta_e \smallsetminus \Delta_I} \mathbb{Z}\, \omega_\alpha.$$

Let us describe the set of invariant and semiinvariant sections with respect to the action of $H$ of a line bundle $\mathcal{L}_\lambda$ on $G/P_I$. Since if $\lambda \in \Lambda_I^+$, we have that $\Gamma(G/P_I,\,\mathcal{L}_\lambda) \simeq V_\lambda^*$ (and is zero if $\lambda$ is not dominant). In this case we can apply directly the result of Vust without introducing any further filtration.

If $\tilde{\alpha} \in \tilde{\Delta} \smallsetminus I$, let $\bar{p}_{\tilde{\alpha}}$ be a non-zero section of $\mathcal{L}_{\tilde{\omega}_{\tilde{\alpha}}}$ on $G/P_I$ invariant under the action of $H^\circ$. Similarly if $\alpha \in \Delta_e \smallsetminus I$ let $\bar{q}_\alpha$ be a non-zero semiinvariant section of $\mathcal{L}_{\omega_\alpha}$ on $G/P_I$. If $\lambda = \sum a_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}}$ is dominant so that $a_{\tilde{\alpha}} \geqslant 0$ for all $\tilde{\alpha}$, we define $\bar{p}^\lambda = \prod_{\tilde{\alpha}} \bar{p}_{\tilde{\alpha}}^{a_{\tilde{\alpha}}}$ and similarly if $\lambda = \sum_{\alpha \in \Delta_e} c_\alpha \omega_\alpha + \sum_{\tilde{\alpha} \in \tilde{\Delta}_{ne}} c_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}}$ is dominant in $\Pi_I$ we define $\bar{q}^\lambda = \prod_{\alpha \in \Delta_e} \bar{q}_\alpha^{c_\alpha} \cdot \prod_{\tilde{\alpha} \in \tilde{\Delta}_{ne}} \bar{p}_{\tilde{\alpha}}^{c_{\tilde{\alpha}}}$.

We notice that up to a non-zero scalar, $\bar{p}^\lambda$ is the unique $H^\circ$ invariant section of $\mathcal{L}_\lambda$ and that it is $H$ invariant if and only $\lambda \in \Omega_H$. Similarly $\bar{q}^\lambda$ is the unique $\bar{H}$ semiinvariant section of $\mathcal{L}_\lambda$.

In particular if we set $R(G/P_I) = \bigoplus_{\lambda \in \Lambda_I} \Gamma(G/P_I,\,\mathcal{L}_\lambda)$, we have that the ring of invariants $R(G/P_I)^{H^\circ}$ is a polynomial ring in $\bar{p}_{\tilde{\alpha}}$ for $\tilde{\alpha} \notin I$ and the ring of semiinvariants $R(G/P_I)_{si}^{\bar{H}}$ is a polynomial ring in $\bar{p}_{\tilde{\alpha}}, \bar{q}_\beta$ for $\tilde{\alpha} \in \tilde{\Delta}_{ne} \smallsetminus I$ and $\beta \in \Delta_e \smallsetminus \Delta_I$.

If $\lambda = \sum c_{\tilde{\alpha}} \tilde{\omega}_{\tilde{\alpha}} \in \Omega$, we define the support of $\lambda$ as $\operatorname{supp}_\Omega \lambda = \{\tilde{\alpha} \colon c_{\tilde{\alpha}} \neq 0\}$. Also if $J \subset \tilde{\Delta} \smallsetminus I$ we define $\bar{p}_J = \prod_{\tilde{\alpha} \in J} \bar{p}_{\tilde{\alpha}}$.

**Proposition 5.1.** *Let $J \subset \tilde{\Delta} \smallsetminus I$ then the equation $\bar{p}_J = 0$ is reduced.*

*Furthermore, the divisor of the section $\bar{p}_{I^c}$ is the complement of the unique open $H^\circ$ orbit in $G/P_I$, and we have $H^\circ P_I = HP_I = \bar{H}P_I$.*

*Proof.* We start with the first assertion. Let $D_J$ denote the divisor of $\bar{p}_J = 0$ with reduced structure. Take $\lambda \in \Lambda_I^+$ with the property that $\mathcal{L}_\lambda \simeq \mathcal{O}(D_J)$ and let $\varphi \in \Gamma(G/P_I, \mathcal{L}_\lambda)$ such that $\operatorname{div}\varphi = D_J$. We claim that $\lambda = \tilde{\omega}_J := \sum_{\tilde{\alpha}\in J} \tilde{\omega}_{\tilde{\alpha}}$.

Notice that by definition $\varphi$ divides $\bar{p}_J$ and for big enough $n$ the section $\bar{p}_J$ divides $\varphi^n$. So $\tilde{\omega}_J = \lambda + \mu$ and $n\lambda = \tilde{\omega}_J + \nu$ with $\mu$ and $\nu$ dominant. Moreover $D_J$ is an $\bar{H}$ invariant so $\varphi$ is an eigenvector under the action of $\bar{H}$. In particular $\lambda$ and also $\mu$ and $\nu$ are quasi spherical. Recall that $\tilde{\Delta}_{ne} = \{\tilde{\alpha} \in \Delta_1 \colon \alpha \text{ is not exceptional}\}$. We can write

$$\lambda = \sum_{\tilde{\alpha}\in\tilde{\Delta}_{ne}\smallsetminus I} c_{\tilde{\alpha}}\tilde{\omega}_{\tilde{\alpha}} + \sum_{\alpha\in\Delta_e\colon \tilde{\alpha}\notin I} c_\alpha \omega_\alpha.$$

Since $\varphi$ divides $\bar{p}_J$, we obtain $c_{\tilde{\alpha}}, c_\alpha \leqslant 1$ and $c_{\tilde{\alpha}} = c_\alpha = 0$ for $\tilde{\alpha} \notin J$. Also since $\bar{p}_J$ divides $\varphi^n$, we obtain $c_{\tilde{\alpha}}, c_\alpha \geqslant 1$ for $\tilde{\alpha} \in J$. So $c_\alpha = c_{\tilde{\alpha}} = 1$ if $\tilde{\alpha} \in J$ and $\lambda = \tilde{\omega}_J$ as claimed.

Now let $U = H^\circ \cdot [P_I]$ denote the open $H^\circ$ orbit in $G/P_I$ and $U'$ the complement of the divisor $D_{I^c}$. Since $U'$ is $H^\circ$ stable, $U \subset U'$.

We claim that $U$ is affine. To see this is enough to prove that the Lie algebra of $H^\circ \cap P_I$ is reductive. Indeed we have that this Lie algebra is equal to the Lie algebra of $L_I^\sigma$, where $L_I$ is the Levi factor of $P_I$ containing $T$.

Since $U$ is affine, by [7, Proposition 3.1, p. 66], $D = G/P_I \smallsetminus U$ has pure codimension one. Let $\mathcal{L}_\lambda \simeq \mathcal{O}(D)$ and let $\varphi \in \Gamma(G/P_I, \mathcal{L}_\lambda)$ be such that $\operatorname{div}\varphi = D$.

Since $U \subset U'$ and $\bar{p}_{I^c} = 0$ is reduced, we have that $\bar{p}_{I^c}$ divides $\varphi$. So $\mathcal{L}_\lambda$ is ample. Moreover the section $\varphi$ must be an eigenvector under the action of $\bar{H}$ so $\lambda$ is quasi spherical. This together with the fact that $\lambda \in \Lambda_I$ easily implies that, up to a non-zero constant,

$$\varphi = \prod_{\tilde{\alpha}\in I^c\cap\tilde{\Delta}_{ne}} \bar{p}_{\tilde{\alpha}}^{c_{\tilde{\alpha}}} \cdot \prod_{\alpha\in\Delta_e\colon \tilde{\alpha}\in I^c} \bar{q}_\alpha^{c_\alpha}$$

with the exponents $c_{\tilde{\alpha}}$ and $c_\alpha$ positive. On the other hand since $\varphi$ is reduced we must have $c_{\tilde{\alpha}} = c_\alpha \leqslant 1$ for all $\tilde{\alpha}$, $\alpha$. So $\lambda = \tilde{\omega}_{I^c}$ and $\varphi = \bar{p}_{I^c}$ as claimed.

Finally since the complement of the section $\bar{p}_J$ is stable by the action of $H$, or $\bar{H}$ and is a single $H^\circ$ orbit, it is also a single $H$ or $\bar{H}$ orbit. □

**5.2. Semistable points in a smooth toroidal compactification.** In this section we prove that the set of semistable points in a smooth toroidal compactification does not depend on the choice of an ample line bundle.

We need to make few remarks on weights and convex functions. We start with a simple and well known lemma on root systems.

**Lemma 5.2.** *Let $\{\alpha_1, \dots, \alpha_r\}$ be a set of simple roots in a root system $R$, and $\{\omega_1, \dots, \omega_r\}$ the corresponding set of fundamental weights. If $K \subset \{1, \dots, r\}$, every $\omega_j$ can be expressed as*

$$\omega_j = \sum_{h\in K} a_h\alpha_h + \sum_{k\notin K} b_k\omega_k$$

*with $a_h$, $b_k$ nonnegative rational numbers.*

*Furthermore, if $K = \Delta$ and $R$ is irreducible, the $a_h$'s are strictly positive.*

*Proof.* If $r = 1$, there is nothing to prove so we can proceed by induction.

Assume $|K| < r$. The space $A$ and $B$ respectively spanned by the $\alpha_h$'s with $h \in K$ and by the $\omega_k$'s with $k \notin K$ are mutually orthogonal. It follows that $\omega_j$ can be uniquely written as

$$\omega_j = \gamma_j + \delta_j$$

with $\gamma_j \in A$ and $\delta_j \in B$.

If $j \notin K$, then $\gamma_j = 0$ and there is nothing to prove.

If $j \in K$, then $\gamma_j$ is a fundamental weight for the root system in $A$ having the $\alpha_h$'s with $h \in K$ as simple roots. Thus by induction $a_h \geqslant 0$ for each $h \in K$. Write $\delta_j = \sum_{k \notin K} b_k \omega_k$. We get $b_k = \langle \delta_j, \alpha_k^\vee \rangle = -\langle \gamma_j, \alpha_k^\vee \rangle \geqslant 0$ as desired.

Assume now $|K| = r$. Write $\omega_j = \sum_{h=1}^{r} a_{j,h} \alpha_h$ and notice that $0 < (\omega_j, \omega_j) = a_{j,j} (\alpha_j, \omega_j)$. Since $(\alpha_j, \omega_j) > 0$, we deduce that $a_{j,j} > 0$. In particular we can write

$$\alpha_j = \frac{\omega_j}{a_{j,j}} + \xi_j,$$

where $\xi_j$ is a linear combination of the elements in $\Delta \setminus \{\alpha_j\}$. It follows that $\omega_k = (a_{k,j}/a_{j,j})\omega_j + x$ with $x$ a linear combination of the elements in $\Delta \setminus \{\alpha_j\}$. Using the positivity of $a_{j,j}$ our claim now follows from the previous analysis applied in the case in which $K = \Delta \setminus \{\alpha_j\}$.

It remains to show that in the irreducible case $a_{j,h} \neq 0$, i. e., $(\omega_j, \omega_h) \neq 0$ for each $j, h = 1, \ldots r$. By contradiction assume that say $(\omega_1, \omega_2) = 0$. This means that $\omega_2$ lies in the span of $\alpha_2, \ldots \alpha_r$ and it is a fundamental weight for the root system having these roots as a set of simple roots. Let $R'$ be the irreducible component of this root system containing $\alpha_2$. We can assume that our ordering of simple roots is such that $\alpha_2, \ldots \alpha_s$ are a set of simple roots for $R'$. By induction $\omega_2 = \sum_{h=2}^{s} a_{2,h} \alpha_h$ with $a_{2,h} > 0$. On the other hand

$$0 = (\omega_2, \alpha_1) = \sum_{h=2}^{s} a_{2,h} (\alpha_h, \alpha_1)$$

so that $(\alpha_h, \alpha_1) = 0$ for each $h = 1, \ldots, s$. This clearly contradicts the irreducibility of $R$. $\square$

Let $Y$ be a smooth toroidal compactification of $G/H$ and let $F_Y$ be the associated decomposition of the Weyl cochamber $C$.

We define the support $\operatorname{supp}_\Omega \rho$ of a face $\rho$ of $F_Y$ as

$$\operatorname{supp}_\Omega \rho := \{\tilde\alpha \in \tilde\Delta \colon \tilde\alpha \text{ is not identically zero on } \rho\}.$$

**Lemma 5.3.** *Let $\underline{\lambda} = (\lambda_\tau)_{\tau \in F_Y(\ell)} \in \operatorname{SPic}_0(Y)$ be such that $\mathcal{L}_{\underline{\lambda}}$ is ample. Let $\rho$ be a face of $F_Y$. Then there exist a positive integer $n$ and $\mu \in \Omega_H^+$ such that $\operatorname{supp}_\Omega \mu = \operatorname{supp}_\Omega \rho$ and $\mu \in \mathcal{A}(n\underline{\lambda})$.*

*Proof.* Let $S = \operatorname{supp}_\Omega \rho$ and $T = \tilde\Delta \setminus \operatorname{supp}_\Omega \rho$. Let $\tau$ be a maximal dimensional face containing $\rho$. Since $\mathcal{L}_{\underline{\lambda}}$ is ample, Corollary 3.9(iv) implies that $\lambda_\tau$ is regular dominant. So by Lemma 5.2 there exists a positive integer $n$ such that we can write $n\lambda_\tau$ as

$$n\lambda_\tau = \sum_{\tilde\alpha\in S} a_{\tilde\alpha}\tilde\omega_{\tilde\alpha} + \sum_{\tilde\alpha\in T} b_{\tilde\alpha}\tilde\alpha$$

with $a_{\tilde\alpha}$ positive integers and $b_{\tilde\alpha}$ nonnegative integers. Set $\mu = \sum_{\tilde\alpha\in S} a_{\tilde\alpha}\tilde\omega_{\tilde\alpha}$. We have $\mu = n\lambda_\tau$ on $\rho$ and $\mu \leqslant n\lambda_\tau \leqslant n\underline{\lambda}$ on the Weyl cochamber $C$ again by Corollary 3.9(iv). □

The following lemma is a sort of converse of Lemma 5.3.

**Lemma 5.4.** *Let $\underline{\lambda} = (\lambda_\tau)_{\tau\in F_Y(\ell)} \in \mathrm{SPic}_0$ be such that $\mathcal{L}_{\underline{\lambda}}$ is ample. Let $\mu \in \Omega^+$ and $n$ a positive integer with $\mu \in \mathcal{A}(n\underline{\lambda})$ and $\mu = n\underline{\lambda}$ on $\rho$. Then $\mathrm{supp}_\Omega\,\mu \supset \mathrm{supp}_\Omega\,\rho$.*

*Proof.* If $\rho$ is the zero face, there is nothing to prove. Assume that $\rho$ has positive dimension. By eventually substituting $\underline{\lambda}$ with $n\underline{\lambda}$, let us also assume that $n = 1$.

Let $\rho(1)$ be the set of 1 dimensional faces contained in $\rho$ and notice that $\mathrm{supp}_\Omega\,\rho = \bigcup_{\vartheta\in\rho(1)} \mathrm{supp}_\Omega\,\vartheta$. So it is enough to prove the claim in the case of one dimensional faces.

Let $\rho$ be one dimensional and choose a non-zero point $v$ in $\rho$.

Take a face $\tau$ of maximal dimension containing $\rho$ and define

$$\tau^\rho = \{u \in \Lambda^\vee_{\mathbb{R}} \colon v + t(v-u) \in \tau \text{ for some positive real number } t > 0\}.$$

Notice that $\mu \geqslant \lambda_\tau$ on $\tau^\rho$. Indeed if $u \in \tau^\rho$ there is a positive $t$ such that $v + t(v-u) \in \tau$. Since $\mu \in \mathcal{A}(\underline{\lambda})$, we have $\mu(v + t(v-u)) \leqslant \lambda_\tau(v + t(v-u))$. But $\underline{\lambda} = \mu$ on $\rho$ so that $\mu(v) = \lambda_\tau(v)$, so $\mu(u) \geqslant \lambda_\tau(u)$.

Since the support of $F_Y$ equals $C$, it is then clear that

$$\bigcup_{\tau\in F_Y(\ell)\colon \tau\supset\rho} \tau^\rho = \{u \in \Lambda^\vee_{\mathbb{R}} \colon \langle\tilde\alpha,\, u\rangle \leqslant 0 \text{ for all } \tilde\alpha \notin \mathrm{supp}_\Omega\,\rho\}.$$

Thus every $\tilde\alpha \in \mathrm{supp}_\Omega\,\rho$ lies in at least one of the sets $\tau^\rho$. It follows that

$$\langle\mu,\, \tilde\alpha\rangle \geqslant \langle\lambda_\tau,\, \tilde\alpha\rangle > 0$$

since $\underline{\lambda}$ is ample. Thus $\tilde\alpha \in \mathrm{supp}_\Omega\,\mu$. □

We can now characterize the set of semistable points with respect to an ample line bundle $\mathcal{L}$ on $Y$.

Consider the $G$-equivariant projection $\phi : Y \to X$ onto the wonderful compactification $X$. For each $\mu \in \Omega$ we pull back the $H^\circ$ invariant section $p^\mu$. This is an $H^\circ$ invariant section of $\Gamma(Y, \phi^*(\mathcal{L}_\mu))$ and we denote it by the same symbol $p^\mu$. For any subset $I \subset \tilde\Delta$ we set $p_I := \prod_{\tilde\alpha\in I} p^{\tilde\omega_{\tilde\alpha}}$. We also remark that the condition of being semistable does not depend on the group $K$ between $H^\circ$ and $\bar H$. Indeed, by the description of invariant sections, if $f$ is a $H^\circ$ invariant section which does not vanish on $x$ there exists an integer $n$ such that $f^n$ is invariant under $\bar H$. In view of this we will speak of semistable points without specifying the group $K$.

**Proposition 5.5.** *Let $Y$ be a smooth projective toroidal embedding of $G/H$ and let $\mathcal{L}$ be an ample line bundle on $Y$. Let $\rho$ be a face of $F_Y$ and let $O_\rho$ be the corresponding $G$ orbit. A point $x \in O_\rho$ is $\mathcal{L}$ semistable if and only if $p_{\mathrm{supp}_\Omega\,\rho}(x) \neq 0$.*

*Proof.* Let $\mathcal{L} = \mathcal{L}_{\underline{\lambda}}$. Let $\rho$ be a face of $F_Y$ and $x \in O_\rho$. Assume that $p_{\mathrm{supp}_\Omega \rho}(x) \neq 0$. By Lemma 5.3 there exist a positive integer $n$ and a dominant weight $\mu$ such that $\mu \in \mathcal{A}(n\underline{\lambda})$, $\mu = n\underline{\lambda}$ on $\rho$ and $\mathrm{supp}_\Omega \mu = \mathrm{supp}_\Omega \rho$. Since $p^\mu$ is a product of the sections $p_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \mathrm{supp}_\Omega \rho$, we have that $p^\mu(x) \neq 0$. Thus $s^{n\underline{\lambda}-\underline{\mu}} p^\mu$ is an $H$ invariant section of $\mathcal{L}^n_{\underline{\lambda}}$ not vanishing on $x$ and $x$ is semistable.

Conversely let $x$ be semistable. Then by the description of invariants given in the proof of Theorem 4.13 there exists a positive integer $n$ and a dominant spherical weight $\mu \in \mathcal{A}(n\underline{\lambda})$ such that $s^{n\underline{\lambda}-\underline{\mu}} p^\mu(x) \neq 0$. The condition $s^{n\underline{\lambda}-\underline{\mu}}(x) \neq 0$ implies $\mu = n\underline{\lambda}$ on $\rho$ so we can apply Lemma 5.4 and we deduce that $\mathrm{supp}_\Omega \mu \supset \mathrm{supp}_\Omega \rho$. In particular $p_{\tilde{\alpha}}(x) \neq 0$ for all $\tilde{\alpha} \in \mathrm{supp}_\Omega \rho$ or equivalently $p_{\mathrm{supp}_\Omega \rho}(x) \neq 0$. □

The above proposition has the following

**Corollary 5.6.** *Let $\mathcal{L}$ and $\mathcal{L}'$ be two ample line bundles on $Y$. Then $Y^{ss}(\mathcal{L}) = Y^{ss}(\mathcal{L}')$.*

In view of this corollary we shall from now on say that a point is semistable if it is semistable with respect to any ample line bundle and we shall denote the set of semistable points by $Y^{ss}$.

We now give a more set theoretic description of semistable points. Take a face $\rho$ of the fan $F_Y$ and denote by $O_\rho$ the corresponding $G$ orbit in $Y$. Set $I = \mathrm{supp}_\Omega \rho$ and consider the $G$-equivariant projection $\phi\colon Y \to X$. Remark that for any $\eta$ in the relative interior of $\rho$ the point $y_\eta$ depends only on $\mathrm{supp}_\Omega \rho$ and not on the choice of $\eta$. Thus we can denote this point by $y_\rho$. By definition for such one parameter subgroup $\eta$ we have $\phi(y_\eta) = x_\eta$. In particular it follows that $\phi(O_\rho) = O_I$ the open orbit in $X_I$ and that the projection $\phi\colon O_\rho \to O_I$ is a $G$-equivariant fibration.

By [4] we have a $G$-equivariant projection $\pi_I\colon O_I \to G/P_{I^c}$ with $I^c = \tilde{\Delta} \setminus I$. Composing we get a fibration

$$\gamma_I := \pi_I \circ \phi\colon O_\rho \to G/P_{I^c}$$

whose fiber over the point $\pi_I \circ \phi(y_\rho)$ we denote by $F_\rho$.

In view of Proposition 5.1 and Proposition 5.5 we immediately get

**Proposition 5.7.** *A point $x$ in $O_\rho$ is semistable if and only if its $H$ orbit intersects $F_\rho$. So we have that $O^{ss}_\rho := Y^{ss} \cap O_\rho$ is equal to $H^\circ F_\rho$ (and also to $KF_\rho$ and $\bar{H}F_\rho$).*

*Proof.* This is clear since by Proposition 5.1 the section $p_{\mathrm{supp}_\Omega \rho}$ does not vanish exactly on the inverse image under $\gamma_I$ of the open $H^\circ$ orbit in $G/P_{I^c}$. □

By Proposition 5.7 we then have that $O^{ss}_\rho = H \times_{H \cap P_{I^c}} F_\rho$.

Let us now take a subset $J \subset \tilde{\Delta}$. We denote by $L$ the standard Levi factor of $P_J$ and recall that $L$ is $\sigma$ stable and that if $U_{P_J}$ is the unipotent radical of $P_J$, $\sigma(U_{P_J}) = (U_{P_J})^-$ the opposite unipotent.

**Lemma 5.8.** *For any subset $J \subset \tilde{\Delta}$ we have*

$$K \cap L = K \cap P_J.$$

*Proof.* It is enough to analyze the case of $K = \bar{H}$. Since our problem is a problem of support, we can work with the associated reduced subgroup $\bar{H}_{\mathrm{red}} = \{x \in G\colon x\sigma(x)^{-1} \in Z(G)\}$.

Take $x = m\,u \in P_J \cap \bar{H}_{\mathrm{red}}$ with $m \in L$ and $u \in U_{P_J}$. Then clearly $u\sigma(u)^{-1} \in L$. It follows that $\sigma(u)^{-1} \in U^-_{P_J} \cap P_J = \{1\}$ thus $u = 1$ and $x \in L$ as desired. □

Going back to the semistable points in the orbit $O_\rho \subset Y$, we get

**Proposition 5.9.** *Let $L$ be the standard Levi factor of $P_{I^c}$. Set $K_L = K \cap L$. We have a $K$-equivariant isomorphism*

$$O_\rho^{ss} \simeq K \times_{K_L} F_\rho.$$

*In particular this induces a closure preserving bijection between $K$ orbits in $O_\rho^{ss}$ and $K_L$ orbits in $F_\rho$.*

Now consider the fiber $F_I$ of $\pi_I$ containing $\phi(y_\rho)$. We know from [4] that the solvable radical of $P_{I^c}$ acts trivially on $F_I$ and that $F_I = \bar{L}/\bar{H}_{\bar{L}}$, where $\bar{L}$ is the adjoint quotient of $L$ and $\bar{H}_{\bar{L}}$ is the subgroup fixed by the involution induced by $\sigma$.

By the description of $Y$ given in Theorem 3.5 it now follows that if we set $L_\rho$ equal to the quotient of $L$ modulo the subgroup in the center of $L$ generated by the one parameter subgroups $\eta$ with $\eta \in \rho$, $F_\rho$ can be identified with $L_\rho/H_\rho$, where $H_\rho$ is isogenous to the subgroup fixed by the involution on $L_\rho$ induced by $\sigma$.

Under this identification $S_{H_\rho}$ coincides with $Y_S \cap O_\rho$. We thus can apply Remark 4.11 and deduce

**Proposition 5.10.** *Let $x \in Y_S \cap O_\rho$. Then the orbit $Kx$ is closed in $O_\rho^{ss}$.*

## 6. Closed Semistable Orbits

In this section we prove that in the case of a smooth toroidal compactification $Y$ of $G/H$ the orbits through the elements of $Y_S$ are the closed semistable orbits. In this way we give a geometric counterpart to Theorem 4.1.

We show first that the closure of $K$ orbits does not interacts with $G$ orbits.

**Proposition 6.1.** *Let $Y$ be a smooth toroidal compactification of $G/H$, let $\mathcal{L}$ be an ample line bundle on $Y$. Let $x$, $y \in Y^{ss}(\mathcal{L})$. If $y \in \bar{K}x$, then $y \in G \cdot x$.*

*Proof.* As we have pointed out in Corollary 5.6, the set of semistable points of $Y$ does not depend on the choice of the ample line bundle $\mathcal{L}$.

Let $\mathcal{L} = \mathcal{L}_{\underline{\lambda}}$. For $n$ big enough we have that $\mathcal{L}_{n\underline{\lambda}+\underline{\alpha}_D}$ is also ample for every $D \in \Delta_Y$. In particular for a large enough positive integer $m$, we can find invariant sections $f_D \in \Gamma(Y, \mathcal{L}_{m(n\underline{\lambda}+\underline{\alpha}_D)})^K$ and $f \in \Gamma(Y, \mathcal{L}_{mn\underline{\lambda}})^K$ such that $f_D(y) \neq 0$ and $f(y) \neq 0$.

Set now

$$U = \{z \in Y \text{ such that } f(z) \neq 0 \text{ and } f_D(z) \neq 0 \text{ for all } D \in \Delta_Y\}.$$

The set $U$ is an open affine $H$ invariant subset of $Y^{ss}$ with the property that if

$$\pi\colon Y^{ss} \to K \mathbin{\backslash\!\backslash} Y$$

is the quotient morphism $U = \pi^{-1}(\pi(U))$. In particular $x \in U$. Furthermore, on $U$ the line bundles $\mathcal{L}_{m(n\underline{\lambda}+\underline{\alpha}_D)}$ and $\mathcal{L}_{mn\underline{\lambda}}$ have an $H$-equivariant trivialization.

It follows that also the line bundle $\mathcal{L}_{m\alpha_D} = \mathcal{L}_{m(n\underline{\lambda}+\underline{\alpha}_D)} \otimes \mathcal{L}^{-1}_{mn\underline{\lambda}}$ has an $H$-equivariant trivialization $\phi_D$ on $U$. Thus we can consider the $H$ invariant function $t_D = \phi_D(s_D^m)$ on $U$. Now by Theorem 3.5 a $G$ orbit in $Y$ is determined by the set of $D$ such that $s_D$ vanishes on the orbit. This implies our claim. □

**Remark 6.2.** The following simple example shows that Proposition 6.1 does not hold for a non-toroidal $Y$. Take the compactification $\mathbb{P}(\mathrm{End}(\mathbb{C}^3))$ of $\mathrm{PSL}(3)$, so $G = \mathrm{SL}(3) \times \mathrm{SL}(3)$ and $K$ is the normalizer of the diagonal copy of $\mathrm{SL}(3)$. The elements

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

give a counterexample to the statement in the proposition.

Let $U$ be any $G$ stable open subset of smooth toroidal projective embedding $Y$. The proof of Proposition 6.1 implies

**Proposition 6.3.** $\pi^{-1}(\pi(U \cap Y^{ss})) = U \cap Y^{ss}$. *Furthermore,*

$$\pi|_{U \cap Y^{ss}} : U \cap Y^{ss} \to \widetilde{W} \backslash U_S$$

*is a well defined quotient map.*

Notice however that, as the following example shows, there are ample line bundles $\mathcal{L}$ on $Y$ such that if we restrict $\mathcal{L}$ to $U$, $U^{ss}(\mathcal{L}) \neq U \cap Y^{ss}$. Indeed, take $Y$ equal to the wonderful embedding of $\mathrm{PSL}(3)$, $\mathcal{L} = \mathcal{L}_{\tilde{\omega}_1 + \tilde{\omega}_2}$ and $U$ equal to the complement of the divisor of equation $s_{\tilde{\alpha}_1} = 0$. Then $U$ is isomorphic to the open set in $\mathbb{P}(\mathrm{End}(\mathbb{C}^3))$ of classes of matrices of rank at least 2 and there is an invariant namely $s_{\tilde{\alpha}_1}^{-1} p_{\tilde{\alpha}_1}^3$, which up to a constant gives the cube of the trace, defined on $U$ and not vanishing on

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

while $x$ is not semistable in $Y$.

From Propositions 5.10 and 6.1 we finally get

**Theorem 6.4.** *Let $Y$ be a smooth toroidal compactification of $G/H$ and let $H^\circ \subset K \subset \bar{H}$. Let $x \in Y^{ss}$ then $Kx$ is closed in $Y^{ss}$ if and only if $Kx \cap Y_S \neq \varnothing$.*

*Proof.* From the proof of Theorem 4.1 we get that $Y_S \subset Y^{ss}$ and the fact that if $x \in Y_S$ its orbit is closed is Proposition 5.10.

On the other hand since $K \mathbin{\backslash\!\backslash} Y = \widetilde{W} \backslash Y_S$, the restriction of the quotient map

$$\pi \colon Y^{ss} \to K \mathbin{\backslash\!\backslash} Y$$

to $Y_S$ is surjective. Given $p \in K \mathbin{\backslash\!\backslash} Y$, $\pi^{-1}(p)$ contains a unique closed orbit and this by the first part is necessarily the orbit of a point in $\pi^{-1}(p) \cap Y_S$. □

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