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Tesi di dottorato

Varietà Quiver

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Introduzione

Uno spettro si aggira nella teoria della rappresentazione. È lo spettro della geometria, il cui crescente utilizzo negli ultimi vent'anni ha portato a risultati straordinari. Non si può ridurre questo fenomeno all'utilizzo dei *fasci perversi* e del *teorema di decomposizione*, che pure ne sono stati gli attori principali e spesso cruciali, sembra che una filosofia generale sia andata affermandosi, filosofia che fa uso di molti altre strutture geometriche e che viene genericamente indicata con *"geometric methods"*, o *"new geometric methods"*, in teoria della rappresentazione. Uno dei suoi aspetti fondamentali consiste nella sorprendente possibilità di descrivere la teoria delle rappresentazioni di alcune algebre di natura geometrica. Può quindi essere utile ed in alcuni casi risolutivo riuscire a dare una descrizione in termini geometrici delle algebre che si vogliono studiare. Malgrado si sia riusciti ad ottenere questa descrizione in vari casi importanti, questa rimane una operazione misteriosa. A lungo sono state utilizzate a questo scopo varietà naturalmente connesse con i gruppi di cui si volevano studiare le rappresentazioni: il gruppo stesso e le sue classi coniugate, la varietà delle bandiere prima totale e poi parziale, la fibra di Springer e la varietà di Steinberg. Poi si è cominciato ad utilizzare la versione "affine" delle varietà appena elencate che pur continuando ad essere varietà naturali sono tecnicamente molto più complicate. Ma ancora manca l'algoritmo per costruire la varietà giusta per ogni problema. In particolare sembrava mancare una varietà adatta per i *quantum groups*: fin dall'inizio di questo decennio si è cominciato a fare dei tentativi, ed è piano piano cresciuto l'interesse verso varietà meno naturali. Le *quiver varieties* sono una nuova di classe di varietà che sembra particolarmente interessante. Queste varietà sono state introdotte intorno al 1993-1994 da Hiraku Nakajima come generalizzazione dello spazio dei moduli degli instantoni su uno spazio ALE, il quale si era già dimostrato particolarmente adatto in connessione con un altro problema che andava in cerca di varietà: la costruzione di strutture hyperKähleriane. Malgrado la loro costruzione possa sembrare più artificiosa queste varietà si sono dimostrate particolarmente interessanti dal punto di vista della teoria della rappresentazione.

1. Quiver varieties e quantum groups

I quantum groups sono stati introdotti da Drinfeld [**3**] e Jimbo [**9**] mediante generatori e relazioni. Un problema naturale che si è subito posto è stato quindi quello di dare una descrizione alternativa in termini di strutture geometriche. Questa ricerca va naturalmente di pari passo con la ricerca di basi speciali di queste algebre e delle loro rappresentazioni.

1.1. La costruzione di $U_q(gl_n)$ di Beilinson, Lusztig e MacPherson. Uno dei primi risultati nella costruzione geometrica dei quantum groups è stato ottenuto nel caso di GL_n da Beilinson, Lusztig e MacPherson in [**1**]. Riassumiamo brevemente questo risultato.

Sia $\mathcal{F}(\Bbbk, d)$ l'insieme delle n-bandiere parziali di uno spazio vettoriale di dimensione d sul campo $\Bbbk$. Le $GL(\Bbbk, d)$ - orbite di $\mathcal{F}(\Bbbk, d) \times \mathcal{F}(\Bbbk, d)$ sono un insieme finito che indichiamo con Θ_d indipendente dal campo $\Bbbk$. Sia inoltre $\hat{\mathbf{K}}_d$ il $\mathbb{Q}(v)$-modulo libero con base gli elementi di Θ_d. Fissiamo ora $\Bbbk = \mathbb{F}_q$ un campo finito e per $A, B, C \in \Theta_d$ e $(F', F'') \in C$ definiamo

$$c_{A,B,C,q} = \text{card}\{F \in \mathcal{F}(\Bbbk, d) \,:\, (F', F) \in A \text{ and } (F, F'') \in B\}.$$

È ovvio che la la definizione non dipende dalla scelta di $(F', F'') \in C$. Osserviamo inoltre che esistono polinomi $c^C_{A,B} \in \mathbb{Z}[v]$ tali che per ogni q si abbia $c_{A,B,C,q} = c^C_{A,B}(\sqrt{q})$. Beilinson, Lusztig e MacPherson dimostravano quindi che l'algebra associativa $\hat{\mathbf{K}}_d$ definita prendendo per costanti di struttura questi polinomi è un quoziente di $U_q(gl_n)$.

Theorem 0.1. *Il prodotto definito da $A * B = \sum_{C \in \Theta_d} c^C_{A,B} C$ definisce su $\hat{\mathbf{K}}_d$ una struttura di algebra associativa e esiste un morfismo di algebre surgettivo da $U_{v^2}(gl_n)$ a $\hat{\mathbf{K}}_d$.*

A questo punto per costruire $U_q(gl_n)$ mandavano d all'infinito. Formalmente la costruzione funziona nel seguente modo. Sia $\Theta = gl(n, \mathbb{N})$, $\tilde{\Theta} \subset gl(n, \mathbb{Z})$ l'insieme delle matrici con coefficienti non negativi fuori della diagonale e $\sigma : \Theta \longrightarrow \mathbb{Z}$ la funzione che associa ad ogni matrice la somma dei suoi coefficienti. Osserviamo che Θ_d è in bigezione con $\sigma^{-1}(d)$. Definiamo inoltre per $A \in \tilde{\Theta}$

$$\begin{aligned} co(A) =& (\Sigma\, a_{i1}, \dots, \Sigma\, a_{in}), \\ ri(A) =& (\Sigma\, a_{1i}, \dots, \Sigma\, a_{ni}). \end{aligned}$$

Sia ora

$$\begin{aligned} \hat{\mathbf{K}} = \Big\{ & \Sigma_{A \in \tilde{\Theta}} f_A\, A \in \mathbb{Q}(v)^{\oplus \tilde{\Theta}} \,:\, \forall x \in \mathbb{Z}^n \text{ gli insiemi degli} \\ & \text{elementi } A \in \tilde{\Theta} \,:\, f_A \neq 0 \text{ and } ri(A) = x \text{ e degli} \\ & \text{elementi } A \in \tilde{\Theta} \,:\, f_A \neq 0 \text{ and } co(A) = x \text{ sono finiti} \Big\} \end{aligned}$$

Vogliamo estendere la struttura di algebra a $\hat{\mathbf{K}}$. Per fare ciò abbiamo bisogno della seguente proposizione ([**1**] Proposition 4.2).

Proposition 0.2. *Sia $r \geq 2$ e siano $A_1, \dots, A_r \in \tilde{\Theta}$ tali che $co(A_i) = ri(A_{i+1})$ per $i = 1, \dots, r-1$. Allora esistono $s, p_0 \in \mathbb{N}$ e $Z_1, \dots, Z_s \in \tilde{\Theta}$ e*

$G_1, \ldots, G_s \in \mathbb{Q}(v)[v']$ *tali che per ogni* $p \geq p_0$ $A_i + pI \in \Theta$ *e*

$$(A_1 + pI) * \cdots * (A_r + pI) = \sum_{i=1}^{s} G_i(v, v^{-p})(Z_i + pI)$$

Possiamo quindi estendere il prodotto $*$ a tutto $\hat{\mathbf{K}}$ mediante la formula:

$$A_1 * \cdots * A_r = \begin{cases} \sum_{i=1}^{s} G_i(v,1) Z_i & \text{se } co(A_i) = ri(A_{i+1}) \text{ for } i = 1, \ldots, r-1\,, \\ 0 & \text{altrimenti.} \end{cases}$$

Definiamo ora il $\mathbb{Q}(v)$-sottomodulo U di $\hat{\mathbf{K}}$ generato dagli elementi

$$A_{d,j} = \sum_{(z_1,\ldots,z_n)\in\mathbb{Z}^n \,:\, \Sigma z_i = d - \sigma(A)} v^{z_1 j_1 + \cdots + z_n j_n} \left(A + diag(z_1, \ldots, z_n)\right)$$

dove $d \in \mathbb{N}$, $A \in \Theta$ è una matrice con la diagonale nulla, $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$ e $diag(z_1, \ldots, z_n)$ è la matrice diagonale che ha i coefficienti lungo la diagonale uguali a $z_1, \ldots, z_n$.

THEOREM 0.3. *Il prodotto* $*$ *definisce una struttura di algebra associativa su* $\hat{\mathbf{K}}$ *e* U *è una sottoalgebra isomorfa a* $U_q(gl_n)$.

Entrambi questi teoremi vengono dimostrati individuando dei generatori di queste algebre che soddisfano le relazioni dei generatori standard dei quantum groups.

1.2. La costruzione geometrica di Ginzburg di sl_n. Prendendo spunto dalla costruzione di Beilinson, Lusztig e MacPherson, Ginzburg diede una costruzione di sl_n che non fa uso della geometria sui campi finiti e che ha rappresentato il modello anche per la costruzione di Nakajima delle algebre di Kac-Moody. In particolare uno dei risultati di questa tesi è la dimostrazione dell'equivalenza della costruzione di Ginzburg con la costruzione di Nakajima nel caso del quiver di tipo A. Riassumiamo quindi il risultato di Ginzburg annunciato in [**6**] e che si può trovare in tutti i dettagli in [**2**]. Nel seguito n è fissato e $\mathcal{N}_N$ è il cono nilpotente in $gl(N)$.

DEFINITION 0.4. Se N è un numero naturale e $a = (a_1, \ldots, a_n)$ una partizione di N, diciamo che una successione $F : \{0\} = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^N$ di sottospazi di $\mathbb{C}^N$ è una *bandiera parziale* di tipo a se $\dim F_i - \dim F_{i-1} = a_i$. Indichiamo con $\mathcal{F}_a$ la $GL(N)$-varietà omogenea delle bandiere parziali di tipo a. Definiamo inoltre

$$\widetilde{\mathcal{N}}_a = T^*\mathcal{F}_a \cong \{(u, F) \in gl(W) \times \mathcal{F}_a \text{ tale che } u(F_i) \subset F_{i-1}\},$$

$$\mu_a : \widetilde{\mathcal{N}}_a \longrightarrow \mathcal{N} \text{ la proiezione sul secondo fattore,}$$

$$\widetilde{\mathcal{N}}_N = \coprod_{a \vdash N} \widetilde{\mathcal{N}}_a \ \text{ e } \ \mu_N = \coprod_{a \vdash N} \mu_a \colon \widetilde{\mathcal{N}}_N \longrightarrow \mathcal{N}_N$$

$$Z_a = \widetilde{\mathcal{N}}_a \times_{\mathcal{N}_N} \widetilde{\mathcal{N}}_a \ \text{ e } \ Z_N = \widetilde{\mathcal{N}}_N \times_{\mathcal{N}_N} \widetilde{\mathcal{N}}_N$$

$$\Delta_a \subset \mathcal{F}_a \times \mathcal{F}_a \text{ la diagonale.}$$

Definiamo inoltre $\mathcal{F}_a = \varnothing$ se $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ e esiste i tale che $a_i < 0$, $\epsilon_i = (\dots, 0, \overset{i}{1}, \overset{i+1}{-1}, 0, \dots)$ e infine per $i = 1, \dots, n-1$ e $a \vdash N$:

$$Y_{i,a}^{+} = \{(F, F') \in \mathcal{F}_{a+\epsilon_i} \times \mathcal{F}_a \,:\, F_j = F'_j \text{ se } j \neq i \text{ e } F_i \supset F'_i\}$$
$$Y_{i,a}^{-} = \{(F, F') \in \mathcal{F}_{a-\epsilon_i} \times \mathcal{F}_a \,:\, F_j = F'_j \text{ se } j \neq i \text{ e } F_i \subset F'_i\}$$

Su $H_*^{BM}(Z_N) = \bigoplus_{a \vdash N} H_*^{BM}(Z_a)$ è definita una struttura di algebra di convoluzione e è facile vedere che in questo caso $H_{top}^{BM}(Z_N) = \bigoplus_{a \vdash N} H_{top}^{BM}(Z_a)$ è una sottoalgebra.

Il risultato principale dell'articolo [**6**] è il seguente:

Theorem 0.5. *Siano e_i, f_i, h_i generatori di Chevalley di $U(sl_n)$ allora possiamo definire un morfismo di algebre $\Theta \colon U(sl_n) \longrightarrow H_{top}^{BM}(Z_N)$ mediante:*

$$h_i \longmapsto \sum_{a \vdash N} (a_i - a_{i+1})[T^*_{\Delta_a}(\mathcal{F}_a \times \mathcal{F}_a)],$$
$$e_i \longmapsto \sum_{a \vdash N} [T^*_{Y^+_{i,a}}(\mathcal{F}_{a+\epsilon_i} \times \mathcal{F}_a)],$$
$$f_i \longmapsto \sum_{a \vdash N} [T^*_{Y^-_{i,a}}(\mathcal{F}_{a-\epsilon_i} \times \mathcal{F}_a)].$$

Inoltre:

(1) *Θ è surgettiva,*
(2) *Se $x \in \mathcal{N}_N$ e consideriamo l'usale struttura di $H_*^{BM}(Z_N)$-modulo sull'omologia $H_*^{BM}(\widetilde{\mathcal{N}}_x)$ della fibra di Springer generalizzata $\widetilde{\mathcal{N}}_x = \mu_N^{-1}(x) \subset \mathcal{N}_N$ abbiamo che:*
   (a) *$H_{top}^{BM}(\widetilde{\mathcal{N}}_x)$ è un $H_{top}^{BM}(Z_N)$ sottomodulo irriducibile*
   (b) *$H_{top}^{BM}(\widetilde{\mathcal{N}}_x)$ visto come rappresentazione di sl_n è il modulo irriducile associato alla partizione $d_1 \geq d_2 \cdots \geq d_n$, con $d_i = \dim\ker x^i - \dim\ker x^{i-1}$.*

1.3. Quiver varieties e costruzione geometrica delle algebre di Kac-Moody. Generalizzando la descrizione dello spazio dei moduli degli istantoni sulle risoluzioni delle singolarità razionali bidimensionali A,D,E data da Kronheimer e Nakajima in [**14**], e ispirato dai lavori di Ginzburg sulla costruzione di $U(sl(n))$ e di Lusztig sulla costruzione geometrica delle basi canoniche [**15, 16**], Nakajima ha introdotto una nuova classe di varietà associate ad un grafo, ad un elemento del reticolo del reticolo dei pesi d dell'algebra associata al grafo e ad un elemento del reticolo delle radici v. Nakajima ha chiamato queste varietà *quiver varieties* ma che forse sarebbe stato meglio chiamare *Nakajima's quiver varieties* perché differiscono da altre varietà assocate al grafo definite in precedenza. Daremo un cenno della costruzione nel seguito di questa tesi, limitiamoci quindi a descrivere con qualche veniale imprecisione uno dei risultati principali di Nakajima. Consideriamo una algebra di Kac-Moody simmetrica $\mathfrak{g}$ dove fissiamo una base di Chevalley e un sistema di radici. Ad un peso d e un elemento del reticolo delle radici v associamo una varietà liscia $M(d,v)$, una singolare $M^0(d)$ e una mappa propria:

$$\pi_{d,v}\colon M(d,v) \longrightarrow M^0(d).$$

Fissiamo ora d e consideriamo l'unione disgiunta $M(d)$ delle varietà $M(d,v)$ al variare di v e il prodotto fibrato $Z(d)$ di $M(d)$ su $M^0(d)$. In virtù di una costruzione molto generale l'omologia di Borel-Moore di $Z(d)$ aquista una struttura di algebra di convoluzione [**2**] e l'omologia delle fibre $M(d)_x = \bigcup_v \pi_{d,v}^{-1}(x)$ aquista una struttura di $H_*(Z(d))$ modulo. Osserviamo infine che in $M^0(d)$ esiste un punto speciale che chiamiamo 0.

Uno dei risultati principali di Nakajima si può cosìriassumere.

THEOREM 0.6. *Esiste un morfismo di algebre $\Phi\colon U(\mathfrak{g}) \longrightarrow H_{top}(Z(d))$ e in particolare $H_*(M(d)_0)$ ha una struttura di $U(\mathfrak{g}$-modulo. e $H_{top}(M(d)_0)$ è un suo $U(\mathfrak{g})$-sottomodulo . Inoltre $H_{top}(M(d)_0)$ è la rappresentazione irriducibile di peso più alto d e la decomposizione*

$$H_{top}(M(d)_0) = \bigoplus_v H_{top}(M(d,v)_0)$$

fornisce la decomposizione in pesi di $H_{top}(M(d)_0)$: più precisamente il sottospazio $H_{top}(M(d,v)_0)$ ha peso $d-v$.

1.4. Quiver varieties e la costruzione geometrica dei quantum groups. Nakajima ha annunciato di poter utilizzare le quiver varieties per dare una costruzione geometrica dei quantum groups. Nel caso di $\widetilde{A}_n$ lo stesso risultato era stato già ottenuto da Varagnolo e Vasserot [**27**].

2. Questa tesi

La geometria può essere spesso una utile guida, quando non è uno strumento insostituibile, nello studio della teoria delle rappresentazioni.

In particolare, il risultato di Nakajima mostra che lo studio della geometria delle varietà quiver può fornire informazioni sulla teoria delle rappresentazioni delle algebre di Kac-Moody: per esempio fornisce immediatamente una base canonica, costituita dalle componenti irriducibili della varietà $M(d,v)_0$, delle rappresentazioni integrali e irriducibili delle algebre di Kac Moody. In questa tesi affronto lo studio di alcune proprietà delle varietà quiver.

Il primo capitolo è principalmente un capitolo di rassegna. Nella prima sezione vengono definite le quiver varieties come varietà hyerkähleriane. Ad un grafo con n vertici e a due ennuple di naturali d, v viene associata una varietà hyperkähleriana $\mathfrak{M}_\zeta(d,v)$ dipendente da un parametro continuo ζ. Questa è la definizione originale che Nakajima da in [**21**]. Ho inserito una dimostrazione alla fine di questa sezione in quanto leggermente diversa da quella originale.

Nella seconda sezione ricordo alcune generalità sul collegamento tra la costruzione dei quozienti via mappe momento e la costruzione del quoziente dei punti semistabili. Il caso che ci interessa è leggermente diverso da quello che si utilizza usualmente ed ho quindi riportato alcune dimostrazioni. Purtroppo non sono riuscito a dimostrare in questa generalità un risultato altrettanto soddisfacente di quello usuale.

Nella terza sezione definisco le quiver varieties come varietà algebriche e introduciamo una serie di risultati e notazioni che saranno necessari nel seguito.

Nella quarta sezione ricordiamo un risultato di Nakajima sull'azione del gruppo di Weyl sulle quiver varieties lisce. Dimostriamo anche un analogo di questo risultato per la varietà singolare M^0 che mi sembra essere nuovo.

Nella quinta sezione enunciamo il risultato di Nakajima sulla costruzione delle rappresentazioni delle algebre di Kac-Moody ricordato sopra.

Questo capitolo può forse apparire un po' disomogeneo. In particolare non tutti i risultati che ricordo saranno effettivamente utilizzati nel seguito della tesi, e altri invece importanti non sono stati ricordati. Questo è dovuto al fatto che ho dato maggior rilievo ai risultati che penso possano essere preliminari nello studio di altri problemi connessi con le varietà quiver. In particolare il quarto paragrafo va interpretato in questo senso. Analogamente può apparire un pò superflua la costruzione delle varietà quiver come varietà hyperKhäleriane. In realtà penso che questa costruzione è utile per costruire una azione del gruppo di Weyl simile a quella costruita da Slodowy nel caso della varietà delle bandiere ([**25**]).

Il secondo capitolo è dedicato ad alcune osservazioni sulla omologia delle quiver varieties. Nell'ultima sezione (§4) viene dimostrata la seguente proprietà dell'omologia delle quiver varities:

$$H_*\left(M(d+d')\right) = H_*\left(M(d)\right) \otimes H_*\left(M(d')\right).$$

Il risultato sulla moltiplicatività avrebbe da una parte lo scopo di ridurre lo studio dell'omologia delle quiver varieties al caso dei pesi fondamentali, dall'altro suggerisce una via per una interpretazione geometrica della formula di Littlewood-Richardson. Soprattutto da questo secondo punto di vista il risultato sulla moltiplicatività si può considerare incompleto. Non viene dimostrato infatti che l'isomorfismo in questione è di $\mathfrak{g}$-moduli. Nell'ultima parte del capitolo dimostro una proprietà da cui la $\mathfrak{g}$-equivarianza potrebbe seguire.

Il risultato sulla moltiplicatività (§4) dell'omologia delle quiver varieties e il calcolo degli indici di contrazione (§2) penso che siano nuovi. Nello stesso capitolo riporto la dimostrazione di un risultato di Nakajima sulla omologia delle varietà quiver che utilizzo e di cui non è disponibile un riferimento bibliografico.

Nel capitolo terzo fornisco una descrizione completa delle quiver varieties in termine delle varietà di Slodowy, nel caso dei grafi di tipo A. Questo risultato era stato congetturato da Nakajima in [**21**] §8 e come è spiegato nello stesso paragrafo non sembra avere una naturale generalizzazione agli altri grafi finiti: D e E. Un'altra distinzione tra il caso A e i casi D, E è dato dalla seguente osservazione: nel caso A_{n-1} se prendiamo $d = (n, 0, \dots, 0)$ e

$v = (n-1, n-2, \ldots, 1)$ otteniamo che la varietà $M(d,v)$ è isomorfa alla varietà delle bandiere complete in $\mathbb{C}^n$. È forse inutile ricordare che questa varietà gioca un rulo importantissimo nella teoria delle rappresentazioni di $SL(n)$ e di S_n. Se C è la matrice di Cartan associata al grafo di tipo A, all'interno del "linguaggio" delle quiver varieties si può caratterizzare questa varietà come una varietà $M(d,v)$ per la quale $d - Cv = 0$ e $M(d,v)_0$ ha una sola componente irriducibile. Ci si può chiedere allora se esiste sempre una varietà del genere e se ha anche negli altri casi un particolare ruolo. E' facile vedere che già nel caso D_4 una varietà di questo tipo non esiste.

Esistono altri aspetti del caso A che invece almeno congetturalmente si potrebbero estendere agli altri casi. La costruzione dell'isomorfismo tra le varietà di Slodowy e le varietà quiver di tipo A data nel terzo capitolo mostra come ogni fibra della mappa

$$\pi : M(d,v) \longrightarrow M^0(d,v)$$

sia isomorfa ad una fibra $M(d',v')_0$ per una varietà quiver per lo stesso grafo e per opportuni d', v'. Nel caso dei grafi di tipo finito questa ipotesi è supportata dalla descrizione delle quiver varieties come spazio dei moduli degli istantoni sulle risoluzioni minimali delle superfici $\mathbb{C}^2/\Gamma$.

Ma l'aspetto che sarebbe più interessante generalizzare è legato al calcolo dell'omologia delle quiver varieties e quindi, grazie al risultato di Nakajima, dei caratteri delle rappresentazioni irriducibili al quale accenniamo nell'osservazione 3.1. L'omologia delle quiver varieties di tipo A_n si può infatti studiare utilizzando un procedimento induttivo. Spaltenstein ha infatti utilizzato questo procedimento per dare una cellurarizzazione delle fibre di Springer generalizzate. Il punto essenziale è che data una fibra di Springer di tipo A_n (cioè una fibra della proiezione di una varietà delle bandiere ad n passi) esiste una mappa in una grassmaniana le cui fibre sono fibre di Springer di tipo A_{n-1}. Questa mappa si può descrivere naturalmente nel linguaggio delle quiver varieties e ci si può chiedere se questo procedimento si possa estendere a situazioni più generali. Nel caso D_4, per esempio, si possono utilizzare metodi simili per calcolare l'omologia delle quiver varieties nel caso dei pesi fondamentali, e quindi grazie ai risultati del capitolo 2 di ogni quiver variety. Non è chiaro però se sia possibile organizzare questi metodi in modo organico. Io penso che questo sia collegato a capire meglio la struttura dell'algebra $\mathcal{R}$ associata al grafo (vedi la definizione 1.17).

3. Ringraziamenti

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General conventions

$\mathbb{N}$**:** $= \{0, 1, 2, \dots\}$

$[\varphi]_{E_i,F_j}$**:** if $E = \bigoplus_i E_i$ and $F = \bigoplus_j F_j$ are complex vector spaces and $\varphi \in \mathrm{Hom}_{\mathbb{C}}(F, E)$ then

$$[\varphi]_{E_i,F_j} = \pi_{E_i} \circ \varphi \circ \imath_{F_j}$$

where π_{E_i} is the projection of E on E_i with kernel equal to $\bigoplus_{h \neq i} E_h$ and $\imath_{F_j}$ is the inclusion of F_j in F. When the decompositions of the sapces will be clear from the contest we will use this convention without specifing the spaces E_i, F_j.

$v \geq u$ $[v > u]$**:** if $v = (v_1, \dots, v_n)$, $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ we say that $v \geq u$ $[v > u]$ if $v_i \geq u_i$ $[v_i > u_i]$ for all i.

dim**:** otherwise stated with dim we mean the complex dimension of a vector space, a manifold or an algebraic variety

$H_i(X), H_i^c(X)$**:** $H_i(X)$ is the Borel-Moore homology group with coefficient in a field of characteristic 0 and $H_i^c(X)$ is the singular homology group with coefficient in a field of characteristic 0 .

CHAPTER 1

General properties of quiver varieties

In this chapter we define quiver varieties and we describe some general properties of these varieties.

1. Notations and definitions

In this section we give the definition of quiver varieties. Except some minor change in section 1.7 all definition, results and proofs are due to Nakajima [**21, 22**].

1.1. The graph. Let (I, H) be a finite oriented graph: I is the set of vertices that we suppose of cardinality n, H the set of arrows and the orientation is given by the two maps

$$h \longmapsto h_0 \text{ and } h \longmapsto h_1$$

from H to I. We suppose also that:

(1) $\forall\, h \in H \quad h_0 \neq h_1$,
(2) an involution $h \mapsto \bar{h}$ of H without fixed points and satisfying $\bar{h}_0 = h_1$ is fixed,
(3) a subset Ω of H is given satisfying:
  (a) $\Omega \cap \overline{\Omega} = \varnothing$ and $\Omega \cup \overline{\Omega} = H$,
  (b) it does not exist $n > 0$ and $h^{(1)}, \dots, h^{(n)} \in \Omega$ such that $h_1^{(i)} = h_0^{(i+1)}$ for $i = 1, \dots, n-1$ and $h_1^{(n)} = h_0^{(1)}$,
  we define $\epsilon : H \longrightarrow \{-1, 1\}$ by

$$\epsilon(h) = \begin{cases} 1 & \text{if } h \in \Omega, \\ -1 & \text{if } h \in \overline{\Omega}. \end{cases}$$

We observe that given a symmetric graph without loops is always possible to define Ω, ϵ and an involution $^-$ as above.

1.2. The Cartan matrix and the Weyl group. Let A be the matrix whose entries are the numbers

$$a_{ij} = card\{h \in H \,:\, h_0 = i \text{ and } h_1 = j\}.$$

We define a generalized symmetric Cartan matrix by $C = 2I - A$. Following [**17**] an X, Y-regular root datum $(I, X, X^\vee, <\,,\,>)$ with Cartan matrix equal to C is defined in the following way:

(1) $X^\vee$ and X are finetely generated free abelian groups,
(2) $<\,,\,>: X \times X^\vee \longrightarrow \mathbb{Z}$ is a perfect bilinear pairing,

(3) two linearly independent sets $\Pi = \{\alpha_i : i \in I\} \subset X$ and $\Pi^\vee = \{\alpha_i^\vee : i \in I\} \subset X^\vee$ are fixed and we set $Q = \langle \Pi \rangle$ and $Q^\vee = \langle \Pi^\vee \rangle$,
(4) $<\alpha_i, \alpha_j^\vee> = c_{ij}$,
(5) (nonstandard) $\operatorname{rank} X = \operatorname{rank} X^\vee = 2n - \operatorname{rank} C$,
(6) (nonstandard) a linearly independent set $\{\omega_i : i \in I\}$ of X such that $<\omega_i, \alpha_j^\vee> = \delta_{ij}$ is fixed.

Once C is given it is easy to construct a data as above. We call $\mathfrak{h}$ the complexification of $X^\vee$ and we observe that through the bilinear pairing $< , >$ we can identify $\mathfrak{h}^*$ with the complexification of X. We observe also that the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of the Cartan matrix C ([**10**] pg.1).

The Weyl group W attached to C is defined as the subgroup of $Aut(X) \subset GL(\mathfrak{h}^*)$ generated by the reflections

$$s_i : x \longmapsto x - <x, \alpha_i^\vee> \alpha_i. \tag{1}$$

We observe that the dual action is given by $s_i(y) = y - <\alpha_i, y> \alpha_i^\vee$ and that the lattices Q and $Q^\vee$ are stable for these actions. So the annihilator $\overset{\circ}{Q^\vee} = \{x \in X : <x, y> = 0 \ \forall y \in Q^\vee\}$ is also stable by W and we can consider the action of Won the lattice $P = X / \overset{\circ}{Q^\vee} \simeq \operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$ and we call $x \mapsto \overline{x}$ the projection from X to P. We observe also that this projection is an isomorphism from the lattice $\widetilde{P}$, that is not W-stable, spanned by $\{\omega_i : i \in I\}$ and P. Finally we observe that

$$\overline{\alpha}_i = \sum_{j \in I} c_{ij} \overline{\omega}_j.$$

1.3. The algebras U and $\widetilde{\mathbf{U}}$. We call $\mathbf{U}$ the enveloping algebra of the Kac-Moody algebra attached to the Cartan matrix C ([**10**] ch.1) and $\widetilde{\mathbf{U}}$ the specialization at $q = 1$ of the algebra introduced in [**17**] ch.23. Since the matrix C is symmetric we can describe the algebra $\mathbf{U}$ as the algebra with unity generated by $\{e_i, f_i : i \in I\}$ and $\mathfrak{h}$ with the following relations ([**10**] Th. 9.11):

$$[h, h'] = 0 \qquad \text{for } h, h' \in \mathfrak{h}, \tag{2a}$$
$$[h, e_i] = <\alpha_i, h> e_i \qquad \text{for } i \in I \text{ and } h \in \mathfrak{h}, \tag{2b}$$
$$[h, f_i] = - <\alpha_i, h> f_i \qquad \text{for } i \in I \text{ and } h \in \mathfrak{h}, \tag{2c}$$
$$[e_i, f_j] = \delta_{ij} \alpha_i^\vee \qquad \text{for } i, j \in I, \tag{2d}$$
$$(\operatorname{ad} e_i)^{1 - c_{ji}} e_j = 0 \qquad \text{for } i \neq j \in I, \tag{2e}$$
$$(\operatorname{ad} f_i)^{1 - c_{ji}} f_j = 0 \qquad \text{for } i \neq j \in I. \tag{2f}$$

We call $\mathbf{U}^+$ the subalgebra with unity of $\mathbf{U}$ generated by $\{e_i : i \in I\}$, $\mathbf{U}^0$ the subalgebra with unity generated by $\mathfrak{h}$ and $\mathbf{U}^-$ the subalgebra with unity generated by $\{f_i : i \in I\}$. We recall that we have the following triangular decomposition of the algebra $\mathbf{U}$:

$$\mathbf{U} = \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+ .$$

and we remark that $\mathbf{U}^0 \simeq S(\mathfrak{h}) \simeq \mathbb{C}[X \otimes_{\mathbb{Z}} \mathbb{C}]$ with multiplication given by the usual multiplication of functions. The algebra $\widetilde{\mathbf{U}}$ is an algebra (without unity) that has the following triangular decomposition:

$$\widetilde{\mathbf{U}} = \mathbf{U}^- \otimes \widetilde{\mathbf{U}}^0 \otimes \mathbf{U}^+ \,,$$

where $\widetilde{\mathbf{U}}^0 = \{f : X \to \mathbb{C} \, : \, f(x) = 0$ for all but a finete number of $x \in X\}$. If $\mathbf{1}_\lambda(x) = \delta_{\lambda x}$ then the set $\{\mathbf{1}_\lambda \, : \, \lambda \in X\}$ is a basis of $\widetilde{\mathbf{U}}^0$ and we can define a product on $\widetilde{\mathbf{U}}$ through the formulas:

$$\mathbf{1}_\lambda e_i = e_i \mathbf{1}_{\lambda-\alpha_i} \qquad \text{for } \lambda \in X \text{ and } i \in I, \tag{3a}$$

$$\mathbf{1}_\lambda f_i = f_i \mathbf{1}_{\lambda+\alpha_i} \qquad \text{for } \lambda \in X \text{ and } i \in I, \tag{3b}$$

$$[e_i, f_j]\mathbf{1}_\lambda = \delta_{ij} <\lambda, \alpha_i^\vee> \mathbf{1}_\lambda \qquad \text{for } \lambda \in X \text{ and } i, j \in I. \tag{3c}$$

To be more precise there is not an element e_i or f_i in $\widetilde{\mathbf{U}}$ but only elements $e_i \mathbf{1}_\lambda$, so the formulas above are a little bit sloppy: for example the first one should be written $\mathbf{1}_\lambda(e_i \otimes \mathbf{1}_\mu \otimes u^-) = \delta_{\lambda-\alpha_i,\mu}(e_i \otimes \mathbf{1}_{\lambda-\alpha_i} \otimes u^-)$.

1.4. d, v **and the space of all matrices.** We begin now to define the varieties attached to the graph which should be the geometric counterparts of the algebra just defined. For the exposition it will be usefull to identify the set I with the set of integers $\{1, \dots, n\}$.

In this thesis $d = (d_1, \dots, d_n)$ and $v = (v_1, \dots, v_n)$ will be two n-tuples of nonnegative integers. We also think of d, v as elements of X in the following way:

$$d = \sum_{i \in I} d_i \omega_i \ \text{ and } \ v = \sum_{i \in I} v_i \alpha_i \, ; \tag{4}$$

and through these identifications we define also an action of W on u, v. We define also $v^\vee = \sum_{i \in I} v_i \alpha_i^\vee \in Q^\vee$. Once d, v are fixed we fix also complex vector spaces D_i and V_i of dimensions d_i and v_i and we define the following spaces of maps:

$$S_\Omega(d, v) = \bigoplus_{i \in I} \operatorname{Hom}(D_i, V_i) \oplus \bigoplus_{h \in \Omega} \operatorname{Hom}(V_{h_0}, V_{h_1}),$$

$$S_{\overline{\Omega}}(d, v) = \bigoplus_{i \in I} \operatorname{Hom}(V_i, D_i) \oplus \bigoplus_{h \in \overline{\Omega}} \operatorname{Hom}(V_{h_0}, V_{h_1}),$$

$$S(d, v) = S_\Omega(d, v) \oplus S_{\overline{\Omega}}(d, v).$$

More often, when it will not be ambiguous we will write $S_\Omega, S_{\overline{\Omega}}$ and S instead of $S_\Omega(d, v), S_{\overline{\Omega}}(d, v)$ and $S(d, v)$.

As a general convention we will call γ_i an element of $\operatorname{Hom}(D_i, V_i)$, δ_i an element of $\operatorname{Hom}(V_i, D_i)$ and B_h an element of $\operatorname{Hom}(V_{h_0}, V_{h_1})$, and we will use γ for $(\gamma_1, \dots, \gamma_n)$, δ for $(\delta_1, \dots, \delta_n)$ and B for $(B_h)_{h \in H}$.

Once D_i, V_i and an element (B, γ, δ) of S are fixed we define also:

$$T_i = D_i \oplus \bigoplus_{h\,:\,h_1=i} V_{h_0}\,, \tag{5a}$$

$$a_i = a_i(B,\gamma,\delta) = (\delta_i, (B_{\bar h})_{h\,:\,h_1=i}) : V_i \longrightarrow T_i\,, \tag{5b}$$

$$b_i = b_i(B,\gamma,\delta) = (\gamma_i, (\epsilon(h)B_h)_{h\,:\,h_1=i}) : T_i \longrightarrow V_i\,. \tag{5c}$$

In this thesis we will identify the dual of space of the $\mathbb{C}$-linear maps $\mathrm{Hom}(E,F)$ between two finite dimensional vector spaces with $\mathrm{Hom}(F,E)$ through the pairing $<\varphi,\psi>= \mathrm{Tr}(\varphi\circ\psi)$. So we can describe S also as $S_\Omega \oplus S_\Omega^* = T^*S_\Omega$ and we observe that a natural symplectic structure ω is defined over S by

$$\omega((s_\Omega, s_{\overline{\Omega}}),(t_\Omega,t_{\overline{\Omega}})) = <s_\Omega, t_{\overline{\Omega}}> - <t_\Omega, s_{\overline{\Omega}}>\,.$$

If we want to describe this structure explicitely in terms of maps (B,γ,δ) we have:

$$\begin{aligned}\omega((B,\gamma,\delta),(\widetilde{B},\widetilde{\gamma},\widetilde{\delta})) &= \sum_{h\in H}\epsilon(h)\,\mathrm{Tr}(B_h\widetilde{B}_{\bar h}) + \sum_{i\in I}\mathrm{Tr}(\gamma_i\widetilde{\delta}_i - \widetilde{\gamma}_i\delta_i) \\ &= \sum_{i\in I}\mathrm{Tr}(b_i\widetilde{a}_i - \widetilde{b}_i a_i)\end{aligned} \tag{6}$$

1.5. Hermitian and hyperKähler structure on S. We suppose now that the spaces D_i, V_i are endowed with hermitian metrics. So we can speak of the adjoint φ^* of a linear map between these spaces, and we have a positive definite hermitian structure h on S with explicit formula:

$$\begin{aligned}h((B,\gamma,\delta),(\widetilde{B},\widetilde{\gamma},\widetilde{\delta})) &= \sum_{h\in H}\mathrm{Tr}(B_h\widetilde{B}_h^*) + \sum_{i\in I}\mathrm{Tr}(\gamma_i\widetilde{\gamma}_i^* + \widetilde{\delta}_i^*\delta_i) \\ &= \sum_{i\in I}\mathrm{Tr}(a_i\widetilde{a}_i^* + \widetilde{b}_i^* b_i)\end{aligned} \tag{7}$$

Moreover we can define a structure of hyperKähler manifold on the real riemannian manifold $(S, \mathrm{Re}\,h)$: that is the datum of three covariant constant orthogonal automorphisms I, J and K of the tangent bundle which satisfy:

$$I^2 = J^2 = K^2 = IJK = -1.$$

In our case I, J and K are defined as follows:

I is the usual multiplication by $\mathrm{i} = \sqrt{-1}$,

$J(B,\gamma,\delta) = (\epsilon(\bar h)B_{\bar h}^*, -\delta_i^*, \gamma_i^*)$, or $J(s_\Omega, s_{\overline{\Omega}}) = (-s_{\overline{\Omega}}^*, s_\Omega^*)$,

$K(B,\gamma,\delta) = (\epsilon(\bar h)\mathrm{i}\,B_{\bar h}^*, -\mathrm{i}\,\delta_i^*, \mathrm{i}\,\gamma_i^*)$ or $K(s_\Omega, s_{\overline{\Omega}}) = (-\mathrm{i}s_{\overline{\Omega}}^*, \mathrm{i}s_\Omega^*)$.

We define also the real symplectic forms $\omega_I, \omega_J, \omega_K$ by the formulas:

$$\omega_I(s,t) = \mathrm{Re}\,h(Is,t), \qquad \omega_J(s,t) = \mathrm{Re}\,h(Js,t), \qquad \omega_K(s,t) = \mathrm{Re}\,h(Ks,t).$$

We observe that $\omega_I, \omega_J, \omega_K$ are closed forms and that

$$\omega_I = -\,\mathrm{Im}\,h \;\text{ and }\; (\omega_J + \mathrm{i}\omega_K)(x,y) = h(y, Jx) = \omega(x,y).$$

In particular $-\omega_I$ is a Kähler form on S.

1.6. Group actions and moment maps. We can define an action of the groups $G = GL(V) = \prod GL(V_i)$ and $GL(D) = \prod GL(D_i)$ on the set S in the following way:

$$g(B_h, \gamma_i, \delta_i) = (g_{h_1} B_h g_{h_0}^{-1}, g_i\gamma_i, \delta_i g_i^{-1}) \qquad \text{for } g = (g_i) \in GL(V), \tag{8}$$

$$g(B_h, \gamma_i, \delta_i) = (B_h, \gamma_i g_i^{-1}, g_i\delta_i) \qquad \text{for } g = (g_i) \in GL(D). \tag{9}$$

We observe that these actions commute, that ω is $GL(V)$ invariant, and that they commute with the action of I. Moreover if $U = U(V) = \prod U(V_i)$ is the group of unitary trasformations in $GL(V)$ we have that $U(V)$ commute with J, K and that $\omega_I, \omega_J, \omega_K, h$ are $U(V)$ invariant.

We want to define moment maps for these actions. We use the definition of moment map given in [**22**], which differs from the one in [**20**] ch.8 for a sign: if (M, η) is a (real or complex) symplectic manifold and $\sigma\colon K\times M \to M$ is an action of a Lie group which preserves the symplectic form η then a map $\mu\colon M \to \mathfrak{k}^*$ is said a moment map if:

$$\mu(km) = \mathrm{Ad}_g^* \mu(m) \qquad \text{for } k \in K \text{ and } m \in M,$$

$$<x, d\mu_m(v)> = \eta(\sigma_x(m), v), \qquad \text{for } m \in M,\ v \in T_m M \text{ and } x \in \mathfrak{k},$$

where $\sigma_x(m) = d\sigma_{(1_G,m)}[x, 0]$.

In the case that (M, η) is a real symplectic vector space we observe that the map $\eta : M \to \mathfrak{k}^*$ defined by:

$$<\eta(m), x>= \tfrac{1}{2}\eta(x \cdot m, m) \qquad \text{for } m \in M \text{ and } x \in \mathfrak{k} \tag{10}$$

is moment map for the action of K. In our case we identify $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$ with $\mathfrak{g} = \oplus gl(V_i)$ through the pairing $<(x_i)\,,(y_i)>= \sum_i \mathrm{Tr}(x_i y_i)$. Since if $x, y \in \mathfrak{u}$ then $<x, y>\in \mathbb{R}$ we can use the same pairing to identify $\mathfrak{u}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{u}, \mathbb{R})$ with $\mathfrak{u}$. Moreover we observe that $\mathfrak{g} = \mathfrak{u} \oplus \mathrm{i}\mathfrak{u}$. The i-component of the moment maps $\mu, \mu_I\colon S \longrightarrow \mathfrak{g} = \oplus gl(V_i)$ defined as in (10) have the following explicit formulas:

$$\mu_i(B, \gamma, \delta) = \sum_{h\in H\,:\,h_1=i} \epsilon(h) B_h B_{\bar h} + \gamma_i\delta_i = b_i a_i\,,$$

$$\mu_{I,i}(B, \gamma, \delta) = \frac{\mathrm{i}}{2}\left(\sum_{h\in H\,:\,h_1=i} B_h B_h^* - B_{\bar h}^* B_{\bar h} + \gamma_i\gamma_i^* - \delta_i^*\delta_i\right) = \frac{\mathrm{i}}{2}(b_i b_i^* - a_i^* a_i),$$

we observe that μ is a moment map for the action of G on the symplectic manifold (S, ω) and that μ_I is a moment map for the action of U on the symplectic manifold (S, ω_I). Since $\omega = \omega_J + \mathrm{i}\omega_K$ we observe that $\mu = \mu_J + \mathrm{i}\mu_K$ splits in the sum of the two moment maps for the action of U on (S, ω_J) and (S, ω_K). It is common to group all these moment maps together and to define an hyperKähler moment map

$$\widetilde{\mu} = (\mu_I, \mu) : S \longrightarrow \mathfrak{u} \oplus \mathfrak{g} = (\mathbb{R} \oplus \mathbb{C}) \otimes_{\mathbb{R}} \mathfrak{u}.$$

Finally we want to identify the $\mathfrak{z} = Z_U \oplus Z_G = Z_U(\mathfrak{u}) \oplus Z_G(\mathfrak{g}) = \oplus_i \mathrm{i}\mathbb{R}\,\mathrm{Id}_{V_i} \oplus \oplus_i \mathbb{C}\,\mathrm{Id}_{V_i}$ with $\mathbb{R}^n \oplus \mathbb{C}^n$ and with $(\mathbb{R} \oplus \mathbb{C}) \otimes_{\mathbb{Z}} P$ through:

$$\sum_{i\in I}(\tfrac{\mathrm{i}}{2}\xi_i, \lambda_i)\mathrm{Id}_{V_i} \longleftrightarrow (\xi_1, \ldots, \xi_n, \lambda_1, \ldots, \lambda_n) \longleftrightarrow \sum_{i\in I}(\xi_i, \lambda_i)\overline{\omega}_i. \tag{11}$$

In particular we consider an action of the Weyl group W on $\mathfrak{z}$ through this identification.

1.7. Quiver varieties as hyperKähler quotients. The equations $\mu_I = \xi$ and $\mu = \lambda$ are called *ADHM equations.* For $\zeta = (\xi, \lambda) \in \mathfrak{z}$, we define:

$$\mathfrak{L}_\zeta(d, v) = \{s \in S \,:\, \widetilde{\mu}(s) = \zeta\}$$

and we observe that it is stable for the action of U, so, at least as a topological Hausdorf space we can define the *quiver variety of type* ζ as

$$\mathfrak{M}_\zeta(d, v) = \mathfrak{L}_\zeta(d, v)/U(V).$$

It will be convenient to define also $\mathfrak{M}_\zeta(d, v) = \varnothing$ if $d, v \in \mathbb{Z}^n$ and there exists i such that $v_i < 0$ or $d_i < 0$. We want now to give a sufficient condition on ζ for the smoothness of $\mathcal{M}_\zeta$.

Lemma 1.1. *Let* $\widetilde{\mu}(s) = \zeta \in \mathfrak{z}$ *then*

$$d\widetilde{\mu}_s \textit{ is surjective} \iff d\mu_s \textit{ is surjective} \iff \mathrm{Stab}_G\{s\} = \{1_G\}$$

Proof. 3) $\Rightarrow$ 2) is a general fact: indeed if $d\mu_s$ is not epi there exists $x \in \mathfrak{g}$ such that $<x, d\mu_s(v)>= 0$ for each $v \in T_sS$. But since μ is a moment map this is equivalent to $\sigma_x(s) = 0$ and this implies $\exp(tx)s = s$ for each $t \in \mathbb{C}$.

2) $\Rightarrow$ 3). Let $g = (g_i) \in G$ such that $gs = s$ and set $X_i = g_i - \mathrm{Id}_{V_i}$ and $x = (X_i) \in \mathfrak{g}$. By explicit formula we can check $\sigma_x(s) = 0$ and $<\mathrm{Im}\, d\mu_s, x>$, so $x = 0$ and $g = 1_G$.

1) $\Rightarrow$ 2) is trivial.

3) $\Rightarrow$ 1) is a general fact: let $\zeta = (\xi, \lambda)$ and set $N = \mu^{-1}(\lambda)$. Since $d\mu_s$ is an epimorphism in a neighborhood of s $(N, \mathrm{Re}\, h)$ is a smooth riemannian manifold and $T_sN = \ker d\mu_s$ is stable by the action of I, so $\omega_I|_{T_sN}$ is a non degenerate symplectic form. Now working as in 3) $\Rightarrow$ 2) we see that $d\mu_{I\,s}|_{T_sN}$ is surjective from which 1) follows. □

Remark 1.2. We observe that for 1) $\Rightarrow$ 2) $\iff$ 3) we don't need $\widetilde{\mu}(s) \in \mathfrak{z}$ and that for 3) $\Rightarrow$ 1) it's enough that $\mu(s) \in Z_G(\mathfrak{g})$.

Definition 1.3. Let now $\mathcal{H} = \{\zeta = (\xi, \lambda) \in \mathfrak{z} \,:\, \exists\, u \in \mathbb{N}^n - \{0\}$ such that $0 \leq u_i \leq v_i$ and $<\xi\,, u^\vee>=<\lambda\,, u^\vee>= 0\}$. $\mathcal{H}$ is a union of a finite number of real subspace of $\mathfrak{z}$ of codimension 3.

Lemma 1.4. *If* $\zeta \in \mathfrak{z} - \mathcal{H}$ *and* $\widetilde{\mu}(s) = \zeta$ *then* $\mathrm{Stab}_G\{s\} = 1_G$.

Proof. For the lemma above it's enough to prove that $d\mu_s$ is an epimorphism. Let $s = (B, \gamma, \delta)$ and suppose that $d\mu_s$ it's not an epimorphism. As we saw above this is equivalent to the existence of a nontrivial $x = (X_i) \in \mathfrak{g}$ such that $\sigma_x(s) = 0$: in our case this is equivalent to the following equations:

$$X_i\gamma_i = 0 \qquad \delta_i X_i = 0 \quad \text{and} \quad X_{h_1}B_h = B_h X_{h_0}. \tag{12}$$

Now we divide the proof in two cases according to the nilpotency of the maps X_i.

First case: X_i nilpotent for each i. Since X_i^s solve the same equations we can assume that $X_i^2 = 0$. We can choose an an orthogonal decomposition $V_i = P_i \oplus Q_i \oplus R_i$ such that $P_i \simeq Q_i \simeq U_i$ and $X_i\big|_{P_i} = X_i\big|_{R_i} = 0$ and $X_i\big|_{Q_i}$ it's the identity map from $Q_i = U_i$ to $P_i = U_i$. From equations (12) it follows that respect this decomposition the maps B_h, γ_i, δ_i have the following shape:

$$B_h = \begin{pmatrix} b_h & b_h^{12} & b_h^{13} \\ 0 & b_h & 0 \\ 0 & b_h^{32} & b_h^{33} \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} \gamma_i^1 \\ 0 \\ \gamma_i^3 \end{pmatrix} \text{ and } \delta_i = \begin{pmatrix} 0 & \delta_i^2 & \delta_i^3 \end{pmatrix}$$

If $u_i = \dim_{\mathbb{C}} U_i$ and $\xi_i \mathrm{Id}_{V_i} = -2\mathrm{i}\mu_{I,i}(s)$ we have that

$$\begin{aligned}
\sum_{i=1}^n \xi_i u_i &= \sum_{i=1}^n \mathrm{Tr}\left(\xi_i \mathrm{Id}_{V_i}\big|_{P_i}\right) \\
&= \sum_{i=1}^n \mathrm{Tr}\left(\sum_{h\,:\,h_1=i} \left[B_h B_h^* - B_{\bar h}^* B_{\bar h}\right]_{P_iP_i} + \left[\gamma_i\gamma_i^* - \delta_i^*\delta_i\right]_{P_iP_i}\right) \\
&= \sum_{h\in H} \mathrm{Tr}\left(b_h b_h^* + b_h^{12} b_h^{12*} + b_h^{13} b_h^{13*} - b_{\bar h}^* b_{\bar h} + \gamma_i^1 \gamma_i^{1*}\right) \\
&= \sum_{h\in H} \mathrm{Tr}\left(b_h^{12} b_h^{12*} + b_h^{13} b_h^{13*} + \gamma_i^1 \gamma_i^{1*}\right) \geq 0
\end{aligned}$$

since $\mathrm{Tr}(\varphi\varphi^*) \geq 0$ for any φ. On the same way we see that

$$\sum_{i=1}^n \xi_i u_i = \sum_{i=1}^n \mathrm{Tr}\left(\xi_i \mathrm{Id}_{V_i}\big|_{Q_i}\right) = \sum_{h\in H} \mathrm{Tr}\left(-b_h^{12*} b_h^{12} - b_h^{32*} b_h^{32} - \delta_i^{2*}\delta_i^2\right) \leq 0,$$

so $\sum_{i\in I} \xi_i u_i = 0$. Finally if $\lambda_i = \mu_i(s)$ we have that

$$\begin{aligned}
\sum_{i=1}^n \lambda_i u_i &= \sum_{i=1}^n \mathrm{Tr}\left(\xi_i \mathrm{Id}_{V_i}\big|_{P_i}\right) = \sum_{i=1}^n \mathrm{Tr}\left(\sum_{h\,:\,h_1=i} \epsilon(h) B_h B_{\bar h} + \gamma_i\delta_i\Big|_{P_i}\right) \\
&= \sum_{i=1}^n \mathrm{Tr}\left(\sum_{h\,:\,h_1=i} \epsilon(h) B_h\Big|_{P_i} B_{\bar h}\Big|_{P_i}\right) = \sum_{h\in H} \mathrm{Tr}\left(\epsilon(h) B_h\Big|_{P_i} B_{\bar h}\Big|_{P_i}\right) = 0.
\end{aligned}$$

So if $u = (u_1, \dots, u_n)$ we have that $u \neq 0$, $0 \leq u_i \leq v_i$ and that $<\xi, u^\vee>=<\lambda, u^\vee>= 0$.

Second case: there exists i_0 such that X_{i_0} is not nilpotent. Let $\{\alpha_1, \dots, \alpha_m\}$ the set of the eigenvalues of the maps X_i and let $\alpha_1 \neq 0$. Taking a power of

X we can assume also that $\alpha_i^2 \neq \alpha_j^2$ if $i \neq j$. Let ξ_i, λ_i as in the first step and $V_i = \oplus_{\alpha \in A} V_{i,\alpha}$ the spectral decomposition of V_i. We observe that:

$$B_h(V_{h_0,\alpha}) \subset V_{h_1,\alpha}, \quad \operatorname{Im} \gamma_i \subset V_{i,0} \text{ and } V_{i,\alpha} \subset \ker \delta_i \text{ if } \alpha \neq 0.$$

We have that

$$\begin{aligned} \sum_{i \in I} \xi_i \operatorname{Tr}(X_i^2) &= \sum_{i \in I} \operatorname{Tr}\left(X_i \left(\sum_{h\,:\,h_1 = i} B_h B_h^* - B_{\bar{h}}^* B_{\bar{h}} + \gamma_i \gamma_i^* - \delta_i^* \delta_i \right) X_i \right) \\ &= \sum_{h \in H} \operatorname{Tr}\left(B_h X_{h_0} B_h^* X_{h_1} \right) - \sum_{h \in H} \operatorname{Tr}\left(X_{h_1} B_{\bar{h}}^* X_{h_0} B_{\bar{h}} \right) = 0 \end{aligned}$$

In the same way we obtain $\sum_{i \in I} \xi_i \operatorname{Tr}(X_i^{2r}) = 0$. Now let $u_{ij} = \dim(V_{i,\alpha_j})$, we have that

$$0 = \sum_{i \in I} \xi_i \operatorname{Tr}(X_i^{2r}) = \sum_{i \in I} \xi_i \left(\sum_{j=1}^m u_{ij} \alpha_j^{2r} \right) = \sum_{j=1}^m \left(\sum_{i \in I} \xi_i u_{ij} \right) \alpha_j^{2r}$$

for any $r > 0$, and since $\alpha_i^2 \neq \alpha_j^2$ if $i \neq j$ we obtain $\sum_{i \in I} \xi_i u_{i1} = 0$. Working as in the first step we obtain also $\sum_{i \in I} \lambda_i u_{i1} = 0$ and so if $u = (u_{11}, \dots, u_{n1})$ we have that $u \neq 0$, $0 \leq u_i \leq v_i$ and that $<\xi, u^\vee> = <\lambda, u^\vee> = 0$. ☐

As a consequence of the above lemma and general results on on hyperKähler manifolds (for example [**7**] or [**8**]) we obtain the following corollary.

Corollary 1.5. *If $\zeta \in \mathfrak{z} - \mathcal{H}$ then if it is not empty $\mathfrak{M}_\zeta(d, v)$ is a smooth hyperKähler manifold of real dimension $2 <2d - v, v^\vee>$.*

2. Geometric invariant theory and moment map

In this section we explain the relation between the moment map and the GIT quotient proved by Kempf, Ness [**12**], Kirwan [**20**] and others. Since I couldn't find a perfect reference for our purpose I prefere to give proofs of these facts.

Let X be an affine variety over $\mathbb{C}$ and G a reductive group acting on X. We can assume that X is a closed subvariety of a vector space V where G acts linearly. Let h be an hermitian form on V invariant by the action of a maximal compact group U of G and define a real U-invariant symplectic form on V by

$$\eta(x, y) = \operatorname{Re} h(\mathrm{i}x, y).$$

Then we can define a moment map $\mu : V \longrightarrow \mathfrak{u}^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{u}, \mathbb{R})$ as in (10):

$$<\mu(x), u> = \tfrac{1}{2} \eta(u \cdot x, x).$$

We observe that the real symplectic form η resctricted to a complex submanifold is always non degenerate and that μ restricted to the non singular locus of X is a moment map for the action of U on X.

Now let χ be a multiplicative character of G. We observe that for all $g \in U$ we have $|\chi(g)| = 1$ so $\mathrm{i}\, d\chi : u \longrightarrow \mathbb{R}$. In particular we can think to

$i\,d\chi$ as an element of $\mathfrak{u}^*$. Morover we observe that it is invariant by the dual adjoint action, hence it makes sense to consider the quotient:

$$\mathfrak{M} = \mu^{-1}(i\,d\chi)/U.$$

On the other side we can consider the GIT quotient. We remind the definition. If φ is a character of G we consider the line bundle $L_\varphi = V \times \mathbb{C}$ on V with the following G-linearization:

$$g(x,z) = (g \cdot x, \varphi(g)z).$$

An invariant section of L_φ is an algebraic function $f : V \longrightarrow \mathbb{C}$ such that $f(gx) = \varphi(g)f(x)$ for all $g \in G$ and $x \in V$. We use the same symbol L_φ also for the restriction of L_φ to X. Given a rational action of G on $\mathbb{C}$-vector space A we define

$$A_{\varphi,n} = \{a \in A \,:\, g \cdot a = \varphi^{-n}(g)a \text{ for all } g \in G\},$$

$$A_\varphi = \bigoplus_{n=0}^{\infty} A_{\varphi,n} \qquad \text{as a graded vector space.}$$

Hence we have that $H^0(X, L_\varphi)^G = \mathbb{C}[X]_{\varphi,1}$. We observe that if I is the ideal of algebraic function on V vanishing on X then

$$H^0(X, L_\varphi)^G = \frac{H^0(V, L_\varphi)^G}{I_{\varphi,1}}.$$

This last fact can be proved easily for example averaging a φ equivariant function f on X in the following way:

$$\tilde{f}(v) = \int_U \varphi^{-1}(u) f(u \cdot v)\, du.$$

Definition 1.6. A point x of X is said to be χ-*semistable* if there exist $n > 0$ and $f \in H^0(X, L_\chi^{\otimes n})^G$ such that $f(x) \neq 0$. We observe that by the remark above a point of X is χ-semistable if and only if is χ-semistable as a point of V. We call X_χ^{ss} (resp. V_χ^{ss}) the open subset of χ-semistable points of X (resp. V).

We observe that we have an isomorphism $\mathbb{Z} \simeq \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*)$ given by $m \longrightarrow \{t \mapsto t^m\}$. Hence we can define a perfect pairing $<,>: \mathrm{Hom}(\mathbb{C}^*, G) \times \mathrm{Hom}(G, \mathbb{C}^*) \longrightarrow \mathbb{Z}$ by $<\lambda, \chi> = \chi \circ \lambda$.

The following lemmas are a consequence of Hilbert-Mumford criterion and they are completely similar to the ones in [**24**].

Lemma 1.7. *1) A point x of X is χ-semistable if and only if $\overline{G(x,1)} \subset L_{\chi^{-1}}$ does not intersect $X \times \{0\} \subset L_{\chi^{-1}}$.*

2) A point x of X is χ-semistable if and only if for all one parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow G$ if there exists the limit $\lim_{t\to 0} \lambda(t) \cdot x$ then $<\lambda, \chi> \geq 0$.

LEMMA 1.8. *Let* $x, y \in X_\chi^{ss}$ *then*

1) Gx *is a closed orbit in* X_χ^{ss} *if and only if* $G(x,1)$ *is closed in* $L_{\chi^{-1}}$ *if and only if for all one parameter subgroup* $\lambda : \mathbb{C}^* \longrightarrow G$ *such that* $<\lambda, \chi>= 0$ *if there exists the limit* $\lim_{t\to 0} \lambda(t)\cdot x = y$ *then* $y \in Gx$.

2) $\overline{Gx} \cap \overline{Gy} \cap X_\chi^{ss} \neq \varnothing$ *if and only if there exists* $\alpha \in \mathbb{C}^*$ *such that* $\overline{G(x,1)} \cap \overline{G(y,\alpha)} \neq \varnothing$.

LEMMA 1.9. *There exists a good quotient of* X_χ^{ss} *by the action of* G *and we have that*

$$X_\chi^{ss}//G = \operatorname{Proj}\mathbb{C}[X]_\chi.$$

Moreover $\operatorname{Proj}\mathbb{C}[X]_\chi$ *is a finetely generated* $\mathbb{C}$*-algebra and a projective map*

$$\pi : X_\chi^{ss}//G \longrightarrow X//G = \operatorname{Spec}\mathbb{C}[X]^G.$$

In the case of $\chi \equiv 1$ the following fact is well known:

$$\operatorname{Proj}\mathbb{C}[X]_\chi = \operatorname{Spec}\mathbb{C}[X]^G = \mu^{-1}(0)/U.$$

We want generalize this result for a general χ. Our construction coincide with the one in [**12**] in the case $\chi \equiv 1$.

If v is an element of V we define a map $p_v : G \longrightarrow \mathbb{R}$:

$$p_v(g) = \|g\cdot v\|^2 - 4\log\bigl|\chi(g)\bigr|.$$

where $\|v\|^2 = h(v,v)$. It is clear that p_v is U-invariant.

LEMMA 1.10.

$$dp_v(g) = 0 \iff \mu(gv) = id\chi.$$

PROOF. Let R_g the right multiplication for g on G and let $x = y + iz$ an element of $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$.

$$\begin{aligned} dp_v(g)[R_{g*}x] &= \frac{d}{d\epsilon} h(exp(\epsilon x)g\cdot v; exp(\epsilon x)g\cdot v) - 4\log\bigl|\chi(g)\bigr|\Bigg|_{\epsilon=0} \\ &= 2\eta(z\,g\,v, g\,v) - 4 <d\chi, iz> \\ &= 4 <\mu(gv) - id\chi, z> \end{aligned}$$

and the thesis follows. □

If $z \in \mathfrak{u}$ we consider the function $a : \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$a(s) = p_v(\exp(izs)).$$

Since h is U-invariant there exist an orthonormal basis and real numbers $b_1, \ldots, b_m$ such that

$$iz = \begin{pmatrix} b_1 & 0 & \ldots & 0 \\ 0 & b_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & b_m \end{pmatrix}.$$

Hence in this basis we have the following explicit formulas:

$$a(s) = \sum_{j=1}^{m} e^{2b_j s}|v_j|^2 - 4 <d\chi, iz> s$$

$$a''(s) = 4\sum_{j=1}^{m} b_j^2 e^{2b_j s}|v_j|^2.$$

From these formulas the following lemma follows easily.

LEMMA 1.11. *1) If g is a critical point of p_v then it is a global minimum. 2) If g, g' are critical point of p_v then*

$$g' \in U \cdot g \cdot \mathrm{Stab}_G\{v\}.$$

3) $\mathrm{Stab}_G\{v\}$ *is the complexification of* $\mathrm{Stab}_U\{v\}$.

PROOF. Let's prove 2): the proof of 1) and 3) are completely similar. We can assume $g = e$ and $g' = k\,exp(iz)$ where $k \in U$ and $z \in \mathfrak{u}$. Moreover since p_v is U-invariant we can also assume $k = e$. Then we have that the function $a(s)$ defined above as a critical point in $s = 0$ and $s = 1$. Since $a'' \geq 0$ for all s it follows that $a'' \equiv 0$ between 0 and 1 . Then $b_j|v_j| = 0$ for all j. Hence $exp(iz) \in \mathrm{Stab}_G\{v\}$. □

LEMMA 1.12. *If p_v has a minimum then Gv is a closed orbit in X_χ^{ss}.*

PROOF. It is clearly enough to study the case $X = V$ and we can assume also that the minimum is obtained in $e \in G$.

First step: $v \in V_\chi^{ss}$. By absurd and lemma 1.7 suppose that $(x, 0) \in \overline{G(v, 1)} \subset L_{\chi^{-1}}$. Then there is a sequence g_n of element of G such that $g_n(v, 1) \longrightarrow (x, 0)$. Then $p_v(g_n) \longrightarrow -\infty$

Second step: if $G = T$ is a torus then the theorem is true. By lemma 1.8 we must prove that if λ is a one parameter subgroup of G such that $<\lambda, \chi>= 0$ and there exists the limit x of $\lambda(t) \cdot v$ for $t \to 0$ then $x \in Gv$. First of all I observe that in T there is a unique maximal compact subgroup $U = \{g \in T : \overline{\{g^n\}}$ is compact $\}$. Hence $\lambda(S^1) \subset U$ and there exist an othonormal basis of V and integers number $b_1, \ldots, b_m$ such that

$$\lambda(t) = \begin{pmatrix} t^{b_1} & 0 & \ldots & 0 \\ 0 & t^{b_2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & t^{b_m} \end{pmatrix}.$$

Since there exists the limit $x = \lim_{t\to 0} \lambda(t) \cdot v$ then $b_i \geq 0$ if the i-component of v is different from 0, Moreover if $x \neq v$ there exist i such that the i-component of v is different from 0 and $b_i > 0$. Hence $h(x, x) < h(v, v)$ and $lim_{t\to 0} p_v(\lambda(t)v) < p_v(e)$ against the minimality of p_v in e.

We can now prove the theorem in the general case. We use the Cartan decomposition $G = UTU$ where T is an algebraic maximal torus in G that is the complexification of a maximal compact subtorus of U. By absurd and lemma 1.8 we can assume that $G(x,\alpha)$ is the closed orbit in $\overline{G(v,1)} \subset L_{\chi^{-1}}$ and that $(x,\alpha) \notin G(v,1)$. Since $G(x,\alpha)$ is closed there exists a T closed orbit in $L_{\chi^{-1}}$ contained in $G(x,\alpha)$. Without loss of generality we can assume that this is precisely the orbit of (x,α). Moreover by the first step we have $\alpha \neq 0$.

Now observe that by U-invariance of p_v for all $z \in Uv$ the function p_z has a minimum in e. By the result claimed in the second step $T(z,\beta)$ is a closed orbit in $L_{\chi^{-1}}$ for all $\beta \neq 0$. Then for all $\tilde{z} \in U(v,1)$ there exists $f_{\tilde{z}} \in \mathbb{C}[L_{\chi^{-1}}]^T$ such that $f_{\tilde{z}}(\tilde{z}) = 1$ and $f_{\tilde{z}}(w,\alpha) = 0$. Let set

$$U_{\tilde{z}} = \left\{\tilde{y} \in L_{\chi^{-1}} \,:\, \left|f_{\tilde{z}}(\tilde{y})\right| > \tfrac{1}{2}\right\}.$$

The collection of open sets $\{U_{\tilde{z}}\}$ covers the compact set $U(v,1)$. Then there exist $\tilde{z}_1,\dots,\tilde{z}_N \in U(v,1)$ such that $U(v,1) \subset U_{\tilde{z}_1} \cup \dots \cup U_{\tilde{z}_N}$. Let now $f(\tilde{y}) = \sum_{i=1}^m \left|f_{\tilde{z}_i}(\tilde{y})\right|$ then by T-invariance $|f(\tilde{y})| > \frac{1}{2}$ on $TU(v,1)$ and $f(w,\alpha) = 0$. Then

$$\overline{TU(v,1)} \cap G(w,\alpha) = \varnothing$$

and since the action $U \times V \longrightarrow V$ is proper and $G(w,\alpha)$ is obviously U stable it follows that

$$\overline{G(v,1)} \cap G(w,\alpha) \subset U\,\overline{TU(v,1)} \cap G(w,\alpha) = \varnothing$$

and we have obtained an absurd. □

Lemma 1.13 (Neeman [**23**]). *Let us consider the map* $\varphi\colon V \longrightarrow V//G \times \mathbb{R}$ *given by*

$$\varphi(v) = (\pi(v), |\mu(v)|).$$

The map φ *is proper.*

Proof. Let $f_1,\dots,f_r$ be homogenous generators of the algebra $\mathbb{C}[V]^G$ of positive degree. Define the map $f = (f_1,\dots,f_r)$ between V and $\mathbb{C}^r$ and the map $\psi = (f,\mu)\colon V \longrightarrow \mathbb{C}^r \times \mathbb{R}$. It is clearly enough to prove that ψ is proper. By absurd suppose that there exists a sequence x_n such that $\|x_n\| \longrightarrow +\infty$ and $\psi(x_n) \longrightarrow y$. Let $y_n = x_n/\|x_n\|$. Since $\|y_n\| = 1$ for all n we can assume (eventually we take a subsequence) that $y_n \longrightarrow y$. By the homogeneity of μ and of the polinomials f_i we have that $\psi(y) = \lim \psi(y_n) = 0$. Hence $\psi(y) = 0$ and by standard properties of $V//G$ we have that $0 \in \overline{Gy}$. Moreover lemmas 1.10,1.12 $\mu(y) = 0$ implies that Gy is a closed orbit in $V_1^{ss} = V$. Hence $\|y\| = 1$, and Gy is a closed orbit in V and $0 \in \overline{Gv}$ which is cleraly an absurd. □

Proposition 1.14. *The inclusion* $\mu^{-1}(\mathrm{id}\chi) \subset X_\chi^{ss}$ *induces a closed embedding*

$$\mu^{-1}(\mathrm{id}\chi)/U \hookrightarrow X_\chi^{ss}//G.$$

Proof. By lemmas 1.10 and 1.11 $Z = \mu^{-1}(\mathrm{id}\chi) = \{x \in X \,:\, p_x$ has a minimum in $e\}$. Moreover by lemma 1.12 $Z \subset \{x \in X_\chi^{ss} \,:\, Gx$ is closed in

$X_\chi^{ss}\}$. Then there exist continous maps π, ψ such tha the following diagram commute:

$$\begin{array}{ccc} Z & \subset & X_\chi^{ss} \\ {\scriptstyle p_0}\downarrow & \searrow^{\pi} & \downarrow{\scriptstyle p_1} \\ Z/U & \xrightarrow{\ \psi\ } & X_\chi^{ss}//G \end{array} .$$

Now we observe that by lemma 1.13, and the fact that the projection the maps π, ψ are proper. So it is enough to prove that ψ is injective.

Let x, y be elements of Z such that $\pi(x) = \pi(y)$. Then $\overline{Gx \cap Gy} = \overline{Gx} \cap \overline{Gy} \cap X_\chi^{ss} \neq \varnothing$. Then $y = gx$ and the function p_x has a minimum in g and by lemma 1.11 $y \in Ux$. Hence $p_0(x) = p_0(y)$.

□

Remark 1.15. In the case $\chi \equiv 1$ it is easy to prove that the map $\varphi : Z/U \longrightarrow X_\chi^{ss}//G$ is surjective. In general the following are equivalent:

(1) φ is surjective,
(2) if Gv is a closed orbit in X_χ^{ss} then p_v has a minimum,
(3) if $v \in X_\chi^{ss}$ then $\mathrm{i}\, d\chi \in \overline{\mu G v}$.

In the case of the 1-dimensional torus it is easy to prove the general statement but I was not able to prove the surjectivity in general.

3. Quiver varieties as algebraic varieties

In this section following [**22**] we introduce the varieties $M^0(d,v)$, $M(d,v)$ and $M^-(d,v)$ that will be the main object of study in this thesis. More in general for $\chi\colon G \to \mathbb{C}^*$ a character and for $\lambda \in Z = Z_G(\mathfrak{g})$ we will define a variety $M_{\chi,\lambda}(d,v)$ and we will set

$$M^0 = M_{1,0} \quad M = M^+ = M_{\det,0} \text{ and } M^- = M_{\det^{-1},0},$$

where det is the character of G defined by $\det(g_i) = \prod \det(g_i)$. As we will shortly explain the construction of $M_{\chi,\lambda}$ is an algebraic version of the construction described in section 1.7.

Definition 1.16. For $\lambda \in Z = Z_G(\mathfrak{g})$ and χ a character of G we define

$$\Lambda_\lambda(d,v) = \mu^{-1}(\lambda) \subset S \text{ with reduced structure,}$$

$$R_{\chi,\lambda,n} = \{f \in \mathbb{C}[\Lambda_\lambda] \,:\, f(gs) = \chi^n(g) f(s)\} \text{ and } \quad R_{\chi,\lambda} = \bigoplus_{n=0}^{\infty} R_{\chi,\lambda,n},$$

$$M_{\chi,\lambda}(d,v) = M_{\chi,\lambda} = \operatorname{Proj} R_{\chi,\lambda} = \big(\Lambda_\lambda\big)_\chi^{ss}//G,$$

$$\pi_{\chi,\lambda} : M_{\chi,\lambda} \longrightarrow M_{1,\lambda} = \Lambda_\lambda//G.$$

Also in this case will be convenient to define $M_{\chi,\lambda} = \varnothing$ if $d, v \in \mathbb{Z}^n$ and there exists i such that $v_i < 0$ or $d_i < 0$.

3.1. The coordinate ring of M^0. As we have just seen M^0 is the affine variety $\Lambda_0//G$. A set of generator of its coordinate ring was given by Lusztig in [**18**] theorem 1.3. In this section we describe his result and we introduce some notation.

DEFINITION 1.17. A *B-path* α in our graph is a sequence $h^{(m)}\dots h^{(1)}$ such that $h^{(i)}\in H$ and $h^{(i)}_1 = h^{(i+1)}_0$ for $i=1,\dots,m-1$. We define also $\alpha_0 = h^{(1)}_0$, $\alpha_1 = h^{(m)}_1$ and we say that the degree of α is m. If $\alpha_0=\alpha_1$ we say that α is a closed B-path. The product of path is defined in the obvious way.

An *admissible path* $[\beta]$ in our graph is a sequence $[i_{m+1}^{r_{m+1}}\alpha^{(m)}i_m^{r_m}\dots\alpha^{(1)}i_1^{r_1}]$, that we write between square brackets such that $i_j\in I$, $\alpha^{(j)}$ are B-path, $r_j\in\mathbb{N}$ and $\alpha^{(j)}_0 = i_j$ and $\alpha^{(j)}_1 = i_{j+1}$ for $j=1,\dots,m$. We consider also the "empty" admissible paths indiced by elements of I: $[\varnothing_i]$. We define $[\beta]_0 = i_1$, $[\beta]_1 = i_{m+1}$ and $[\varnothing_i]_0 = [\varnothing_i]_1 = i$. The degree of $[\beta]$ is $2+\sum_{j=1}^{m+1} r_j + \sum_{j=1}^{m} degree(\alpha^j)$ and the product of paths is defined by:

$$[\beta]\cdot[\beta'] = \begin{cases} 0 & \text{if } [\beta']_1 \neq [\beta]_0 \\ [\beta i \beta'] & \text{if } [\beta']_1 = [\beta]_0 = i \end{cases}$$

Given a B-path $\alpha = h^{(m)}\dots h^{(1)}$ and an admissible path $\beta = [i_{m+1}^{r_{m+1}}\alpha^{(m)}\dots \dots i_1^{r_1}]$ we define an evaluation of α and β on S in the following way: if $s=(B,\gamma,\delta)\in S$ then

$$\begin{aligned} \alpha(s) &= B_{h^{(m)}}\circ\dots\circ B_{h^{(1)}} \in \operatorname{Hom}(V_{\alpha_0},V_{\alpha_1}) \\ \beta(s) &= \delta_{i_{m+1}}\circ(\gamma_{i_{m+1}}\circ\delta_{i_{m+1}})^{r_{m+1}}\circ\alpha^{(m)}(s)\circ(\gamma_{i_m}\circ\delta_{i_m})^{r_m}\circ\dots\circ \\ &\qquad \circ\dots\circ\alpha^{(1)}(s)\circ(\gamma_{i_1}\circ\delta_{i_1})^{r_1}\circ\gamma_{i_1}\in\operatorname{Hom}(D_{\beta_0},D_{\beta_1}). \end{aligned}$$

For this reason sometimes instead of writing $\alpha = h^{(m)}\dots h^{(1)}$ we will write $\alpha = B_{h^{(m)}}\cdots B_{h^{(1)}}$ and if $\beta=[i_{m+1}^{r_{m+1}}\dots i_1^{r_1}]$ we will write $\delta_{i_{m+1}}(\gamma_{i_{m+1}}\delta_{i_{m+1}})^{r_{m+1}}\dots\gamma_{i_1}$.

The algebra $\mathcal{R}$ is the vector space spanned by the admissible path with the product induced by the product of path described above. Finally once $\lambda=(\lambda_1,\dots,\lambda_n)$ is fixed we define the associative *algebra $\mathcal{R}_\lambda = \mathcal{R}/\mathcal{I}_\lambda$ of admissible polynomials* where $\mathcal{I}_\lambda$ is the bisided ideal generated by the elements $[\alpha\theta_i\alpha'] - \lambda_i[\alpha\alpha']$ where α,α' are B-path such that $\alpha_0 = i = \alpha'_1$ and

$$\theta_i = \sum_{h\,:\,h_1=i}\epsilon(h)[h\bar{h}] + [i].$$

If f is an element of $\mathcal{R}$ or $\mathcal{R}_\lambda$ and there exist $i,j\in I$ such that:

$$f = \sum_{\beta\,:\,\beta_0=i \quad \beta_1=j} a_\beta[\beta]$$

we say that f is of type (i,j).

Remark 1.18. We observe that the evaluation on S is a morphism of algebra from $\mathcal{R}$ to the algebra defined by the morphisms of the category of vector spaces. We observe also that the evaluation of $\mathcal{R}_\lambda$ on elements of Λ_λ is well defined. Moreover if f is of type (i, j) we observe that $f(s) \in \mathrm{Hom}(D_i, D_j)$.

Theorem 1.19 (Lusztig, [**18**] theorem 1.3). *The ring $\mathbb{C}[S]^G$ is generated by the polynomials:*

$$s \longmapsto \mathrm{Tr}\,(\alpha(s)) \qquad \text{for } \alpha \text{ a closed } B\text{-path}$$
$$s \longmapsto \varphi\,(\beta(s)) \qquad \text{for } \beta \text{ an admissible path and } \varphi \in \big(Hom(D_{\beta_0}, D_{\beta_1})\big)^*$$

3.2. GIT description of $M_{\chi,\lambda}$. Given a character χ of G we say that a point $s \in S$ is χ-*semistable* if there exist $n > 0$ and $f \in R_{\chi,\lambda,n}$ such that $f(s) \neq 0$. If λ is a central element λ we define:

$$\Lambda_{\chi,\lambda} = \{s \in \Lambda_\lambda \,:\, s \text{ is } \chi - \text{semistable}\}$$

For $\chi = det, det^{-1}$ we will use the following notation:

$$\Lambda^+_\lambda = \Lambda_{\det,\lambda} \text{ and } \Lambda^-_\lambda = \Lambda_{\det^{-1},\lambda}.$$

As we explained in section 2 we have that $M_{\chi,\lambda}$ is a good quotient of the set $\Lambda_{\chi,\lambda}$ of χ-semistable points of Λ_λ. In this section we want to give a more explicit description of these points. We call

$$p_{\chi,\lambda} \colon \Lambda_{\chi,\lambda} \longrightarrow M_{\chi,\lambda}$$

the quotient map and we set $p^0 = p_{1,p}$, $p = p^+ = p_{\det,0}$ and $p^- = p_{\det^{-1},0}$.

Definition 1.20. Let $s = (B, \gamma, \delta) \in S$.

Let for each $i \in I$ $U_i \subset V_i$ a linear subspace. We say that $U = (U_1, \dots, U_n)$ is B-stable if $B_h(U_{h_0}) \subset U_{h_1}$.

We define $V^+ = V^+(s)$ as the smaller B-stable subspace of V containing $\mathrm{Im}\,\gamma$. It is easy to see that

$$V^+_i = \sum_{\substack{\alpha \text{ a } B-\text{path} \\ \alpha_1 = i}} \mathrm{Im}\,\alpha(s)\gamma_{\alpha_0}.$$

We define define $V^- = V^-(s)$ as the bigger B-stable subspace of V contained in $\ker\delta$. It is easy to see that

$$V^-_i = \ker\Big\{ \prod_{\substack{\alpha \text{ a } B-\text{path} \\ \alpha_0 = i}} \delta_{\alpha_1}\alpha(s) \colon V_i \longrightarrow \prod_{\substack{\alpha \text{ a } B-\text{path} \\ \alpha_0 = i}} D_{\alpha_1} \Big\}.$$

We say that s is stable if its G-orbit is closed in S and $Stab_G\{s\} = \{1\}$ and we define

$$\Lambda^{reg}_\lambda = \{s \in \Lambda_\lambda \,:\, s \text{ is stable }\}.$$

We say that s is $+$stable if $V^+ = V$.

We say that s is $-$stable if $V^- = 0$.

LEMMA 1.21. *Let* $s = (B, \gamma, \delta) \in S$ *then:*
1) s *is* $+$*stable* $\iff$ s *is det-semistable,*
2) s *is* $-$*stable* $\iff$ s *is* det^{-1}*-semistable,*
3) s *is stable* $\iff$ s *is* $+$ *and* $-$*stable,*
4) If s *is* $+$ *or* $-$ *stable then* $\mathrm{Stab}_G\{s\} = 1$

DEFINITION 1.22. Let $H = \{\lambda \in Z \,:\, \exists u \in \mathbb{N}^n - \{0\}$ such that $0 \leq u_i \leq v_i$ and $<\lambda\,, u^{\vee}>= 0\}$. H is a union of a finite number of complex hyperplanes of $Z = Z_G(\mathfrak{g})$.

As for the hyperKähler case it follows that if $\lambda \in Z - H$ then $M^+_{\lambda}(d, v)$ and $M^-_{\zeta}(d, v)$ are two smooth variety of complex dimension

PROPOSITION 1.23 ([**21, 22**]). *1) Let* $s = (B, \gamma, \delta) \in \Lambda_{\lambda}$ *then* Gs *is a closed orbit in* S *if and only if there exists a deconposition* $V = \bigoplus_{j=0}^{m} V^{(j)}$ *such that*

(1) $V^{(j)}$ *is* B*-stable for any* j*,*
(2) $\mathrm{Im}\,\gamma \subset V^{(0)}$ *and* $V^{(j)} \subset \ker \delta$ *for* $j \neq 0$*,*
(3) $(B\big|_{V^{(0)}}, \gamma, \delta\big|_{V^{(0)}}) \in \Lambda^{reg}_{\lambda}(d, v^{(0)})$*,*
(4) $s_j = (B\big|_{V^{(j)}}, 0, 0) \in \Lambda_{\lambda}(0, v^{(j)})$ *for* $j \neq 0$ *and describes a closed orbit in* $\Lambda_{\lambda}(0, v^{(j)})$*, moreover*
  (a) $\mathrm{Stab}_{G(v^{(j)})}\{s_j\} = \mathbb{C}^*$*,*
  (b) $\sum_{h\,:\,h_1 = i} \mathrm{Im}\, B_h = V^{(j)}_i$ *for any* $i \in I$*.*

2) Let $s = (B, \gamma, \delta) \in \Lambda^+_{\lambda}$ *and* $s' \in \Lambda_{\lambda}$ *and suppose that* Gs' *is closed in* S*, then* $\pi(p(s)) = p^0(s')$ *if and only if there exist* $g \in GL(v)$ *and a* B*-stable filtration with respect to* s*:* $0 = V^{(m)} \subset V^{(m-1)} \subset \ldots V^{(1)} \subset V^{(0)} = V$ *such that* $V^{(1)} \subset \ker \delta$ *and* $s' = g \cdot gr(s)$*, where* $gr(s) \in \Lambda_{\lambda}$ *is defined in the following way:*

$$gr(s) = (s^{(0)}, s^{(1)}, \ldots, s^{(m-1)}) \in \Lambda^{reg}_{\lambda}(d, V^{(0)}/V^{(1)}) \times \prod_{j=1}^{m-1} \Lambda_{\lambda}(0, V^{(j)}/V^{(j+1)})$$

and $s^{(0)}, s^{(j)}$ *are enduced by* s*.*

REMARK 1.24. In the case of graph of finite type it is not difficult to see that the only closed orbit in $\Lambda_0(0, v)$ is 0 (see [**18**] or [**22**]). The same result is true also for $\Lambda_{\lambda}(0, v)$ (see [**19**]).

REMARK 1.25 (De Concini). In the case $d = 0$ we can look at the space $S(0, v)$ as representation of the graph (I, H). It is easy to see that Gs is a closed orbit if and only if s is the direct sum of simple representation of this graph.

3.3. HyperKähler description of $M_{\chi,\lambda}$**.** If we apply the results of section 2 we obtain:

$$\mathfrak{M}_{0,\lambda} \simeq M_{1,\lambda},$$
$$\mathfrak{M}_{id\chi,\lambda} \hookrightarrow M_{\chi,\lambda}.$$

Moreover proposition 3.5 in [**21**] proved that we have isomorphisms also in the case of $\chi = det$ and $\chi = det^{-1}$.

4. Weyl group action and reduction to the dominant case

It is easy to check that if $v \in Q$ and $d \in \widetilde{P}$ then for all $w \in W$ we have that $w(v-d)+d \in Q$. So it makes sense to consider the variety $\mathfrak{M}_{w\zeta}(d, w(v-d)+d)$. In [**21**] §9 Nakajima proved the following theorem in the case of graph of finite type (we say that a graph is of finite type if C is the Cartan matrix of a Dynkin diagram of type A,D,E).

THEOREM 1.26. *If $\zeta \in \mathfrak{z} - \mathcal{H}$ and $w \in W$ then there exists an isomorphism of differential manifolds:*

$$\Phi_{w,\zeta} \colon \mathfrak{M}_\zeta(d,v) \longrightarrow \mathfrak{M}_{w\zeta}(d, w(v-d)+d).$$

Moreover $\Phi_{w',w\zeta} \circ \Phi_{w,\zeta} = \Phi_{w'w,\zeta}$.

The proof of Nakajima is purely analytic but he suggests that the same result can be achieved for a general graph using reflections functors. These methods were also used by Lusztig in [**19**] to obtain a very similar result.

4.1. Reduction to the dominant case for M^0. As we saw above we can reduce the study of topological properties of $M(d,v)$ to the case $d-v$ dominant. Although we don't have an action of the Weyl group in the case of M^0, is still true that we can reduce the study of M^0 to the dominant case. Moreover in this case our construction will be algebraic and not only of C^∞-manifold.

To prove this result we consider the following general construction: let $v' \leq v$ (that is $v'_i \leq v_i$ for each i) and fix an embedding $V'_i \hookrightarrow V_i$ and a complement U_i of V' in V_i, then we can define a map $\widetilde{\jmath} \colon S(d,v') \longrightarrow S(d,v)$ through:

$$\widetilde{\jmath}(B', \gamma', \delta') = \left(\begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \gamma' \\ 0 \end{pmatrix}, \begin{pmatrix} \delta' & 0 \end{pmatrix} \right) \tag{13}$$

where the matrices of the new triple represents the maps described through the decomposition $V_i = V'_i \oplus U_i$. It is easy to see that this map enduces a map $\jmath_v^{v'} = \jmath \colon M^0(d,v') \longrightarrow M^0(d,v)$.

LEMMA 1.27. *$\jmath$ is a closed immersion*

PROOF. We prove that the map $\jmath^\sharp : \mathbb{C}[\Lambda_0(d,v)]^{G(v)} \longrightarrow \mathbb{C}[\Lambda_0(d,v')]^{G(v')}$ is surjective. By proposition 1.19 this follows by the following two identities:

$$\operatorname{Tr}(\alpha(\jmath(s))) = \operatorname{Tr}(\alpha(s)) \quad \text{and} \quad \beta(\jmath(s)) = \beta(s)$$

for each B-path α and for each admissible path β. □

LEMMA 1.28. *If $2v_i > d_i + \sum_{j \in I} a_{ij} v_j$ and $v' = v - \alpha_i$ then $\jmath$ is an isomorphism of algebraic varieties*

PROOF. It's enough to prove that $\jmath$ is surjective. Let $s = (B, \gamma, \delta) \in \Lambda_0(d, v)$ and consider the sequence (see (5) for the notation) :

$$T_i \xrightarrow{b_i} V_i \xrightarrow{a_i} T_i.$$

Since $b_i a_i = 0$ and $2 \dim V_i > \dim T_i$ we have that b_i is not surjective or that a_i is not injective.

Suppose that b_i is not surjective, then up to the action of $G(V)$ we can assume that $\operatorname{Im} b_i \subset V_i'$. Then, for $t \in \mathbb{C}^*$ consider $g_t = (g_{j,t}) \in G(V)$ with

$$g_i = \begin{pmatrix} \mathrm{Id}_{V_i'} & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{and} \quad g_j = \mathrm{Id}_{V_j} \text{ for } j \neq i.$$

Then

(1) $g_{i,t} B_h = B_h$ if $h_1 = i$ and $g_i \gamma_i = \gamma_i$, since $\operatorname{Im} B_h, \operatorname{Im} \gamma_i \subset \operatorname{Im} b_i \subset V_i'$,
(2) $\exists \lim_{t \to 0} B_h g_{i,t}^{-1} = B_h$ if $h_0 = i$ and $\delta_i g_i^{-1} = \delta_i$

So $\exists \lim_{t \to 0} g_t s = s'$ and it is clear that $s' \in \widetilde{\jmath}(\Lambda_0(d, v'))$ and that $p_0(s) = p_0(s') \in \operatorname{Im} \jmath$.

If b_j is surjective and a_i is not injective the argument is similar. □

Applyng this lemma repetedly we obtain the following result.

PROPOSITION 1.29. *$\forall v\ \exists v'$ such that $M^0(d, v) \simeq M^0(d, v')$ and $d - v'$ is dominant.*

REMARK 1.30. This result seems to be new.

5. Nakajima's construction

In this section we describe Nakajima's construction of integrable highest weight representation of the algebra $\widetilde{U}$. All the results of this section are due to Nakajima [**22**].

5.1. Some remark on M^0. We observe that if $v \leq v' \leq v''$ then the embeddings $j^v_{v'} : M^0(d, v) \hookrightarrow M^0(d, v')$ defined in section 4.1 satisfy $j^{v'}_{v''} \circ j^v_{v'} = j^v_{v''}$. So, at least as a set we can define:

$$M^0(d) = \varinjlim_{v} M^0(d, v).$$

We observe that in the case of graphs of finite type (A,D,E), as a consequence of lemma 1.28 there exists v such that $M^0(d, v) = M^0(d)$. In the general case although this limit has not a structure of an algebraic variety we will use for example the notation $M(d, v) \times_{M^0(d)} M(d, v')$ to mean $M(d, v) \times_{M^0(d, v'')} M(d, v')$ with $v'' \geq v, v'$.

We observe also that working as in [**22**] Lemma 3.27 it is easy to prove that $\forall\, d$ and $\forall x \in M^0(d)$ there exists a smallest v such that $x \in M^0(d, v)$. We call this minimal dimension vector $vmin(x)$.

If we fix d, v we define also $M^0_{reg}(d, v)$ as the (geometric!) quotient of Λ_0^{reg} by G. It is an open set (possibly empty) of $M^0(d, v)$. We define also $M^0_{sreg}(d) = \bigcup_v M^0_{reg}(d, v)$ and we observe that the last union is a disjoint union and that more precisely: $x \in M^0_{sreg}(d) \iff x \in M^0_{reg}(d, vmin(x))$. By remark 1.24 in the case of a graph of finite type we have $M^0(d) = M^0_{sreg}(d)$.

5.2. The convolution algebra. We fix the vector d. Given v and v' we define:

$$\Delta(d,v) \text{ is the diagonal in } M(d,v) \times M(d,v),$$
$$Z(d,v,v') = M(d,v) \times_{M^0(d)} M(d,v'),$$
$$Z_{reg}(d,v,v') = \overline{Z - \overline{\{(x_1,x_2) \in Z \,:\, \pi(x_1) \notin M_{sreg}(d)\}}},$$

using the convention explained in the previous section.

Definition 1.31. Since we can have an infinite number of components, in order to define the convolution algebra $H_*(Z)$ some special care has to be taken. We define

$$H_*(Z(d)) = \widetilde{\prod_{v,v'}} H_*(Z(d,v,v'))$$

where $\widetilde{\prod}_{v,v'} A(v,v') = \{ (a_{v,v'}) \in \prod_{v,v'} A(v,v') \,:\, \forall\, v \; a_{v,v'} = 0$ for all but finitely many v' and $\forall\, v' \; a_{v,v'} = 0$ for all but finetely many $v\}$. If $\alpha \in H_i(Z(d,v,v'))$ and $\beta \in H_i(Z(d,v',v''))$ we can define

$$\alpha * \beta = p_{13\,*}\,(p^*_{12}(\alpha) \cap p^*_{23}(\beta)) \in H_{i+j-2\dim M(d,v')}(Z(d,v,v''))$$

with the usual convention (see for example [**2**]). We observe now that

$$(\alpha_{v,v'})_{v,v'} * (\beta_{v,v'})_{v,v'} = \left(\sum_{v''} \alpha_{v,v''} * \beta_{v'',v'} \right)_{v,v'} \tag{14}$$

defines an associative algebra structure with unity on $H_*(Z(d))$.

If $x \in M^0(d,v)$ we define $M(d,v)_x = \pi^{-1}(x) \subset M(d,v)$ and

$$H_*(M(d)_x) = \bigoplus_v H_*(M(d,v)_x)$$

and we observe that the usual convolution

$$H_i(Z(d,v,v')) \times H_j(M(d,v')_x) \longrightarrow H_{i+j-2\dim M(d,v')}(M(d,v)_x)$$

extend to an action of $H_*(Z(d))$ on $H_*(M^+(d)_x)$. In the case $x = 0$ we will use $L(d,v)$ for the fiber $M(d,v)_0$. We define also

$$H_{top}(Z(d)) = \widetilde{\prod_{v,v'}} H_{\dim M(d,v) + \dim M(d,v')}(Z(d,v,v'))$$
$$H_{top}(M(d)_x) = \bigoplus_v H_{\dim M(d,v) - \dim M(d,vmin(x))}(M(d,v)_x)$$
$$H_{top-i}(M(d)_x) = \bigoplus_v H_{\dim M(d,v) - \dim M(d,vmin(x)) - i}(M(d,v)_x)$$

Remark 1.32. If $x \notin M_{sreg}(d)$ then the right definition of *top* should be different from the one given above (but in this thesis we are not interested in this case).

PROPOSITION 1.33 ([**22**]). *1) Let* $x \in M^0_{reg}(d,v^0) \subset M^0(d,v)$ *then if it is not empty* $M(d,v)_x$ *is of pure dimension* $\frac{1}{2}(\dim M(d,v) - \dim M(d,v^0))$ *and it is a lagrangian subvariety of* $M(d,v)$.

2) Z *is lagrangian.*

3) $Z_{reg}(d,v,v')$ *is of pure dimension* $\frac{1}{2}(\dim M(d,v) + \dim M(d,v'))$.

4) $L(d,v) \times L(d,v') \subset Z_{reg}(d,v,v')$.

5) $\Delta(d,v)$ *the diagonal of* $M(d,v) \times M(d,v)$ *is a component of* $Z(d,v,v)$ *of dimension* $\dim M(d,v)$ *and* $\mathbf{1}_d = \sum_v [\Delta(d,v)]$ *is the unit element of* $H_*(Z(d))$.

6) $H_{top}(Z(d))$ *is a subalgebra of* $H_*(Z(d))$ *and* $H_{top}(M(d))$ *and* $H_{top-i}(M(d))$ *are* $H_{top}(Z(d))$*-submodules of* $H_*(M(d))$.

5.3. The action of the enveloping algebra. To define the action of $\mathbf{U}$ or $\widetilde{\mathbf{U}}$ we have to define some special subvarieties of $Z(d)$ called *Hecke correspondences*. In the following we fix the vector d.

Let $v' = v - \alpha_i$ and set

$$P_i(d,v) = \Big\{ \big((B',\gamma',\delta'),(B,\gamma,\delta)\big) \in M(d,v') \times M(d,v) \,:\, \exists \varphi_j : V_j \longrightarrow V'_j \\ \text{such that } \varphi_{h_1} B_h = B'_h \varphi_{h_0} \quad \varphi_j \gamma_j = \gamma'_j \quad \delta_j = \delta'_j \varphi_j \Big\}$$

$$\widetilde{P}_i(d,v) = \big\{ (s,s') \in M(d,v) \times M(d,v') \,:\, (s',s) \in P_i(d,v) \big\}.$$

LEMMA 1.34. $P_i(d,v)$ *is a closed nonsingular lagrangian subvariety of* $M(d,v-\alpha_i) \times M(d,v)$ *contained in* $Z(d,v,v')$.

We define

$$E_i = \sum_v [P_i(d,v)] \in H_{top}(Z(d))$$

$$F_i = \sum_v (-1)^{<d-v,\alpha_i^\vee>+1} [\widetilde{P}_i(d,v)] \in H_{top}(Z(d))$$

for $i \in I$ and

$$A_\lambda = \begin{cases} [\Delta(d,v)] & \text{if } v = d - \lambda \in Q \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda \in X$.

THEOREM 1.35. *There exists a unique algebra homomorphism* $\mathbf{U} \longrightarrow H_{top}(Z(d))$ *such that :*

$$e_i \longmapsto E_i \qquad f_i \longmapsto F_i \qquad h \longmapsto \sum_v <d-v,h> [\Delta(d,v)]$$

for $i \in I$ *and* $h \in \mathfrak{h}$.

THEOREM 1.36. *There exists a unique algebra homomorphism* $\widetilde{\mathbf{U}} \longrightarrow H_{top}(Z(d))$ *such that :*

$$e_i \mathbf{1}_\lambda \longmapsto E_i * A_\lambda \qquad \mathbf{1}_\lambda f_i \longmapsto A_\lambda * F_i \qquad \mathbf{1}_\lambda \longmapsto A_\lambda$$

for $i \in I$ *and* $\lambda \in X$.

5.4. The construction of the integrable highest weight modules. By the results described in the section above, for any d, and for any $x \in M^0(d,v)$ we have an action of the algebra $\mathbf{U}$ (or $\widetilde{\mathbf{U}}$) on $H_*(M(d)_x)$. In the case $x = 0$ we call this module *Nakajima's module*. We observe that we have the following natural decompositions of this module:

$$H_*(M(d)_x) = \bigoplus_v H_*(M(d,v)_x) \quad \text{and} \quad H_*(M(d)_x) = \bigoplus_j H_{top-j}(M(d)_x).$$

The first decomposition is the weight space decomposition and more precisely $H_*(M(d,v)_x)$ is the weight space of weight $d-v$. The second decomposition is a decomposition in $\mathbf{U}$ (or $\widetilde{\mathbf{U}}$) submodules.

In [**22**] the following theorem is proved.

THEOREM 1.37. *If $x \in M^0_{reg}(d,v^0) \subset M(d)$ then $H_{top}(M(d)_x)$ is the irreducible module of heighest weight space $d - v^0$ and with highest vector $[M^0(d,v^0)_x]$.*

CHAPTER 2

The homology of quiver varieties

In this chapter I describe some results on the homology of quiver varieties.

1. The dot action

In this section I follow [**21**]. Let's set on S the following $\mathbb{C}^*$-action:

$$t \cdot (B_h, \gamma_i, \delta_i) = (t^{\frac{1-\epsilon(h)}{2}} B_h, \gamma_i, t\delta_i).$$

We call this action the *dot action*. This action commutes with the action of $GL(v)$ and leaves Λ_0 and Λ_λ stable, so it induces actions on M^0 and M commuting with the projection π. Recall that $L(d,v) = \pi^{-1}(0) \subset M(d,v)$ and denote by $Fix(d,v)$ the subvariety of the fixed points under this action. The following lemmas are easy to prove ([**21**]).

Lemma 2.1. *1)* $\forall\, p \in M^0(d,v)$ *there exists* $\lim_{t\to 0} t \cdot p = 0$.
2) For all $p \in M(d,v)$ *there exists* $\lim_{t\to 0} t \cdot p \in L(d,v)$.
3) If $p \in M^0(d,v)$ *and there exists* $\lim_{t\to\infty} t \cdot p$ *then* $p = 0$.
4) For all $p \in M(d,v)$ *there exists* $\lim_{t\to\infty} t \cdot p$ *if and only if* $p \in L(d,v)$.

Corollary 2.2. *There is a deformation retraction of* $M(d,v)$ *on* $L(d,v)$*, hence* $H_i^c(M(d,v)) = H_i^c(L(d,v))$.

We consider now the connected component $Fix_\tau(d,v)$ of the variety $Fix(d,v)$ of fixed points under the $\mathbb{C}^*$-action on $M(d,v)^{\mathbb{C}^*}$ (here τ is just an index). Since $\mathbb{C}^*$ is reductive $Fix_\tau(d,v)$ is a smooth subvariety of $L(d,v)$. Let

$$\mathcal{F}_\tau^+ = \{p \in M(d,v) \,:\, \lim_{t\to 0} t \cdot p \in Fix_\tau\},$$
$$\mathcal{F}_\tau^- = \{p \in M(d,v) \,:\, \lim_{t\to \infty} t \cdot p \in Fix_\tau\}.$$

By lemma 2.1 $\{\mathcal{F}_\tau^+\}_\tau$ and $\{\mathcal{F}_\tau^-\}_\tau$ are respectively partitions of $M(d,v)$ and $L(d,v)$ in locally closed algebraic subvarieties. If $p \in Fix_\tau$ we have a $\mathbb{C}^*$ action on $T_pM(d,v)$ and a decomposition $T_pM(d,v) = (T_pM)^- \oplus (T_pM)^0 \oplus (T_pM)^+$ in $\mathbb{C}^*$ submodules such that $\mathbb{C}^*$ acts with negative weights on $(T_pM)^-$, trivially on $(T_pM)^0$ and with positive weights on $(T_pM)^+$. Then

$$r_\tau^- = \dim (T_pM)^- \qquad r_\tau^0 = \dim (T_pM)^0 \qquad r_\tau^+ = \dim (T_pM)^+ \qquad (15)$$

are independent of the choice of p in Fix_τ and we have that

$$r^0_\tau = \dim Fix_\tau,$$
$$p^+_\tau = \lim_{t\to 0} : \mathcal{F}^+_\tau \longrightarrow Fix_\tau \text{ is a vector bundle of rank } r^+_\tau \text{ on } Fix_\tau,$$
$$p^-_\tau = \lim_{t\to \infty} : \mathcal{F}^-_\tau \longrightarrow Fix_\tau \text{ is a vector bundle of rank } r^-_\tau \text{ on } Fix_\tau.$$

We will use these partitions in order to get information on the homology of $M(d,v)$.

2. The contraction index for the dot action

In this section we will compute the numbers r^-_τ, r^0_τ and r^+_τ. We approach the problem from a more general point of view which will prove usefull later on. Let (X,ω) be a smooth complex symplectic manifold and $\sigma\colon G\times X \longrightarrow X$ a free action of a reductive group G on X, with respect to which ω is invariant and with moment map $\mu : X \longrightarrow \mathfrak{g}^*$. Let $Z = \mu^{-1}(0)$ and $p\colon Z \longrightarrow M = Z/G$ the canonical projection. Since the action is free, as we saw in lemma 1.1, Z and M are two smooth varieties of dimension $\dim X - \dim \mathfrak{g}$ and $\dim X - 2\dim \mathfrak{g}$ respectively. Suppose now that another reductive group H acts on X and that the actions of the two groups commute. If Z is H-stable then the action of H on X induces a similar action on M (hence on TM). Let now $m = p(z) \in M$ is fixed by this action. Then we have a linear action of H on T_mM which is the object of our interest. In case we can find a group homomorphism $\varphi\colon H \longrightarrow G$ such that $\varphi(h)h\cdot z = z$ for all $h\in H$, then we can define a new action $\tau\colon H\times X \longrightarrow X$ as $\tau(h,p) = \varphi(h)h\cdot p$. The H action on M induced by τ is equal to the one enduced by the original action of H. Moreover $\tau(h,z) = z$ for all h, hence we get a linear action of H on T_zX. To avoid possible confusion in the future computation we emphasize the role of φ and we write T_zX_φ to denote this representation.

LEMMA 2.3. *Let $X, Z, p, M, G, H, \omega, z, m$ as above. Let $\chi\colon H \longrightarrow \mathbb{C}^*$ be a character of H and $\varphi\colon H \longrightarrow G$ a group homomorphism such that*

(1) $\omega(h\cdot u, v) = \chi(h)\omega(u, h^{-1}\cdot v)$ *for any $h\in H$ and for any $u, v\in TM$,*
(2) $\varphi(h)h\cdot z = z$ *for any $h\in H$.*

Then in the representation ring of H we have

$$[T_mM] = [T_zX_\varphi] - [\chi\otimes\varphi * (\mathfrak{g}^*)] - [\varphi^*(\mathfrak{g})].$$

PROOF. Consider the following two exact sequences:

$$0 \longrightarrow T_zZ \longrightarrow T_zX_\varphi \xrightarrow{d\mu_z} \chi\otimes\mathfrak{g}^* \longrightarrow 0,$$
$$0 \longrightarrow \mathfrak{g} \xrightarrow{\sigma_z} T_zZ \longrightarrow T_mM \longrightarrow 0.$$

where $\sigma_z(x) = d\sigma_(1_G, z)[x,0]$. If we prove that σ_z and $d\mu_z$ are H-equivariant we have proved the lemma. Let $x \in \mathfrak{g}$, $u \in T_z X_\varphi$ and $h \in H$ then:

$$\begin{aligned}
\sigma_z(h \cdot x) &= \frac{d}{d\epsilon} \exp(\epsilon \operatorname{Ad}_{\varphi(h)} x) z\Big|_{\epsilon=0} = \\
&= \frac{d}{d\epsilon} \varphi(h) h \exp(\epsilon x) \varphi(h^{-1}) h^{-1} z\Big|_{\epsilon=0} = \varphi(h)_* h_* \sigma_z(x) \\
<d\mu_z(h(u)), x> &= \omega(\sigma_z(x), h(u)) = \omega(\sigma_z(x), \varphi(h)_* h_*(u)) = \\
&= \chi(h)\omega(\varphi(h^{-1})_* h^{-1}{}_* \sigma_z(x), u) = \chi(h)\omega(\sigma_z(h \cdot x), u) = \\
&= <\chi(h) Ad^*_{\varphi(h)} d\mu_z(h(u)), x>
\end{aligned}$$

□

2.1. Computation of the index. Now we want to use lemma 2.3 to compute the numbers $r^-_\tau, r^0_\tau, r^+_\tau$.

DEFINITION 2.4. If $\alpha = h^{(m)} \dots h^{(1)}$ is a B-path we define:

$$\ell(\alpha) = card\{j \,:\, h^{(j)} \in \overline{\Omega}\}.$$

IF $s = (B, \gamma, \delta) \in \Lambda_0^+(d, v)$, $p(s) \in Fix_\tau$, $j \in \mathbb{N}$ and $i \in I$ we define

$$V_i^{(j)} = \sum_{\substack{\alpha \text{ a } B\text{-path} \\ \ell(\alpha)=j \ \alpha_1=i}} \operatorname{Im} \alpha(s) \gamma_{\alpha_0}.$$

We observe that by the stability condition for s we have $\sum_j V_i^{(j)} = V_i$ for any i.

If $t \in \mathbb{C}^*$ then $\exists_1 g \in G(V)$ such that $t \cdot s = g \cdot s$. We observe also that if $v \in V_i^{(j)}$ then:

$$\begin{aligned}
g \cdot v = g \cdot \Bigg(\sum_{\substack{\alpha \text{ a } B\text{-path} \\ \ell(\alpha)=j \ \alpha_1=i}} \alpha(s)\gamma_{\alpha_0}(d_\alpha) \Bigg) &= \sum_{\substack{\alpha \text{ a } B\text{-path} \\ \ell(\alpha)=j \ \alpha_1=i}} \alpha(g \cdot s) g_{\alpha_0} \gamma_{\alpha_0}(d_\alpha) = \\
= \sum_{\substack{\alpha \text{ a } B\text{-path} \\ \ell(\alpha)=j \ \alpha_1=i}} \alpha(t \cdot s)\gamma_{\alpha_0}(d_\alpha) &= t^j \sum_{\substack{\alpha \text{ a } B\text{-path} \\ \ell(\alpha)=j \ \alpha_1=i}} \alpha(s)\gamma_{\alpha_0}(d_\alpha) = t^j v
\end{aligned}$$

Hence $V = \bigoplus_j V^{(j)}$ and

(1) $\operatorname{Im} \gamma \subset V^{(0)}$,
(2) $B_h(V_{h_0}^{(j)}) \subset V_{h_1}^{(j)}$ if $h \in \Omega$,
(3) $B_h(V_{h_0}^{(j)}) \subset V_{h_1}^{(j+1)}$ if $h \in \overline{\Omega}$,
(4) $\delta = 0$.

We have just proved 1), 2), 3). To prove 4) we observe that for any $s \in \Lambda_0^+$ such that $p^0(s) = 0$ we have $\delta = 0$. Indeed if $\delta_i(v_i) \neq 0$ then by the stability condition there exist elements $d_\alpha \in D_{\alpha_0}$ such that $v_i = \sum_{\alpha_1 = i} \alpha(s)\gamma_{\alpha_0}(d_{\alpha_0})$.

So we have $\delta_i(v_i) = \sum_{\alpha_1=i} \delta_i\alpha(s)\gamma_{\alpha_0}(d_{\alpha_0})$ and $\delta_i\alpha(s)\gamma_{\alpha_0} = [\alpha](s) = 0$ (since $p^0(s) = 0$). Now for $t \in \mathbb{C}^*$ we define $\varphi(t) \in G(V)$ by:

$$(\varphi(t))\Big|_{V_i^{(j)}} = t^{-j}\mathrm{Id}_{V_i^{(j)}}$$

and we observe that $\varphi \colon \mathbb{C}^* \longrightarrow G(V)$ is a group homomorphism and that

$$\varphi(t)t \cdot s = s \qquad \text{for any } t \in \mathbb{C}^*. \tag{16}$$

We also set $\chi(t) = t$ and we observe that $\omega(t \cdot s', s'') = \chi(t)\omega(s', t^{-1} \cdot s'')$.

Now we apply lemma 2.3 to this situation. In this case S is a vector space on which $G, \mathbb{C}^*$ acts lineraly and $\mathfrak{g}^*$ is canonically identified with $\mathfrak{g}$. To avoid confusion we call S_φ the representation of $\mathbb{C}^*$ defined by $\tau(h, s) = \varphi(h)h \cdot s$.

The decomposition $V = \bigoplus_j V^{(j)}$ induces the following decomposition on $S, \mathfrak{g}$:

$$\begin{aligned} S_\varphi = &\bigoplus_{i\in I,k} \mathrm{Hom}\left(D_i, V_i^{(k)}\right) \oplus \bigoplus_{i\in I,k} \mathrm{Hom}\left(V_i^{(k)}, D_i\right) \oplus \\ &\bigoplus_{h\in\Omega,k,l} \mathrm{Hom}\left(V_{h_0}^{(k)}, V_{h_1}^{(l)}\right) \oplus \bigoplus_{h\in\overline{\Omega},k,l} \mathrm{Hom}\left(V_{h_0}^{(k)} V_{h_1}^{(l)}\right) \\ \mathfrak{g} = &\bigoplus_{i\in I,k} \mathrm{Hom}\left(V_i^{(k)}, V_i^{(l)}\right) \end{aligned}$$

Moreover this decompositions result to be a weight decompositions with respect the $\mathbb{C}^*$-action and

$$\begin{array}{lll} \mathrm{Hom}\left(D_i, V_i^{(k)}\right) & \text{has weight } t^{-k} & \forall i \in I \text{ and } \forall k, \\ \mathrm{Hom}\left(V_i^{(k)}, D_i\right) & \text{has weight } t^{k+1} & \forall i \in I \text{ and } \forall k, \\ \mathrm{Hom}\left(V_{h_0}^{(k)}, V_{h_1}^{(l)}\right) & \text{has weight } t^{k-l} & \forall h \in \Omega \text{ and } \forall k, l, \\ \mathrm{Hom}\left(V_{h_0}^{(k)}, V_{h_1}^{(l)}\right) & \text{has weight } t^{k-l+1} & \forall h \in \overline{\Omega} \text{ and } \forall k, l, \\ \mathrm{Hom}\left(V_i^{(k)}, V_i^{(l)}\right) & \text{has weight } t^{k-l} & \forall i \in I \text{ and } \forall k, l. \end{array}$$

Now set $v^{(k)}$ the dimension vector of $V^{(k)}$ and:

$$\begin{aligned} a_{ij}^{\Omega} &= card\{h \in \Omega \,:\, h_0 = i \text{ and } h_1 = j\}, \quad \text{and} \quad A_\Omega = \left(a_{ij}^{\Omega}\right)_{i,j\in I}, \\ a_{ij}^{\overline{\Omega}} &= card\{h \in \overline{\Omega} \,:\, h_0 = i \text{ and } h_1 = j\}, \quad \text{and} \quad A_{\overline{\Omega}} = \left(a_{ij}^{\overline{\Omega}}\right)_{i,j\in I}. \end{aligned}$$

Applyng lemma 2.3 we obtain:

$$\begin{aligned} r^0_\tau &= \sum_{i\in I} d_i v_i^{(0)} + \sum_{h\in\Omega,k} v_{h_0}^{(k)} v_{h_1}^{(k)} + \sum_{h\in\overline{\Omega},k} v_{h_0}^{(k)} v_{h_1}^{(k+1)} - \sum_{i\in I,k} v_i^{(k)} v_i^{(k)} - \sum_{i\in I,k} v_i^{(k)} v_i^{(k+1)} \\ &= {}^t\mathrm{d}\, v^{(0)} + \sum_k {}^t\mathrm{v}^{(\mathrm{k})} \left(A_\Omega v^{(k)} + A_{\overline{\Omega}} v^{(k+1)} - v^{(k)} - v^{(k+1)}\right) \\ r^+_\tau &= {}^t\mathrm{d}\, v + \sum_{k>l} {}^t\mathrm{v}^{(\mathrm{k})} \left(A_\Omega v^{(l)} - v^{(l)}\right) + \sum_{k\ge l} {}^t\mathrm{v}^{(\mathrm{k})} \left(A_{\overline{\Omega}} v^{(l)} - v^{(l)}\right) \\ r^-_\tau &= {}^t\mathrm{d}(v - v^{(0)}) + \sum_{k<l} {}^t\mathrm{v}^{(\mathrm{k})} \left(A_\Omega v^{(l)} - v^{(l)}\right) + \sum_{k<l-1} {}^t\mathrm{v}^{(\mathrm{k})} \left(A_{\overline{\Omega}} v^{(l)} - v^{(l)}\right) \end{aligned}$$

3. A result of H.Nakajima

In a lecture given at the IAS in Princeton Nakajima proved the following result on the homology of $M(d,v)$. Here $A_i(X)$ is the Chow group of i-dimensional cycle and $cl_X\colon A_i(X) \longrightarrow H_{2i}(X)$ is the cycle map ([**5**] ch. 19).

Theorem 2.5. *1)* $H_{2i+1}\big(L(d,v),\mathbb{Z}\big) = 0$ *for all* i.
2) $cl\colon A_i(X) \longrightarrow H_{2i}(X,\mathbb{Z})$ *is an isomorphism and* $H_{2i}(X,\mathbb{Z})$ *has no torsion for all* i.

Since a proof is not yet available in literature we will prove this result. In the sequel we will need only point 1), but since the result seems to me very nice and the proof furnishes the two properties almost together we will prove also point 2). We need the following lemma of Ellingsrud and Strømme [**4**].

Lemma 2.6. *Let* X *be a smooth projective variety,* $\Delta\colon X \hookrightarrow X\times X$ *the diagonal embedding and* X_Δ *its image,* p_1, p_2 *the projection from* $X\times X$ *to* X *and* q *the projection from* X *to a point.*
Suppose there exist $\alpha_i, \beta_i \in A_*(X)$ *such that:*

$$[X_\Delta] = \sum_i p_1^*(\alpha_i)\cdot p_2^*(\beta_i) \tag{17}$$

and suppose also that $\{\alpha_i\}$ *is a set of minimal cardinality with this property, then:*

(1) α_i *generates* $A_*(X)$,
(2) $A_*(X)$ *is without torsion and* α_i *is a* $\mathbb{Z}$*-basis of* $A_*(X)$,
(3) $q_*(\alpha_i\cdot\beta_j) = \delta_{ij}$,
(4) $cl\colon A_*(X) \longrightarrow H_*(X,\mathbb{Z})$ *is an isomorphism (in particular* $H_{odd}(X,\mathbb{Z}) = 0$*).*

Proof. Let $\alpha\in A_*(X)$. We observe that

$$[X_\Delta]\cdot p_2^*\alpha = \Delta_*\Delta^* p_2^*(\alpha) = \Delta_*\Delta^* p_1^*(\alpha) = [X_\Delta]\cdot p_1^*\alpha.$$

Hence

$$\alpha = [X]\cdot\alpha = p_{1*}\Delta_*([X])\cdot\alpha = p_{1*}([X_\Delta]\cdot p_1^*(\alpha)) = p_{1*}([X_\Delta]\cdot p_2^*(\alpha)) =$$
$$= \sum_i p_{1*}(p_1^*(\alpha_i)\cdot p_2^*(\beta_i)\cdot p_2^*(\alpha)) = \sum_i \alpha_i\cdot p_{1*}(p_2^*(\beta_i\cdot\alpha)).$$

Now by base chage we observe that $p_{1*}p_2^*(\sigma) = q^*q_*(\sigma) = q_*(\sigma)[X]$ where $q_*(\sigma)\in\mathbb{Z} = A_*(\{\bullet\})$, hence

$$\alpha = \sum_i \alpha_i\cdot[X]q_*(\beta_i\cdot\alpha) = \sum_i q_*(\beta_i\cdot\alpha)\alpha_i. \tag{18}$$

So we proved 1). To prove 2) we observe that if $m\,\alpha = 0$ then $mq_*(\beta_i\cdot\alpha) = 0$ for all i hence $q_*(\beta_i\cdot\alpha) = 0$ and $\alpha = 0$ by (18). Now by the condition of minimal cardinality it follows that α_i is a $\mathbb{Z}$-basis of $A_*(X)$. Moreover applying formula (18) to $\alpha = \alpha_i$ we obtain 3).

To prove 4) we observe that if apply the cycle map to (17) we obtain

$$cl[X_\Delta] = \sum_i p_1^*(cl(\alpha_i))\cdot p_2^*(cl(\beta_i)) \tag{19}$$

and using the same argument as before we obtain that for all $\alpha\in H_*(X,\mathbb{Z})$ we have $\alpha = \sum_i q_*(cl(\beta_i)\cap\alpha)cl(\alpha_i)$. Hence $H_*(X,\mathbb{Z})$ has no torsion, $cl(\alpha_i)$ generates $H_*(X,\mathbb{Z})$ and by point 3) $q_*(cl(\alpha_i)\cap cl(\beta_j)) = \delta_{ij}$. Moreover if $\sum a_i cl(\alpha_i) = 0$ then $0 = \sum_i a_i q_*(cl(\alpha_i)\cap cl(\beta_j) = a_i$. Hence $cl(\alpha_i)$ is a basis for $H_*(X,\mathbb{Z})$ and cl is an isomorphism. □

Remark 2.7. The same result holds in K-theory.

Proof of theorem 2.5. *First step.* If $H_{2i+1}(Fix_\tau,\mathbb{Z}) = 0$ for all i and all τ, $H_i(Fix_\tau,\mathbb{Z})$ has no torsion for all i and all τ, and and $cl\colon A_i(Fix_\tau)\longrightarrow H_{2i}(Fix_\tau,\mathbb{Z})$ is an isomorphism for all i and all τ then the theorem is true.

We prove this claim by induction in the following way. We observe that by equivariance the closures of the strata $\mathcal{F}^-_\tau$ is a union of strata $\mathcal{F}^-_{\tau'}$ of smaller dimension. So we can give a complete order $\preccurlyeq$ to the index set $\{\tau\}$ in such a way that $Z_\tau = \bigcup_{\tau'\preccurlyeq\tau}\mathcal{F}^-_{\tau'}$ is closed. Since by properness of $L(d,v)$ the number of connected component of the fixed point locus is finite the claim in the first step is a consequence of the following two remarks:

(1) $p_\tau^{-*}\colon H_j(Fix_\tau,\mathbb{Z})\longrightarrow H_{j+2r^-_\tau}(\mathcal{F}^-_\tau,\mathbb{Z})$ and $p_\tau^{-*}\colon A_j(Fix_\tau)\longrightarrow A_{j+r^-_\tau}(\mathcal{F}^-_\tau)$ are isomorphisms commuting with the cycle map
(2) If Z is a closed subvarieties of Y and $U = Y - Z$ then if $H_{odd}(Z,\mathbb{Z}) = H_{odd}(U,\mathbb{Z}) = 0$, cl_U, cl_Z are isomorphisms and $H_*(Z,\mathbb{Z}) = H_*(U,\mathbb{Z})$ are without torsion then the same is true for Y.

1. is a consequence of Proposition 2.7.8 in [**11**] and Theorem 3.3.1 in [**5**]. To prove 2. we observe that if i_Z and i_U are the immersions of Z,U in Y then we have a long exact sequence: $H_i(Z,\mathbb{Z})\longrightarrow H_i(Y,\mathbb{Z})\longrightarrow H_i(U,\mathbb{Z})\longrightarrow H_{i+1}(Z,\mathbb{Z})\dots$. Hence if $H_{odd}(Z,\mathbb{Z}) = H_{odd}(U,\mathbb{Z}) = 0$ then $H_{odd}(Y,\mathbb{Z}) = 0$.

Moreover we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & A_i(Z) & \xrightarrow{i_{Z*}} & A_i(Y) & \xrightarrow{i_U^*} & A_i(U) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle cl_Z} & & \downarrow{\scriptstyle cl_Y} & & \downarrow{\scriptstyle cl_U} & & \\
0 & \longrightarrow & H_{2i}(Z,\mathbb{Z}) & \xrightarrow{i_{Z*}} & H_{2i}(Y,\mathbb{Z}) & \xrightarrow{i_U^*} & H_{2i}(U,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

from which point 2. follows.

Second step. We call $p_1, p_2 \colon M(d,v)\times M(d,v) \longrightarrow M(d,v)$ the projections on the first and the second factor. There exist an equivariant complex of $\mathbb{C}^*$-equivariant vector bundles on $M(d,v) \times M(d,v)$:

$$L^{(1)} \xrightarrow{\ \rho\ } L^{(2)} \xrightarrow{\ \psi\ } L^{(3)} \tag{20}$$

such that:

(1) $L^{(j)} = \bigoplus_k p_1^* A_k^{(j)} \otimes p_2^* B_k^{(j)}$ and $A_k^{(j)}, B_k^{(j)}$ are $\mathbb{C}^*$ equivariant vector bundles on $M(d,v)$,
(2) ρ_p is injective and ψ_p is surjective for all p, and $\psi\rho = 0$,
(3) there exists an equivariant section ν of $F = \ker\psi / \operatorname{Im}\rho$ such that $Zero(\nu)$ is the diagonal $\Delta(d,v) \subset M(d,v) \times M(d,v)$ and $d\nu_p$: $T_pM(d,v) \times T_pM(d,v) \longrightarrow F_p$ is surjective for all $p \in \Delta(d,v)$.

The existence of these object is proved in [**22**] §5.

Now we apply this fact and Lemma 2.6 to prove that $H_{odd}(Fix_\tau, \mathbb{Z}) = 0$, that $H_*(Fix_\tau, \mathbb{Z})$ has no torsion and that $cl : A_*(Fix_\tau) \longrightarrow H_*(Fix_\tau, \mathbb{Z})$ is an isomorphism.

We restrict the complex (20) to $Fix_\tau \times Fix_\tau$. If $p \in Fix_\tau \times Fix_\tau$ (resp. Fix_τ) and E is a $\mathbb{C}^*$-equivariant vector bundle on $M(d,v) \times M(d,v)$ (resp. $M(d,v)$) then there is an action of $\mathbb{C}^*$ on E_p, so it makes sense to consider the trivial part with respect to the $\mathbb{C}^*$ action of the restriction of E to $Fix_\tau \times Fix_\tau$ (resp. Fix_τ): we call this bundle E^0. Hence we have a sequence of vector bundles on $Fix_\tau \times Fix_\tau$:

$$\big(L^{(1)}\big)^0 \xrightarrow{\ \rho^0\ } \big(L^{(2)}\big)^0 \xrightarrow{\ \psi^0\ } \big(L^{(3)}\big)^0.$$

moreover we observe that

(1) $\big(L^{(j)}\big)^0 = \bigoplus_k \big(p_1^* A_k^{(j)}\big)^0 \otimes \big(p_2^* B_k^{(j)}\big)^0$,
(2) ρ_p^0 is injective and ψ_p^0 is surjective for all p, and $\psi^0\rho^0 = 0$,
(3) $\big(F\big)^0 = \ker\psi^0 / \operatorname{Im}\rho^0$
(4) $\nu\big|_{Fix_\tau \times Fix_\tau} : Fix_\tau \times Fix_\tau \longrightarrow F^0$,
(5) $Zero(s^0)$ is the diagonal Δ_τ of $Fix_\tau \times Fix_\tau$,
(6) $d\nu_0 \colon T_pFix_\tau \times T_pFix_\tau \longrightarrow F^0$ is surjective.

1, 2 , 3 are clear. To prove 4. we observe that if $p \in Fix_\tau \times Fix_\tau$ then $t \cdot \nu(p) = \nu(t \cdot p) = \nu(p)$. Now 5. follows from 4. and to prove 6. we observe that since $d\nu$ is equivariant it respects the weight decomposition: $d\nu = \bigoplus_{l\in\mathbb{Z}} d\nu_p \colon \bigoplus_{l\in\mathbb{Z}} \Big(T_pM(d,v) \oplus T_p(M(d,v))\Big)^l \longrightarrow F_p^l$ and since it is surjective it follows that: $d\nu_p\colon \Big(T_p(M(d,v)\times M(d,v))\Big)^0 = T_p(Fix_\tau \times Fix_\tau \longrightarrow F^0_p$ is also surjective.

As a consequence of 4, 5, 6 we obtain that

$$c_{top}(F^0) = [\Delta_\tau] \in A_*(Fix_\tau \times Fix_\tau).$$

As a consequence of 1, 2, 3, we obtain that there exist $\alpha_i, \beta_i \in A_*(Fix_\tau)$ such that:

$$c_{top}(F^0) = \sum_i p_1^*(\alpha_i)\cdot p_2^*(\beta_i)$$

If we apply Lemma 2.6 the Theorem follows. □

If we analyze the proof we observe that we proved the following usefull and well known result.

LEMMA 2.8. *Let $\{Y_\tau\}_\tau$ be a finite partition of the variety X in locally closed subvariety such that:*

1. $Y_{\tau'} \cap \overline{Y_\tau} \neq \varnothing \Rightarrow Y_{\tau'} \subset \overline{Y_\tau}$,
2. *for each τ there exist Z_τ a closed subvariety of Y_τ and an affine bundle omomorphism $p_\tau\colon Z_\tau \longrightarrow Y_\tau$ of rank r_τ that is the identity map on Z_τ.*

Then:

1. $H_{odd}(X,\mathbb{Z}) = 0 \iff H_{odd}(Z_\tau,\mathbb{Z}) = 0$ *for all* τ,
2. $H_{odd}(X) = 0 \iff H_{odd}(Z_\tau) = 0$ *for all* τ,
3. *If the condition in 1) is verified then $H_*(X,\mathbb{Z})$ has no torsion if and only if $H_*(Z_\tau,\mathbb{Z})$ has no torsion for all τ,*
4. *If conditions 1) and 3) are verified then*
$$H_i(X,\mathbb{Z}) = \bigoplus_\tau H_{i-r_\tau}(Z_\tau,\mathbb{Z}),$$
5. *If condition 2) is verified then*
$$H_i(X) = \bigoplus_\tau H_{i-r_\tau}(Z_\tau).$$

As a corollary of Nakajima's theorem we obtain also:

COROLLARY 2.9.

$$H_i^c(M(d,v),\mathbb{Z}) = H_i(L(d,v),\mathbb{Z}) = \bigoplus_\tau H_{i-r_\tau^-}(Fix_\tau,\mathbb{Z}),$$
$$H_i(M(d,v),\mathbb{Z}) = \bigoplus_\tau H_{i-r_\tau^+}(Fix_\tau,\mathbb{Z}).$$

4. Multiplicativity

Let $d', d'' \in \mathbb{N}^n$ and $d = d' + d''$; in this section we prove the following multiplicativity formula:

$$H_*(L(d)) = H_*(L(d')) \otimes H_*(L(d'')). \tag{21}$$

I wish to tank I. Mirkovic for suggesting to me that such a property could hold and that it should be proved using a $\mathbb{C}^*$ action.

4.1. The embedding. Let $d', d'', v', v'' \in \mathbb{N}^n$ and $d = d'+d''$, $v = v'+v''$. We fix vector spaces D_i', D_i'', V_i', V_i'' of dimensions d_i', d_i'', v_i', v_i'' respectively and we set $D_i = D_i' \oplus D_i''$ and $V_i = V_i' \oplus V_i''$. We define

$$\widetilde{\eta} = \widetilde{\eta}_{v',v''} \colon \Lambda_0(d', v') \times \Lambda_0(d'', v'') \longrightarrow \Lambda_0(d, v)$$

through the formula

$$\widetilde{\eta}\,((B', \gamma', \delta'), (B'', \gamma'', \delta'')) = \left(\begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}, \begin{pmatrix} \gamma' & 0 \\ 0 & \gamma'' \end{pmatrix}, \begin{pmatrix} \delta' & 0 \\ 0 & \delta'' \end{pmatrix} \right).$$

It's clear that this map enducuce maps η, η^0 as in the following commutative diagram

$$\begin{array}{ccc} M(d', v') \times M(d'', v'') & \xrightarrow{\ \eta\ } & M(d, v) \\ {\scriptstyle \pi_{d',v'} \times \pi_{d'',v''}} \big\downarrow & & {\scriptstyle \pi_{d,v}} \big\downarrow \\ M^0(d', v') \times M^0(d'', v'') & \xrightarrow{\ \eta^0\ } & M^0(d, v) \end{array}$$

Lemma 2.10. *1) η is injective.*

2) If $G(v')s'$ is a closed orbit in $\Lambda_0(d', v')$ and $G(v'')s''$ is a closed orbit in $\Lambda_0(d'', v'')$ then $G(v)\widetilde{\eta}(s', s'')$ is a closed orbit in $\Lambda_0(d, v)$.

3) $\eta^0(s', s'') = 0 \Rightarrow (s', s'') = (0, 0) \in M^0(d', v') \times M^0(d'', v'')$.

4) $\widetilde{\eta}(t \cdot x) = t \cdot \widetilde{\eta}(x)$, $\eta(t \cdot p) = t \cdot \eta(p)$ and $\eta^0(t \cdot p) = t \cdot \eta^0(p)$.

5) If $\eta(s', s'') \in Fix(d, v) \Rightarrow (s', s'') \in Fix(d', v') \times Fix(d'', v'')$.

Proof. 1) It's enough to prove that if $x = (s', s''), \bar{x} = (\bar{s}', \bar{s}'') \in \Lambda_0^+(d', v') \times \Lambda_0^+(d'', v'')$ and if $g \in G(V)$ is such that $g \cdot \widetilde{\eta}(x) = \widetilde{\eta}(\bar{x})$, then $g(V') \subset V'$ and $g(V'') \subset V''$. By the $+$-stability condition for s' we have that for all $u' \in V_i'$ there exist elements $d_\alpha \in D'_{\alpha_0}$ such that $u' = \sum_{\alpha_1 = i} \alpha(s')\gamma'_{\alpha_0}(d_\alpha)$. Hence

$$g(u') = \sum_{\alpha_1 = i} \alpha(\bar{s}')\bar{\gamma}'_{\alpha_0}(d_\alpha) \in V_i'.$$

The same arguments works for V''.

2) and 3) are clear by proposition 1.23.

4) is trivial.

5) is a consequence of 4) and 1). □

Remark 2.11. We want to do some remark about the injectivity of η^0. Let

$$\eta^\sharp \colon \mathbb{C}[\Lambda_0(d, v)]^{G(v)} \longrightarrow \mathbb{C}[\Lambda_0(d', v')]^{G(v')} \otimes \mathbb{C}[\Lambda_0(d'', v'')]^{G(v'')}$$

the adjoint map of η^0. We observe that we have

$$\mathrm{Tr}(\alpha(\cdot)) \overset{\eta^\sharp}{\longmapsto} \mathrm{Tr}(\alpha(\cdot)) \otimes 1 + 1 \otimes \mathrm{Tr}(\alpha(\cdot)) \quad \forall\, \alpha \text{ closed } B\text{-path },$$
$$\varphi(\beta(\cdot)) \overset{\eta^\sharp}{\longmapsto} \varphi_0(\beta(\cdot)) \otimes 1 + 1 \otimes \varphi_1(\beta(\cdot)) \quad \forall\, \beta \text{ admissible path },$$

and $\forall\, \varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_4) \in \Big(\mathrm{Hom}(D'_{\beta_0}, D'_{\beta_1})\Big)^* \oplus \Big(\mathrm{Hom}(D''_{\beta_0}, D''_{\beta_1})\Big)^* \oplus$ $\oplus \Big(\mathrm{Hom}(D'_{\beta_0}, D''_{\beta_1})\Big)^* \oplus \Big(\mathrm{Hom}(D''_{\beta_0}, D'_{\beta_1})\Big)^*$. In the case of graph of type A $\eta^\sharp$ is surjective since the invariants of type $\varphi(\beta(\cdot))$ generates the rings of invariants (see ch. 2), but in general this is not true. In the case of graph of finite type (A,D,E) it follows by a result of Lusztig ([**18**]) on the rings of invariants that η^0 is injective and finite. The injectivity is also clear by point 2) in lemma 2.10 and remark 1.24. Finally, still using 2.10 point 2), it is easy to show that in the general case η^0 is not even injective.

4.2. Big dot $\mathbb{C}^*$ action. Let d, d', d'', D, D', D'' as above. If $t \in \mathbb{C}^*$ then we define an element $g_t \in G(D)$ through:

$$(g_t)_i = \begin{pmatrix} \mathrm{Id}_{D'_i} & 0 \\ 0 & t\,\mathrm{Id}_{D''_i}. \end{pmatrix}$$

We can define the *big dot action* of $\mathbb{C}^*$ on S: $t \bullet s = g_t \cdot s$. We observe that this action commutes with the action of $GL(v)$ and leaves $\Lambda_0(d, v)$ and $\Lambda_0^+(d, v)$ stable. Hence it induces action on $M(d, v)$ and $M^0(d, v)$ commuting with the map π.

Proposition 2.12. *If $(v', v'') \neq (u', u'')$ then*

$$\eta_{v',v''}(M(d', v') \times M(d'', v'')) \cap \eta_{u',u''}(M(d', u') \times M(d'', u'')) = \varnothing, \qquad (22)$$

and moreover the set of the point fixed by the big dot action decomposes as below:

$$M(d, v)^{\mathbb{C}^*, \bullet} = \coprod_{v'+v''=v} \eta_{v',v''}(M(d', v') \times M(d'', v'')). \qquad (23)$$

Proof. We prove (22) and (23) together. First of all we observe that it is clear that the image of the maps η is contained in the fixed point set of the big dot $\mathbb{C}^*$ action.

Now let $s = (B, \gamma, \delta) \in \Lambda_0(d, v)$ and $p(s) \in M(d, v)^{\mathbb{C}^*, \bullet}$, we define

$$V_i'^s = \sum_{\substack{\alpha \text{ a } B\text{-path} \\ \alpha_1 = i}} \alpha(s)\gamma_{\alpha_0}(D'_i) \qquad (24a)$$

$$V_i''^s = \sum_{\substack{\alpha \text{ a } B\text{-path} \\ \alpha_1 = i}} \alpha(s)\gamma_{\alpha_0}(D''_i) \qquad (24b)$$

We have $V_i'^s + V_i''^s = V_i$ by the stability condition and V' and V'' $B-stable$ by definition. We want to prove $V_i'^s \cap V_i''^s = 0$. Let $x \in V_i'^s \cap V_i''^s$ then there exist $y_\alpha \in D'_{\alpha_0}$ and $z_\alpha \in D''_{\alpha_0}$ such that:

$$x = \sum_{\alpha\,:\,\alpha_1=i} \alpha(s)\gamma_{\alpha_0}(y_\alpha) = \sum_{\alpha\,:\,\alpha_1=i} \alpha(s)\gamma_{\alpha_0}(z_\alpha).$$

Now for any $t \in mC^*$ there exists $g \in GL(V)$ such that $t \bullet s = g \cdot s$. Hence

$$\begin{aligned} g_i x &= \sum_{\alpha\,:\,\alpha_1=i} g_i \alpha(s)\gamma_{\alpha_0}(y_\alpha) = \sum_{\alpha\,:\,\alpha_1=i} \alpha(g\cdot s) g_{\alpha_0}\gamma_{\alpha_0}(y_\alpha) \\ &= \sum_{\alpha\,:\,\alpha_1=i} \alpha(t\bullet s)\gamma_{\alpha_0}(g_t^{-1} y_\alpha) = \sum_{\alpha\,:\,\alpha_1=i} \alpha(s)\gamma_{\alpha_0}(y_\alpha) = x. \end{aligned}$$

In the same way using $x = \sum_{\alpha\,:\,\alpha_1=i} \alpha(s)\gamma_{\alpha_0}(z_\alpha)$ we obtain $g_i x = t^{-1}x$. Hence $x = 0$ and $V = V'^s \oplus V''^s$ and if v', v'' are the dimension vector of V'^s and V''^s it is clear that s is in the image of $\widetilde{\eta}_{v',v''}$ and $p(s)$ in the image of $\eta_{v',v''}$.

Finally we observe that once $s \in \widetilde{\eta}(\Lambda_0^+(d', v') \times \Lambda_0^+(d'', v'')$ we have $V'^s \simeq V'$ and $V''^s \simeq V''$ so (22) and (23) are proved. □

Since $M(d, v)^{\mathbb{C}^*,\bullet}$ is a smooth variety we have proved also the following fact.

Corollary 2.13. *$\eta_{v',v''}$ is an isomorphism with its image.*

4.3. Contraction index for the big dot action. Let d, d', d'', D, D', D'' as above. We will use Proposition 2.12 to describe the homology of $M(d, v)$ in terms of the homology of $M(d', v')$ and of $M(d'', v'')$. By Lemma 2.8 and Proposition 2.12 we have the following sequence of isomorphisms which do not preserve the degree and implies (21):

$$\begin{aligned} H_*(L(d,v)) &= \bigoplus_{\tau} H_*(Fix_\tau(d,v)) = \\ = \bigoplus_{\substack{v'+v''=v \\ \tau',\tau''}} & H_*(Fix_{\tau'}(d',v')) \otimes H_*(Fix_{\tau''}(d'',v'')) = \qquad (25) \\ &= \bigoplus_{v'+v''=v} H_*(L(d',v')) \otimes H_*(L(d'',v'')) \end{aligned}$$

In this section we want understand the behaviour of the degrees through these isomorphisms.

One way to compute the shift in the degrees in the isomorphism 25 is to compute the dimension of the cells in $Fix_{\tau'\times\tau''}(d, v)$ which contract to a point of $\eta(Fix_{\tau'}(d', v') \times Fix_{\tau''}(d'', v''))$. We procees in a sligtly different but equivalent way.. We define a new $\mathbb{C}^*$-action on S:

$$t \centerdot s = t^2 \cdot (t \bullet s)).$$

We call this action the *square action* and we observe that it commutes with the $G(V)$ action, leaves $\Lambda_0(d, v), \Lambda_0^+(d, v)$ stable and therefore induces actions on $M^0(d, v)$ and $M(d, v)$ commuting with the projection π.

LEMMA 2.14. *1)* $\forall\, p \in M^0(d,v)$ *there exists* $\lim_{t\to 0} t \centerdot p = 0$.
2) For all $p \in M(d,v)$ *there exists* $\lim_{t\to 0} t \centerdot p \in L(d,v)$.
3) If $p \in M^0(d,v)$ *and there exists* $\lim_{t\to\infty} t \centerdot p$ *then* $p = 0$.
4) For all $p \in M(d,v)$ *there exists* $\lim_{t\to\infty} t \centerdot p$ *if and only if* $p \in L(d,v)$.
5) $\widetilde{\eta}(t^2 \cdot s', t^2 \cdot s'') = t \centerdot \widetilde{\eta}(s', s'')$ *for all* $s' \in S(d',v')$ *and* $s'' \in S(d'',v'')$*, hence* $\eta(Fix_{\tau'}(d',v') \times Fix_{\tau''}(d'',v''))$ *is in the fixed point locus of the square action.*
6) If p *is fixed by the square action then there exist* $v', v''\,:\, v'+v'' = v$*,* τ', τ'' *and* $(p',p'') \in Fix_{\tau'}(d',v') \times Fix_{\tau''}(d'',v'')$ *such that* $p = \eta(p',p'')$*. Hence*

$$M(d,v)^{\mathbb{C}^*,\centerdot} = \coprod_{\substack{v'+v''=v\\ \tau',\tau''}} \eta_{v',v''}(Fix_{\tau'}(d',v') \times Fix_{\tau''}(d'',v'')). \tag{26}$$

PROOF. 1), 2), 3), 4), 5) are easy. We prove 6). If $t \centerdot p(s) = p(s)$ for all s we define:

$$V_i^{(j)'} = \sum_{\substack{\alpha \text{ a } B\text{-path}\\ \alpha_1 = i\ \ell(\alpha)=j}} \alpha(s)\gamma_{\alpha_0}\left(D_i'\right),$$
$$V_i^{(j)''} = \sum_{\substack{\alpha \text{ a } B\text{-path}\\ \alpha_1 = i\ \ell(\alpha)=j}} \alpha(s)\gamma_{\alpha_0}\left(D_i''\right).$$

Let $g \in G(V)$ such that $g \cdot s = t \centerdot s$ then if $g \in V_i^{(j)'}$ (resp. $V_i^{(j)''}$) then $g \cdot s = t^{2j} v$ (resp. $g \cdot s = t^{2j+1} v$). Hence if $V_i' = \bigoplus_j V_i^{(j)'}$ and $V_i'' = \bigoplus_j V_i^{(j)''}$ we have $V = V_i' \oplus V_i''$ and we conclude as in Proposition 2.12. $\square$

Let now

$$\mathcal{G}^+_{\tau'\tau''} = \{p \in M(d,v) \,:\, \lim_{t\to 0} t \centerdot p \in \eta(Fix_{\tau'} \times Fix_{\tau''})\},$$
$$\mathcal{G}^-_{\tau',\tau''} = \{p \in M(d,v) \,:\, \lim_{t\to \infty} t \centerdot p \in \eta(Fix_{\tau'} \times Fix_{\tau''})\}.$$

If $p \in \eta(Fix_{\tau'} \times Fix_{\tau''})$ we define $m^-_{\tau',\tau''} = \dim\left(T_pM(d,v)\right)^-$, $m^0_{\tau',\tau''} = \dim\left(T_pM(d,v)\right)^0$, and $m^+_{\tau',\tau''} = \dim\left(T_pM(d,v)\right)^+$ as in (15).

Let $v = v'+v''$ and $V = V' \oplus V''$ as in 4.1. Let $p(s') \in Fix_{\tau'}(d',v')$, $p(s'') \in Fix_{\tau''}(d'',v'')$ and $s = \widetilde{\eta}(s',s'')$. Let $\varphi' \colon \mathbb{C}^* \longrightarrow GL(V')$ and $\varphi'' \colon \mathbb{C}^* \longrightarrow GL(V'')$ the homomorphisms constructed as in (16), then we define $\varphi \colon \mathbb{C}^* \longrightarrow GL(V)$ by

$$\big(\varphi(t)\big)_i = \begin{pmatrix} \varphi_i'(t^2) & 0 \\ 0 & t\,\varphi_i''(t^2) \end{pmatrix}$$

with respect to the decomposition $V = V_i' \oplus V_i''$. We observe that $\varphi(t)t \centerdot s = s$ for all $t \in \mathbb{C}^*$. We set also $\chi(t) = t^2$ and we observe that $\omega(t \centerdot u, u') =$

$\chi(t)\omega(u,t^{-1}\centerdot u')$. for all $u,u'\in S$. If we apply Lemma 2.3 we obtain

$$m^-_{\tau',\tau''}=r^-_{\tau'}+r^-_{\tau''}+d''v'+d'(v''-v''^{(0)})+v''Av'-2v''v'+ \\ -\sum_k v''^{(k)}A_{\overline{\Omega}}v'^{(k)}-\sum_k v'^{(k)}A_{\overline{\Omega}}v''^{(k+1)}+\sum_k v''^{(k)}v'^{(k)}+\sum_k v'^{(k)}v''^{(k+1)}$$

$$m^0_{\tau',\tau''}=r^0_{\tau'}+r^0_{\tau''},$$

$$m^+_{\tau',\tau''}=r^+_{\tau'}+r^+_{\tau''}+d''v'+d'(v''+v''^{(0)})+v''Av'-2v''v'+ \\ +\sum_k v''^{(k)}A_{\overline{\Omega}}v'^{(k)}+\sum_k v'^{(k)}A_{\overline{\Omega}}v''^{(k+1)}-\sum_k v''^{(k)}v'^{(k)}-\sum_k v'^{(k)}v''^{(k+1)}.$$

Hence we obtain the followin refinement of isomorphism (25):

$$H_i(L(d,v))=\bigoplus_{\substack{v'+v''=v\\ \tau',\tau''\\ j}} H_{i-m^-_{\tau',\tau''}-j}(Fix_{\tau'}(d',v'))\otimes H_j(Fix_{\tau''}(d'',v'')).$$

4.4. U-equivariance. We would like to prove that the isomorphism (21) is an isomorphism of **U**-modules. I believe that the **U**-equivariance should follow from the following fact:

$$P_i(d,v)\cap \operatorname{Im}\eta=\eta(P_i(d',v')\times M(d'',v'')).$$

We explain the notation and we give a proof of this formula.

Let $d,d',d'',v,v',v'',D,D',D'',V,V',V''$ as in section 4.1. Let also $\bar v=v-\alpha_i$, $\bar v'=v'-\alpha_i$, $\bar v''=v''-\alpha_i$ and fix vector spaces $\overline{V}'$ and $\overline{V}''$ of dimension $\bar v'$, $\bar v''$ respectively. We have maps

$$\widetilde{\eta}_1\colon S(d',\overline{V}')\times S(d',V')\times S(d'',V'')\longrightarrow S(d,\overline{V}'\oplus V'')\times S(d,V'\oplus V''),$$
$$\widetilde{\eta}_2\colon S(d',V')\times S(d'',\overline{V}'')\times S(d'',V'')\longrightarrow S(d,V'\oplus \overline{V}'')\times S(d,V'\oplus V'')$$

defined by

$$\widetilde{\eta}_1(\bar s',s',s'')=(\widetilde{\eta}_{\bar v',v''}(\bar s',s'),\widetilde{\eta}_{v',v''}(s',s'')),$$
$$\widetilde{\eta}_2(s',\bar s'',s'')=(\widetilde{\eta}_{v',\bar v''}(s',\bar s''),\widetilde{\eta}_{v',v''}(s',s'')).$$

These two maps define similar maps $\eta_1,\eta_2,\eta_1^0,\eta_2^0$ between varieties of type M^+ and M^0. In particular it makes sense to consider

$$\eta_1(P_i(d',v')\times M(d'',v''))\subset M(d,\bar v)\times M(d,v)$$
$$\eta_2(M(d',v')\times P_i(d'',v''))\subset M(d,\bar v)\times M(d,v).$$

Lemma 2.15. *Let $i\in I$ then*
1) $P_i(d,v)\cap \text{Image}\,\eta_1=\eta_1(P_i(d',v')\times M(d'',v''))$,
2) $P_i(d,v)\cap \text{Image}\,\eta_2=\eta_2(M(d',v')\times P_i(d'',v''))$.

Proof. We prove only 1). The $\supset$ part is easy. We prove $\subset$ Let $\bar s'\in\Lambda_0^+(d',\bar v')$, $s'\in\Lambda_0^+(d',v')$, $\bar s''\in\Lambda_0^+(d'',\bar v'')$, $s''\in\Lambda_0^+(d'',v'')$, $\bar s=\widetilde{\eta}(\bar s',s'')=$

$(\overline{B}, \bar{\gamma}, \bar{\delta})$ and $s = \widetilde{\eta}(s', s'') = (B, \gamma, \delta)$. Suppose that $(p(\bar{s}, s''), p(s', s'')) \in P_i(d, v)$ and let $\varphi_i \colon V_i' \oplus V_i'' \longrightarrow \overline{V}_i' \oplus V_i''$ be such that

$$\varphi_{h_1} B_h = \overline{B}_h \varphi_{h_0} \qquad \varphi_i \gamma_i = \bar{\gamma}_i \qquad \delta_i = \bar{\delta}_i \varphi_i$$

Now we observe that the thesis follows from

$$\varphi_i(V_i') = \overline{V}_i' \ \text{ and } \ \varphi_i(V_i'') = V_i''.$$

Since the proof of this fact is very similar to other proofs in this chapter we skip it. □

CHAPTER 3

Quiver varieties of type A

In this chapter we prove a conjecture of Nakajima describing the relation between the geometry of quiver varieties of type A and the geometry of partial flags varieties and of the nilpotent variety. I want to thank Corrado De Concini who pointed out to me this problem and Hiraku Nakajima who pointed out an error and the solution to it in the original proof.

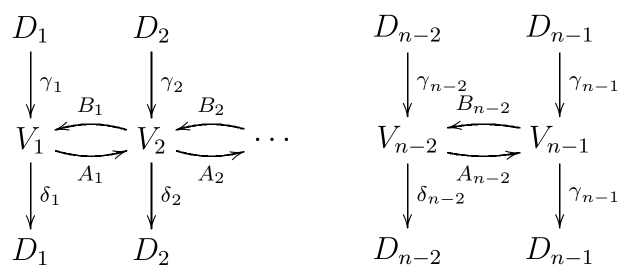
1. Nakajima's conjecture

We recall some definition and fix some notation on quiver varieties of type A_{n-1} and on partial flags varieties.

1.1. Convention for quiver varieties of type A_{n-1}. We choose Ω and the numbering of the vertices and of the edges of Ω of the (doubled) graph of type A_{n-1} in the following way:

$$1 \xrightarrow[1]{} 2 \xrightarrow[2]{} 3 \qquad \cdots \qquad n-2 \xrightarrow[n-2]{} n-1.$$

If $h \in \Omega$ we call A_h the map associated to h and B_h the map associated to $\bar{h}$. So our conventions can be summarized in the following diagram:



and instead of speaking of the triple (B, γ, δ) we will speak of the quadruple (A, B, γ, δ) where $A = (A_1, \dots, A_{n-2})$, $B = (B_1, \dots, B_{n-2})$, $\gamma = (\gamma_1, \dots, \gamma_{n-1})$ and $\delta = (\delta_1, \dots, \delta_{n-1})$. With these conventions the ADHM equation $\mu = 0$ can be written in the following way:

$$\begin{aligned} B_1 A_1 &= \gamma_1 \delta_1, \\ B_i A_i &= A_{i-1} B_{i-1} + \gamma_i \delta_i \quad \text{ for } 2 \leq i \leq n-2, \\ 0 &= A_{n-2} B_{n-2} + \gamma_{n-1} \delta_{n-1}. \end{aligned}$$

In this chapter we will call $M^1(d, v)$ the image of π, which is closed since π is projective, with the reduced structure.

We will use also the following notations: $\gamma_{j\to i} = B_i \dots B_{j-1}\gamma_j$ and $\delta_{j\to i} = \delta_i A_{i-1}\dots A_j$ and we consider the admissible path $[\delta_{\ell\to j}\gamma_{i\to\ell}]$ with the notations expalined in definition 1.17.

LEMMA 3.1. *1) The algebra of admissible path is generated by the following set*

$$\mathcal{P} = \{[\delta_{l\to j}\gamma_{i\to l}] \,:\, i,j \in \{1,\dots,n-1\} \text{ and } l \le \min(i,j)\}. \tag{27}$$

2) $\mathbb{C}[\Lambda_0]^G$ *is generated by the polynomials:*

$$s \longmapsto \varphi(\beta(s)) \qquad \text{for } \beta \in \mathcal{P} \text{ and } \varphi \in \big(Hom(D_{\beta_0}, D_{\beta_1})\big)^*.$$

3) If $(A,B,\gamma,\delta) \in \Lambda_0$ *then it is an element of* Λ_0^+ *iff for all* $1 \le i \le n-1$

$$\operatorname{Im} A_{i-1} + \sum_{j=i}^{n-1} \operatorname{Im} \gamma_{j\to i} = V_i.$$

PROOF. 1) and 3) are easy. 2) is a consequence of 1) and Proposition 1.19 □

LEMMA 3.2. *If* $\sigma \in W$ *is such that* $\sigma(d-v)$ *is dominant and* $v' = \sigma(v-d)+d$ *then*

$$M(d,v) \neq \varnothing \iff v'_i \ge 0 \quad \text{for } i = 1,\dots,n.$$

PROOF. This follows from Nakajima's theorem 1.37. □

1.2. The Slodowy's variety. In this section we recall some definitions on the nilpotent variety and on the partial flag variety.

DEFINITION 3.3. If N is a natural number and D is a vector space of dimension N we define $\mathcal{N} = \mathcal{N}_N$ to be the variety of nilpotent elements in $gl(D)$. Counting the dimensions of the Jordan blocks of an element of $\mathcal{N}$ we obtain a partion of N, and this give us a parametrization of the orbits O_λ, for λ a partion of N, of the action of $GL(D)$ on $\mathcal{N}$. If $x \in \mathcal{N}$ and x, y, h is a sl_2 triple in $gl(D)$ we define the *transversal slice* to the orbit of x in $\mathcal{N}$ in the point x as:

$$\mathcal{S}_x = \{u \in \mathcal{N} \text{ such that } [u-x, y] = 0\}.$$

Here and in the sequel, using a non standard convention, we admit $0,0,0$ as an sl_2 triple, so that in the case of $x = 0$ we have $\mathcal{S}_0 = \mathcal{N}$.

DEFINITION 3.4. For N an natural number , $a = (a_1,\dots,a_n)$ a vector of nonnegative integers such that $a_1 + \dots + a_n = N$, D a vector space of dimension N we define a *partial flag* of type a of D to be an increasing sequence $F : \{0\} = F_0 \subset F_1 \subset \dots \subset F_n = D$ of subspaces of D such that $\dim F_i - \dim F_{i-1} = a_i$. We define $\mathcal{F}_a$ to be the $GL(D)$-homogenous variety of partial flags of type a. We define also

$$\widetilde{\mathcal{N}}_a = T^*\mathcal{F}_a \cong \{(u,F) \in gl(D) \times \mathcal{F}_a \text{ such that } u(F_i) \subset F_{i-1}\},$$

$\mu_a : \widetilde{\mathcal{N}}_a \longrightarrow \mathcal{N}$ the projection onto the first factor, and

$\mathcal{F}_a^x = \mu_a^{-1}(x)$ for $x \in \mathcal{N}$.

For N, a, D as above let $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n)$ be a permutation of a and define the partition $\lambda_a = 1^{\alpha_1-\alpha_2}2^{\alpha_2-\alpha_3}\cdots n^{\alpha_n}$. λ_a is a partition of N and it is known that if (u,F) is in $\widetilde{\mathcal{N}}_a$ then u is in the closure of O_{λ_a}. Moreover the map

$$\mu_a : \widetilde{\mathcal{N}}_a \longrightarrow \overline{O}_{\lambda_a}$$

is a resolution of singularity and it is an isomorphism over O_{λ_a}. We define

$$\mathcal{S}_{a,x} = \mathcal{S}_x \cap \overline{O}_{\lambda_a}, \qquad \widetilde{\mathcal{S}}_{a,x} = \mu_a^{-1}(\mathcal{S}_{a,x}).$$

We call $\widetilde{\mathcal{S}}_{a,x}$ the *Slodowy's variety.*

The following proposition is well known.

PROPOSITION 3.5. *Let $x \in \mathcal{N}_N$ of type $1^{d_1}2^{d_2}\ldots(n-1)^{d_{n-1}}$ and $a = (a_1,\ldots,a_n)$ a partition of N then:*

1) $\widetilde{\mathcal{S}}_{a,x} \neq \varnothing \iff x \in \overline{O}_{\lambda_a} \iff \forall\, 1 \leq k \leq n$ and $\forall\, 1 \leq i_1 < i_2 < \cdots < i_k \leq n$ the following enequality holds:

$$d_1 + 2d_2 + \cdots + kd_k + \cdots + kd_{n-1} \geq a_{i_1} + \cdots + a_{i_k} \tag{28}$$

2) If $\widetilde{\mathcal{S}}_{a,x} \neq \varnothing$ then it is a smooth variety of dimension $\dim Z_{gl}(x) - \dim Z_{gl}(u_a)$, *where u_a is an element of O_{λ_a}.*

1.3. Nakajima's conjecture. If $d = (d_1,\ldots,d_n)$ and $v = (v_1,\ldots,v_n)$ are two $n-1$-tuples of integers we define the n-tuple $a = a(d,v) = (a_1,\ldots,a_n)$ by:

$$a_1 = d_1 + \cdots + d_{n-1} - v_1, \qquad a_n = v_{n-1},$$
$$\text{and } a_i = d_i + \cdots + d_{n-1} - v_i + v_{i-1} \ \text{ for } i = 2,\ldots,n-1.$$

We observe that $\sum_{i=1}^n a_i = N = \sum_{i=1}^{n-1} id_i$. Moreover we observe that once d is fixed the map a gives a bijection between $n-1$-tuples of integers v and n-tuples of integers a such that $\sum a_i = N$. Indeed we have that

$$v_{n-1} = a_n \quad v_i = a_n + \cdots + a_{i+1} - d_{i+1} - 2d_{i+2} \cdots - (n-i-1)d_{n-1} \text{ for } i = 1,\ldots,n-2.$$

Now we can state the main proposition of this chapter. We recall that $M(d,v) = M^1(d,v) = \varnothing$ if $v_i < 0$ for some i and $\widetilde{\mathcal{S}}_{a,x} = \mathcal{S}_{a,x} = \varnothing$ if $a_i < 0$ for some i (see definition 1.16). The theorem was conjectured by Nakajima in [**21**].

THEOREM 3.6. *Let v, d, N, $a = a(d,v)$ as above. Let $x \in \mathcal{N}$ be a nilpotent element of type $1^{d_1}\cdot\cdots\cdot(n-1)^{d_{n-1}}$ then there exist isomorphisms of algebraic varieties $\widetilde{\varphi}$ between $M(d,v)$ and $\widetilde{\mathcal{S}}_{a,x}$, and φ_1 between $M^1(d,v)$ and $\mathcal{S}_{a,x}$ such that $0 \in M^1(d,v)$ goes to $x \in \mathcal{S}_{d,x}$ and the following diagram commutes:*

$$\begin{array}{ccc} M(d,v) & \xrightarrow{\widetilde{\varphi}} & \widetilde{\mathcal{S}}_{d,x} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\mu_d}\downarrow \\ M^1(d,v) & \xrightarrow{\varphi_1} & \mathcal{S}_{d,x} \end{array} \tag{29}$$

REMARK 3.7. If $M(d,v) \neq \varnothing$ then it is easy to see that $0 \in M^1(d,v)$: anyway this will be a consequence of the proof.

We begin the proof of the theorem with some remarks on the degenerate cases and on the dimension of the varieties $M(d,v)$ and $\widetilde{\mathcal{S}}_{a,x}$. $W = S_n$ is the Weyl group.

LEMMA 3.8. *Let d, v, N, a as above and let $\sigma \in D$ such that $\sigma(d-v)$ is dominant and $v' = \sigma(v-d)+d$ then:*
1) If there exists i such that $v_i < 0$ then there exists i such that $v'_i < 0$.
2) If there exists i such that $a_i < 0$ then $M(d,v) = \varnothing$.
3) If there exists i such that $v_i < 0$ then $\widetilde{\mathcal{S}}_{a,x} = \varnothing$.
4) If $\widetilde{\mathcal{S}}_{a,x} \neq \varnothing$ then $M(d,v) \neq \varnothing$ and they are two smooth varieties of the same dimension.

PROOF. 1) This is an easy consequence of the following well known property: if u, d are dominant and $d - u \geq 0$ (that is $d - u \in \sum \mathbb{Z}_{\geq 0}\alpha_i$) then for any $\sigma \in W$ we have $d - \sigma u \geq 0$ (Ilaria Damiani).
2) Is an easy consequence of lemma 3.1 point 3).
3) If $v_i < 0$ then we have

$$N - (a_1 + \cdots + a_i) = a_n + \cdots + a_{i+1} <$$
$$< d_{i+1} + 2d_{i+2} \cdots + (n-i-1)d_{n-1} = N - (d_1 + \cdots + id_i + \cdots + id_n).$$

So $a_1 + \cdots + a_i > d_1 + \cdots + id_i + \cdots + id_n$ and $\widetilde{\mathcal{S}}_{a,x}$ is empty by lemma 3.5.
4) We observe that the Weyl group S_n acts by permutation on the n-tuple a and that :
(1) $\widetilde{\mathcal{S}}_{\sigma(a),x} \neq \varnothing \iff \widetilde{\mathcal{S}}_{a,x} \neq \varnothing$,
(2) $a(d, \sigma(v-d)+d) = \sigma(a(d,v))$.
The first property is clear from proposition 3.5 (indeed with a little more effort can be checked that $\widetilde{\mathcal{S}}_{\sigma(a),x} \simeq \widetilde{\mathcal{S}}_{a,x}$ but we don't need this result). The second property is a computation that can easily checked for $\sigma = (i, i+1)$. So by proposition it is enough to prove that $\widetilde{\mathcal{S}}_{a,x} \neq \varnothing \Rightarrow M(d,v) \neq \varnothing$ when $d - v$ is dominant. If we set $i_1 = 1, \ldots, i_k = k$ in the inequality (28) we obtain $v_k \geq 0$ for $k = 1, \ldots, n-1$ and by lemma 3.2 $M(d,v) \neq \varnothing$.
The equality of dimensions is an easy computations using proposition 1.5 and proposition 3.5. □

2. Definition of the map

In this section we will define the maps φ_1 and $\widetilde{\varphi}$ in the case $v_i, a_i \geq 0$ for each i.

LEMMA 3.9 (Nakajima, [**21**]). *If $N \geq v_1 \geq \cdots \geq v_{n-1}$ and if $d = (N, 0, \ldots, 0)$ then the conjecture is true. In this case we have $D = D_1$ and $M(d,v) \simeq \widetilde{\mathcal{N}}_a$ and $M^1(d,v) \simeq \overline{O}_{\lambda_a}$ and $M^0(d,v)$ is the closure of a nilpotent orbit.*

PROOF. The proof is given in [**21**], but in that case Nakajima took the inverse condition of stability so we remind the definition of the isomorphism in our case and we give a sketch of the proof. The isomorophism between $M(d,v)$ and $\mathcal{F}_a$ is given by:

$$(A,B,\gamma,\delta) \longmapsto (\delta_1\gamma_1, \{0\} \subset \ker\gamma_1 \subset \ker A_1\gamma_1 \subset \cdots \subset \ker A_{n-1}\cdots A_1\gamma_1)$$

The map between $M^1(d,v)$ and $\overline{O}_{\lambda_a}$ or between $M^0(d,v)$ and $\mathcal{N}$ is given by $(A,B,\gamma,\delta) \mapsto \delta_1\gamma_1$. Once the map on $M(d,v)$ is defined it is easy to check that it is bijective and that it is $GL(D)$ equivariant. Now we know that the map μ_a is a resolution of singularity and that it is an isomorphism over O_{λ_a} which is a homogenous space. Now by bijectivity and equivariance we see that the map we have defined must be a isomorphism over this set. Now we can proove the lemma by Zarisky main theorem and the normality of the closures of nilpotent orbits proved by Kraft and Procesi [**13**] . □

Now to treat the general case we use the lemma above in the following way. Let d,v,a,λ_a be given as in theorem 3.6 and define $\widetilde{d}_i = 0$ if $i > 1$ and $\widetilde{d}_1 = N = \sum_{j=1}^{n-1} jd_j$, $\widetilde{v}_i = v_i + \sum_{j=i+1}^{n-1}(j-i)d_j$. We observe that by the lemma above $M(\widetilde{v},\widetilde{d}) = T^*\mathcal{F}_a$ and $M^1(\widetilde{v},\widetilde{d}) = \overline{O}_{\lambda_a}$. So we can think $\mathcal{S}_{a,x}$ and $\widetilde{\mathcal{S}}_{a,x}$ as subvarieties of $M^1(\widetilde{v},\widetilde{d})$ and $M(\widetilde{v},\widetilde{d})$:

$$\mathcal{S}_{a,x} = p_0\Big(\big\{(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta}) \in \Lambda_0(\widetilde{v},\widetilde{d}) \,:\, [\widetilde{\delta}_1\widetilde{\gamma}_1 - x, y] = 0\big\}\Big) \cap M^1(\widetilde{v},\widetilde{d}),$$
$$\widetilde{\mathcal{S}}_{a,x} = p\Big(\big\{(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta}) \in \Lambda_0^+(\widetilde{v},\widetilde{d}) \mid [\widetilde{\delta}_1\widetilde{\gamma}_1 - x, y] = 0\big\}\Big).$$

So we can construct our map by giving a map from $\Lambda_0(d,v)$ to $\Lambda_0(\widetilde{v},\widetilde{d})$. Let us begin with the definition of $\widetilde{V}$ and $\widetilde{D}$. Let $D_i^{(j)}$ be an isomorphic copy of D_i. We define:

$$\widetilde{D_1} = \widetilde{D} = \bigoplus_{1\le k\le j\le n-1} D_j^{(k)}$$
$$\widetilde{V}_i = V_i \oplus \bigoplus_{1\le k\le j-i\le n-i-1} D_j^{(k)}$$

So We will use also the following conventions: $\widetilde{V}_0 = \widetilde{D_1}$, $\widetilde{A}_0 = \widetilde{\gamma}_1$, $\widetilde{B}_0 = \widetilde{\delta}_1$ and we define the following subspaces of $\widetilde{V}_i$:

$$D_i' = \bigoplus_{\substack{i+1\le j\le n-1\\ 1\le k\le j-i}} D_j^{(k)} \qquad D_i^+ = \bigoplus_{\substack{i+2\le j\le n-1\\ 2\le k\le j-i}} D_j^{(k)} \qquad D_i^- = \bigoplus_{\substack{i+2\le j\le n-1\\ 1\le k\le j-i-1}} D_j^{(k)}$$

We consider the group $GL(V)$ as the subgroup of $GL(\widetilde{V})$ acting as the identity map on D_i' and mapping V_i into V_i. We will always think at the maps $\widetilde{A}_i, \widetilde{B}_i$ as a block-matrix with respect to the given decomposition of $\widetilde{V}$, $\widetilde{D}$ and when we use a projection on one of our subspaces, it will be a projection with respect to the given decompositions (30). We give also a name to the blocks:

$$\begin{aligned}
\pi_{D_j^{(h)}}\widetilde{A}_i|_{D_{j'}^{(h')}} &= t_{i,j,h}^{j',h'} & \pi_{D_j^{(h)}}\widetilde{B}_i|_{D_{j'}^{(h')}} &= s_{i,j,h}^{j',h'} \\
\pi_{D_j^{(h)}}\widetilde{A}_i|_{V_i} &= t_{i,j,h}^{V} & \pi_{D_j^{(h)}}\widetilde{B}_i|_{V_{i+1}} &= s_{i,j,h}^{V} \\
\pi_{V_{i+1}}\widetilde{A}_i|_{D_{j'}^{(h')}} &= t_{i,V}^{j',h'} & \pi_{V_i}\widetilde{B}_i|_{D_{j'}^{(h')}} &= s_{i,V}^{j',h'} \\
\pi_{V_{i+1}}\widetilde{A}_i|_{V_i} &= a_i & \pi_{V_i}\widetilde{B}_i|_{V_{i+1}} &= b_i
\end{aligned} \tag{31}$$

We define also $(x_i, y_i, [x_i, y_i])$ to be the following special sl_2 triples of $sl(D_i')$:

$$\begin{aligned}
x_i|_{D_j^{(1)}} &= 0, \\
x_i|_{D_j^{(h)}} &= Id_{D_j} : D_j^{(h)} \to D_j^{(h-1)}, \\
y_i|_{D_j^{(j-i)}} &= 0, \\
y_i|_{D_j^{(h)}} &= h(j-i-h)Id_{D_j} : D_j^{(h)} \to D_j^{(h+1)},
\end{aligned}$$

and we observe that $x = x_0$ $y = y_0$, $[x, y]$ is an sl_2 triple in $sl(\widetilde{D})$ of the type required in the theorem.

We want now to introduce a subset of $\Lambda_0(\widetilde{v}, \widetilde{d})$. To do it we give a formal degree to the block of our matrices. Indeed we define two different kind of degrees, *deg* and *grad*, in the following way:

$$\begin{aligned}
\deg(t_{i,j,h}^{j',h'}) &= \min(h - h' + 1, h - h' + 1 + j' - j), \\
\mathrm{grad}(t_{i,j,h}^{j',h'}) &= 2h - 2h' + 2 + j' - j, \\
\deg(s_{i,j,h}^{j',h'}) &= \min(h - h', h - h' + j' - j), \\
\mathrm{grad}(s_{i,j,h}^{j',h'}) &= 2h - 2h' + j' - j.
\end{aligned}$$

Definition 3.10. An element $(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta})$ of $\Lambda_0(\widetilde{v}, \widetilde{d})$ is called *transversal* if it satisfies the following relations for $0 \leq i \leq n-2$:

$$\begin{aligned}
t^{j',h'}_{i,j,h} &= 0 && \text{if } \deg(t^{j',h'}_{i,j,h}) < 0 \\
t^{j',h'}_{i,j,h} &= 0 && \text{if } \deg(t^{j',h'}_{i,j,h}) = 0 \text{ and } (j',h') \neq (j,h+1) \\
t^{j',h'}_{i,j,h} &= Id_{D_j} && \text{if } \deg(t^{j',h'}_{i,j,h}) = 0 \text{ and } (j',h') = (j,h+1) \\
t^{V}_{i,j,h} &= 0 && \\
t^{j',h'}_{i,V} &= 0 && \text{if } h' \neq 1 \\
s^{j',h'}_{i,j,h} &= 0 && \text{if } \deg(s^{j',h'}_{i,j,h}) < 0 \\
s^{j',h'}_{i,j,h} &= 0 && \text{if } \deg(s^{j',h'}_{i,j,h}) = 0 \text{ and } (j',h') \neq (j,h) \\
s^{j',h'}_{i,j,h} &= Id_{D_j} && \text{if } \deg(s^{j',h'}_{i,j,h}) = 0 \text{ and } (j',h') = (j,h) \\
s^{V}_{i,j,h} &= 0 && \text{if } h \neq j-i \\
s^{j',h'}_{i,V} &= 0 &&
\end{aligned} \tag{32}$$

and finally if for each $0 \leq i \leq n-2$:

$$[\pi_{D'_i} \widetilde{B}_i \widetilde{A}_i|_{D'_i} - x_i, y_i] = 0.$$

We call T the set of transversal data and we call T^+ the set of +stable data which are also transversal.

We observe that $p(T^+) \subset \widetilde{\mathcal{S}}_{a,x}$ and $p_0(T) \cap M^1(\widetilde{v}, \widetilde{d}) \subset \mathcal{S}_{a,x}$ and we observe also that T and T^+ are $GL(V)$ invariant closed subset of Λ_0 and Λ_0^+ respectively.

We will define our maps $\widetilde{\varphi}$, φ by giving a $GL(V)$ equivariant map Φ from $\Lambda_0(d, v)$ to T. If $(A, B, \gamma, \delta) \in \Lambda(d, v)$ its image under Φ is an element $(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta})$ of T such that:

$$a_i = A_i \qquad b_i = B_i \tag{33a}$$

$$t^{j',1}_{i,V} = \gamma_{j' \to i+1} \qquad s^{V}_{i,j,j-i} = \delta_{i+1 \to j} \tag{33b}$$

$$t^{j',h'}_{i,j,h} = T^{j',h'}_{i,j,h}(A, B, \gamma, \delta) \qquad s^{j',h'}_{i,j,h} = S^{j',h'}_{i,j,h}(A, B, \gamma, \delta) \tag{33c}$$

where $S^{j',h'}_{i,j,h}$ and $T^{j',h'}_{i,j,h}$ are admissible polinomials of type (j', j) (see definition 1.17).

Remark 3.11. The conditions $t^{j',1}_{i,V} = \gamma_{j' \to i+1}$ for $j' > i+1$ and $s^{V}_{i,j,j-i} = \delta_{i+1 \to j}$ for $j > i+1$ are reduntant. Indeed it is easy to see that if $(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta}) \in T$ and $a_i = A_i$, $b_i = B_i$ and $t^{i+1,1}_{i,V} = \gamma_{i+1}$, $s^{V}_{i,i+1,i} = \delta_{i+1}$ then (33b) is satisfied. We do not give the details of this simple fact becouse the argument is comptely similar (but much more simple) to the proof of the next lemma.

Lemma 3.12. *There exist uniquely determined admissible polinomials* $T^{j',h'}_{i,j,h}$ *and* $S^{j',h'}_{i,j,h}$ *in* (A, B, γ, δ) *such that* $(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta}) \in T$. *Moreover for* deg > 0 *they* *result to be homogeneous polinomials of degree equal to* grad *of the following form:*

$$T^{j',h'}_{i,j,h} = \lambda^{j',h'}_{i,j,h}\delta_{r\to j}\gamma_{j'\to r} + Q^{j',h'}_{i,j,h}$$
$$S^{j',h'}_{i,j,h} = \mu^{j',h'}_{i,j,h}\delta_{r\to j}\gamma_{j'\to r} + R^{j',h'}_{i,j,h}$$

where $r = j + h' - h$ *and* P *and* Q *are admissible polinomials that can be expressed as a linear combination of products of admissible polinomials of degree strictly less than* grad *(at least each monomial of* P *and* Q *is a product of two admissible polinomials of positive degree) and* $\lambda^{j',h'}_{i,j,h}$*,* $\mu^{j',h'}_{i,j,h}$ *are rational numbers.*

2) Moreover for $i = 0, \dots, n-2$ *and* $\deg > 0$ *the following inequalities hold:*

$$\lambda^{j',h'}_{i,j,h} > 0$$

for $h' = 1, \quad i+2 \le j' \le n-1$ *and* $1 \le h \le j-i-1 \le n-i-2,$

$$\lambda^{j',h'}_{i,j,h} + \mu^{j',h'-1}_{i,j,h} > 0$$

for $1 < h' \le j'-i-1 \le n-i-2$ *and* $1 \le h \le j-i-1 \le n-i-2,$

$$\mu^{j',h'}_{i,j,h} > 0$$

for $1 \le h' \le j'-i-1 \le n-i-2, \quad h = j-i$ *and* $i+1 \le j \le n-1.$

Proof. We prove this lemma by decreasing induction on i. To be more precise we prove something slightly stronger of what claimed above. We prove that once $(A, B, \gamma, \delta) \in \Lambda_0(d, v)$ is fixed then there exist a unique element $(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta}) \in T$ such that (33a) and (33b) are satisfied. Moreover we give an inductive formula for the computation of this element and from this formula will be clear that there exist admissible polynomials as claimed in the lemma.

For $i = n-2$ we have that $\widetilde{A}_{n-2}$ and $\widetilde{B}_{n-2}$ are already completely defined by relations (33) and they verify the relation $\widetilde{A}_{n-2}\widetilde{B}_{n-2} = 0$. Now we assume to have constructed $T^{*,*}_{j,*,*}$ and $S^{*,*}_{j,*,*}$ for $j \ge i+1$ as stated in the lemma such that $\widetilde{A}_j, \widetilde{B}_j$ verify the relations requested to be in T. We prove that there exist unique $T^{*,*}_{i,*,*}$ and $S^{*,*}_{i,*,*}$ such that:

$$[\pi_{D'_i}\widetilde{B}_i\widetilde{A}_i|_{D'_i} - x_i, y_i] = 0 \text{ and } \widetilde{A}_i\widetilde{B}_i = \widetilde{B}_{i+1}\widetilde{A}_{i+1}, \tag{34}$$

and we prove also that they have the required form. First we observe that the following equations are satisfied by relations (32) and (33):

$$\pi|_{V_{i+1}}\widetilde{A}_i\widetilde{B}_i|_{V_{i+1}} = A_iB_i + \gamma_{i+1}\delta_{i+1} = B_{i+1}A_{i+1} = \pi|_{V_{i+1}}\widetilde{B}_{i+1}\widetilde{A}_{i+1}|_{V_{i+1}}$$
$$\pi|_{V_{i+1}}\widetilde{A}_i\widetilde{B}_i|_{D^{(h)}_j} = \delta_{h,1}\gamma_{j\to i+1} = B_{i+1}\gamma_{j\to i+2} = \pi|_{V_{i+1}}\widetilde{B}_{i+1}\widetilde{A}_{i+1}|_{D^{(h)}_j}$$
$$\pi|_{D^{(h)}_j}\widetilde{A}_i\widetilde{B}_i|_{V_{i+1}} = \delta_{h,1}\delta_{i+1\to j} = \delta_{i+2\to j}A_{i+1} = \pi|_{D^{(h)}_j}\widetilde{B}_{i+1}\widetilde{A}_{i+1}|_{V_{i+1}}$$

Let $L = \widetilde{B}_{i+1}\widetilde{A}_{i+1}$, $M = \widetilde{A}_i\widetilde{B}_i$ and $N = \pi_{D'_i}\widetilde{B}_i\widetilde{A}_i|_{D'_i} - x_i$, and as we have done in (31) we define the blocks $L_{j,h}^{j',h'}$, $M_{j,h}^{j',h'}$ and $N_{j,h}^{j',h'}$. So we can give the following formulation to equations (34):

$$M_{j,h}^{j',h'} = L_{j,h}^{j',h'} \tag{35}$$

for $1 \le h' \le j' - i - 1 \le n - i - 2$ and $1 \le h \le j - i - 1 \le n - i - 2$,

$$N_{j,h}^{j',j'-i} = 0 \tag{36}$$

for $1 + i \le j' \le n - 1$ and $1 \le h \le j - i - 1 \le n - i - 2$,

$$N_{j,1}^{j',h'} = 0 \tag{37}$$

for $1 + i \le j \le n - 1$ and $2 \le h' \le j' - i \le n - i - 1$, and

$$h'(j' - i - h')N_{j,h+1}^{j',h'+1} = h(j - i - h)N_{j,h}^{j',h'} \tag{38}$$

for $1 \le h' \le j' - i - 1 \le n - i - 2$ and $1 \le h \le j - i - 1 \le n - i - 2$.

Now we give a degree deg and a degree grad also to these new blocks, in the following way:

$$\deg(L_{j,h}^{j',h'}) = \deg(M_{j,h}^{j',h'}) = \deg(N_{j,h}^{j',h'}) = \min(h - h' + 1, h - h' + 1 + j' - j),$$

$$\mathrm{grad}(L_{j,h}^{j',h'}) = \mathrm{grad}(M_{j,h}^{j',h'}) = \mathrm{grad}(N_{j,h}^{j',h'}) = 2h - 2h' + 2 + j' - j.$$

Since $\min(m - h' + 1, m - h' + 1 + j' - j) + \min(h - m, h - m + l - j) \le \min(h - h' + 1, h - h' + 1 + j' - j)$ and $\min(m - h', m - h' + j' - j) + \min(h - m + 1, h - m + 1 + l - j) \le \min(h - h' + 1, h - h' + 1 + j' - j)$ we have that deg and grad behaves well under composition; that is:

$$\begin{aligned}
\deg(S_{i+1,j,h}^{l,m}) + \deg(T_{i+1,l,m}^{j',h'}) &\le \deg(L_{j,h}^{j',h'})\\
\deg(T_{i,j,h}^{l,m}) + \deg(S_{i,l,m}^{j',h'}) &\le \deg(M_{j,h}^{j',h'})\\
\deg(S_{i,j,h}^{l,m}) + \deg(T_{i,l,m}^{j',h'}) &\le \deg(N_{j,h}^{j',h'})\\
\mathrm{grad}(S_{i+1,j,h}^{l,m}) + \mathrm{grad}(T_{i+1,l,m}^{j',h'}) &= \mathrm{grad}(L_{j,h}^{j',h'})\\
\mathrm{grad}(T_{i,j,h}^{l,m}) + \mathrm{grad}(S_{i,l,m}^{j',h'}) &= \mathrm{grad}(M_{j,h}^{j',h'})\\
\mathrm{grad}(S_{i,j,h}^{l,m}) + \mathrm{grad}(T_{i,l,m}^{j',h'}) &= \mathrm{grad}(N_{j,h}^{j',h'})
\end{aligned}$$

(We observe that in the N-case the term $-x_i$ respects these rules). So if the blocks have deg strictly less than 0 they vanish identically, and if deg $= 0$ then to be different from zero we must have $j = j'$ and $h = h' - 1$ and it is straight forward that also in this case all the equations are satisfied. In this way we see that the equations (36) and (37) are always satisfied.

Now we argue by induction on $d = \deg > 0$ in the following way: we assume to have constructed $T_{i,j,h}^{j',h'}$ and $S_{i,j,h}^{j',h'}$ for the blocks with deg $< d$ such that all the relations (35) and (38) for blocks with deg $< d$ are satisfied and we prove that $T_{i,j,h}^{j',h'}$ and $S_{i,j,h}^{j',h'}$ for blocks of deg $= d$ are uniquely determined by the equations (35) and (38) for blocks of deg $= d$. So we have the following relations:

$$M_{j,h}^{j',h'} = L_{j,h}^{j',h'}$$
$$h'(j'-i-h')N_{j,h+1}^{j',h'+1} = h(j-i-h)N_{j,h}^{j',h'}$$

for $1 \leq h' \leq j'-i-1 \leq n-i-2$ and $1 \leq h \leq j-i-1 \leq n-i-2$ and $\min(h-h'+1, h-h'+1+j'-j) = d > 0$. By induction hypothesis we have under this assumptions on j, j', h, h', d the following formulas:

$$L_{j,h}^{j',h'} = \nu_h \delta_{r\to j}\gamma_{j'\to r} + C_{j,h}^{j',h'}$$
$$M_{j,h}^{j',h'} = S_{i,j,h+1}^{j',h'} + T_{i,j,h}^{j',h'} + D_{j,h}^{j',h'}$$
$$N_{j,h}^{j',h'} = E_{j,h}^{j',h'} + \begin{cases} T_{i,j,h}^{j',h'} & \text{if } h'=1 \\ T_{i,j,h}^{j',h'} + S_{i,j,h}^{j',h'-1} & \text{if } 1 < h' \leq j'-i-1 \end{cases}$$
$$N_{j,h+1}^{j',h'+1} = F_{j,h}^{j',h'} + \begin{cases} S_{i,j,h+1}^{j',h'} & \text{if } h = j-i-1 \\ T_{i,j,h+1}^{j',h'+1} + S_{i,j,h+1}^{j',h'} & \text{if } 1 \leq h < j-i-1 \end{cases}$$

where $r = j+h'-h$, $C_{j,h}^{j',h'}$, $D_{j,h}^{j',h'}$, $E_{j,h}^{j',h'}$, $F_{j,h}^{j',h'}$ are admissible polinomials that by induction we already know and that are a linear combination of products of admissible polinomials of degree strictly less than grad, and

$$\nu_h = \begin{cases} 1 & \text{if } h'=1 \text{ and } h=j-i-1 \\ \lambda_{i+1,j,h}^{j',h'} & \text{if } h'=1 \text{ and } h<j-i-1 \\ \mu_{i+1,j,h}^{j',h'-1} & \text{if } h=j-i-1 \text{ and } h'>1 \\ \lambda_{i+1,j,h}^{j',h'} + \mu_{i+1,j,h}^{j',h'-1} & \text{if } h'>1 \text{ and } h<j-i-1 \end{cases}$$

In any case by induction hypothesis we see that ν_h is a positive rational number. We observe that also the numbers $h'(j'-i-h') = \alpha_h$, and $h(j-i-h) = \beta_h$ are positive rational numbers. Now we group together all the equations with the same j and the same j' and we solve them altogether. Once we have fixed j and j' the relations between indeces can be written in this form: $h_0 \leq h \leq h_1$ and $h' = k+h$, where $h_1 = j-i-1$ and:

$$h_0 = \begin{cases} d & \text{if } j' \geq j \\ d+j-j' & \text{if } j' < j \end{cases}, \qquad k = \begin{cases} 1-d & \text{if } j' \geq j \\ 1+j'-j-d & \text{if } j' < j \end{cases}$$

We observe also that once j, j', d are fixed also $r = j+h'-h$ is fixed. Now to write our sistems of equations in a more readable way we introduce the following variables: $X_h = T_{i,j,h}^{j',h'}$ and $Y_h = S_{i,j,h+1}^{j',h'}$; and we observe that variables involved exhaust all the unknown blocks of type $T_{i,j,*}^{j',*}$ and $S_{i,j,*}^{j',*}$ of deg $= d$ that is what we we want to construct. So we can write the equations (35) and (38) in the following way:

$$\begin{aligned} X_{h_0} + Y_{h_0} &= \nu_{h_0}\delta_{r\to j}\gamma_{j'\to r} + P_{1,h_0} \\ &\cdots \\ X_{h_1} + Y_{h_1} &= \nu_{h_1}\delta_{r\to j}\gamma_{j'\to r} + P_{1,h_1} \end{aligned} \tag{39}$$

and

$$\begin{aligned} \alpha_{h_0}(Y_{h_0} + X_{h_0+1}) &= \beta_{h_0} X_{h_0} + P_{2,h_0} \\ \alpha_{h_0+1}(Y_{h_0+1} + X_{h_0+2}) &= \beta_{h_0+1}(Y_{h_0} + X_{h_0+1}) + P_{2,h_0+1} \\ &\dots \\ \alpha_{h_1-1}(Y_{h_1-1} + X_{h_1}) &= \beta_{h_1-1}(Y_{h_1-2} + X_{h_1-1}) + P_{2,h_1-1} \\ \alpha_{h_1} Y_{h_1} &= \beta_{h_1}(Y_{h_1-1} + X_{h_1}) + P_{2,h_1} \end{aligned} \tag{40}$$

where $P_{*,*}$ are known polinomials of degree equal to the grad $= 2\deg + |j - j'|$ of our blocks and that are a linear combination of products of polinomials that have degree strictly less than grad. This system has a unique solution: first we use the equations (40) to give an expression of $Y_{h_1} + X_{h_1+1}$ in terms of X_{h_0} then we sum all the equations (39) and we obtain a formula for X_{h_0} and then we see that we can determine all the others X_h and Y_h. We observe also that equations (39) and (40) give an inductive formula for the coefficients $\lambda^{j',h'}_{i,j,h}$ and $\mu^{j',h'}_{i,j,h}$. Indeed they are the coefficient of the term $\delta_{r\to j}\gamma_{j'\to r}$ in the polinomials $T^{j',h'}_{i,j,h}$ and $S^{j',h'}_{i,j,h}$ above so they solve the same systems (39) and (40) but with the costant coefficients $P_{*,*}$ equal to zero. So if we use the same variables X and Y for λ and μ, we obtain from system (40) the following formulas:

$$\begin{aligned} Y_{h_0} + X_{h_0+1} &= \rho_{h_0} X_{h_0} \\ &\dots \\ Y_{h_1-1} + X_{h_1} &= \rho_{h_1-1} X_{h_0} \\ Y_{h_1} &= \rho_{h_1} X_{h_0} \end{aligned}$$

where ρ_h are positive rational numbers. We observe that the coefficients of the point 2) of the lemma are just, with our convention $X_{h_0}, (Y_{h_0} + X_{h_0+1}), \dots, Y_{h_1}$. So it is enough to prove that $X_{h_0} > 0$. But summing the equations in system (39) we obtain:

$$X_{h_0} = \frac{\nu_{h_0} + \dots + \nu_{h1}}{1 + \rho_{h_0} + \dots + \rho_{h_1}}$$

which is a positive rational number and we have proved the lemma. □

REMARK 3.13. The lemma above show, how is possible to define the map Φ from $\Lambda_0(d,v)$ to T. An inverse of Φ is given in the following way. Take $(\widetilde{A}, \widetilde{B}, \widetilde{\gamma} = \widetilde{A}_0, \widetilde{\delta} = \widetilde{B}_0) \in T$ and define $\Phi^{-1}(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta}) = (A, B, \gamma, \delta)$, where $A_i = \pi_{V_{i+1}} \widetilde{A}_i|_{V_i}$, $B_i = \pi_{V_i} \widetilde{B}_i|_{V_{i+1}}$, $\gamma_i = \pi_{V_i} \widetilde{A}_{i-1}|_{D_i^{(1)}}$ and $\delta_i = \pi_{D_i^{(1)}} \widetilde{B}_{i-1}|_{V_i}$. It is clear that the new data is in $\Lambda_0(d,v)$ and it also clear that $\Phi^{-1} \circ \Phi = Id_{\Lambda_0(d,v)}$. The relation $\Phi \circ \Phi^{-1} = Id_T$ follows from the unicity proved in the lemma. To be more precise it follows from the unicity explained at the beinning of the proof of the lemma and remark 3.11.

LEMMA 3.14. *1)* $\Phi : \Lambda_0(d,v) \longrightarrow T$ *is a* $GL(V)$*-equivariant isomorphism.*
2) $\Phi(z) \in T^+ \iff z \in \Lambda_0^+(d,v)$ *and* $\Phi|_{SZ} : \Lambda_0^+(d,v) \longrightarrow T^+$ *is a* $GL(V)$*-equivariant isomorphism*

PROOF. We have just proved 1). To prove 2) we observe that in the case of $(\widetilde{v},\widetilde{d})$ the stability condition is equivalent to $\widetilde{A}_i$ is an epimorphism for $i=0,\dots,n-2$. We observe also that if $(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})=\Phi(A,B,\gamma,\delta)\in T$ we have that $\widetilde{A}_i|_{D_i^+}$ is an isomorphism onto D'_{i+1}. Since $V_i\oplus D^{(1)}_{i+1}\oplus\cdots\oplus D^{(1)}_{n-1}$ is a complementary space of D_i^+ and that V_{i+1} is a complementary space of D'_{i+1}, we conclude, by (32) and (33), that the stability condition in our case is equivalent to $A_i\oplus\gamma_{i+1}\oplus\cdots\oplus\gamma_{n-1\to i+1}: V_i\oplus D^{(1)}_{i+1}\oplus\cdots\oplus D^{(1)}_{n-1}\longrightarrow V_{i+1}$ is an epimorhpism for $i=0,\dots,n-2$; which is exactly the condition of lemma 3.1 point 3) for the stability of (A,B,γ,δ). $\square$

DEFINITION 3.15. As observed Φ is a $GL(V)$-equivariant morphism, so we can define φ_0 and $\widetilde{\varphi}$ as the maps making the following diagrams commute:

$$\begin{array}{ccc} \Lambda_0(d,v) & \xrightarrow{\Phi} & T \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle p_0}\downarrow \\ M_0(d,v) & \xrightarrow{\varphi_0} & M_0(\widetilde{v},\widetilde{d}) \end{array} \qquad \begin{array}{ccc} \Lambda_0^+(d,v) & \xrightarrow{\Phi} & T^+ \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ M(d,v) & \xrightarrow{\widetilde{\varphi}} & \widetilde{\mathcal{S}}_{a,x} \end{array}$$

and if we set $\varphi_1=\varphi_0|_{M^1(d,v)}$ we observe that by definition the diagram (29) commutes, and that i$\varphi_1\subset\mu_d(\widetilde{\mathcal{S}}_{a,x})=\mathcal{S}_{a,x}$.

COROLLARY 3.16. *Let a,d,v,N as in section 1.3 then*

$$M(d,v)=\varnothing \iff \widetilde{\mathcal{S}}_{a,x}=\varnothing$$

PROOF. After lemma 3.8 we have only to prove that $M(d,v)\neq\varnothing\Rightarrow\widetilde{\mathcal{S}}_{a,x}\neq\varnothing$ but this is clear since we have constructed a map from $M(d,v)$ to $\widetilde{\mathcal{S}}_{a,x}$. $\square$

3. Proof of Theorem 3.6

LEMMA 3.17. *Let $(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})\in T$ and $\widetilde{g}\in GL(\widetilde{V})$ then*

$$\widetilde{g}(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})\in T\Longrightarrow\exists g\in GL(V) \text{ such that } \widetilde{g}(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})=g(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})$$

PROOF. We prove first that $\widetilde{g}_i(V_i)=V_i$ and $\widetilde{g}_i(D'_i)=D'_i$. To prove it we introduce for $i=0,\dots,n-2$, $l=0,\dots,n-2-i$ and $h=0,\dots,n-2-i-l$ the following subspaces of $\widetilde{V}_i$:

$$D_i^{l,(h)}=\bigoplus_{\substack{0\leq h'\leq h\\ i+1+l+h'\leq j\leq n-1}} D_j^{(j-i-h')}.$$

We prove that $\widetilde{g}_i(D_i^{l,(h)})=D_i^{l,(h)}$. Indeed we observe that if $(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})\in T$ then $\widetilde{A}_i|_{D_i^{l,(h)}}$ is an isomorphism onto $D_{i+1}^{l-1,(h)}$ for $l\geq 1$. So we can argue by induction on i, taking as first step the trivial case $i=0$, that $\widetilde{g}_i(D_i^{l,(h)})=D_i^{l,(h)}$. We observe that $D_i^{0,(n-i-2)}=D'_i$ and so the we have proved $\widetilde{g}_i(D'_i)=D'_i$. Now we observe that if $(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})\in T$ then $\pi_{D_i^-}\widetilde{B}_i|_{D'_{i+1}}$ is an isomorphism and that $\widetilde{B}_i(V_{i+1})\subset D_i^{0,(0)}\oplus V_i$. Since $D_i^-\oplus V_i$ is the complementary subspace, respect our decomposition, of $D_i^{0,(0)}$ and $\widetilde{g}_i(D_i^{0,(0)}) = D_i^{0,(0)}$ we can conclude that $\widetilde{g}_{i+1}(V_{i+1}) = V_{i+1}$.

Now we consider $g_i = \widetilde{g}_i|_{V_i}$ and we prove that $\widetilde{g}(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta}) = g(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta})$. Arguing as in remark 3.13 we see that it is enough to prove that the a_i, b_i and $t_{i,V}^{i+1,1}$ and $s_{i,i,1}^{V}$ of the two elements of T are equal. By construction we have already proved the equality of the a_i and b_i block. To prove the equality for the t and the s block we observe that it's enough to prove that $\widetilde{g}_i|D_{i+1}^{(1)} = Id_{D_{i+1}^{(1)}}$. To prove it we observe that $\widetilde{A}_i|_{D_i^{l,(0)}}$ is the identity map from $D_i^{l,(0)}$ to $D_{i+1}^{l-1,(0)}$. So arguing by induction as above we conclude that $\widetilde{g}_i|_{D_i^{l,(0)}}$ is the identity map, and finally we observe that $D_{i+1}^{(1)} \subset D_i^{l,(0)}$. □

REMARK 3.18. By direct computation we can prove that $\widetilde{g}(\widetilde{A}, \widetilde{B}, \widetilde{\gamma}, \widetilde{\delta}) \in T \iff \widetilde{g} \in GL(V)$ but we don't need this result.

LEMMA 3.19. *φ_0 and φ_1 are closed immersions.*

PROOF. It is enough to prove that φ_0 is a closed immersion. We observe that $M^0(d,v)$ and $M^0(\widetilde{v}, \widetilde{d})$ are affine varieties whose coordinate ring is described in lemma 3.1. We will prove that the associate map ${\varphi_0}^\sharp$ between these rings is surjective by showing that is it possible to obtain the polinamials in $\mathcal{P}(d,v)$ from the admissible polinomials for $(\widetilde{d}, \widetilde{v})$ through the map φ_0. Let us introduce the following *deg* on the set $\mathcal{P}(v,d)$:

$$\deg(\delta_{r\to j'}\gamma_{j\to r}) = \min(j-r+1, j'-r+1)$$

and de observe that usual degree is given by $\operatorname{grad}(\delta_{r\to j'}\gamma_{j\to r}) = 2\deg + |j'-j|$. We will prove the statement by induction on $d = \deg$. If $d \le 0$ then $r \ge j+1$ or $r \ge j'+1$ and so there are no polinomial in the set $\mathcal{P}$ in this case, and the statement is proved. If $d > 0$ we consider the following blocks of degree d:

$$(\widetilde{\delta}_1\widetilde{\gamma}_1)_{j,h}^{j',1} = (\widetilde{B}_0\widetilde{A}_0)_{j,h}^{j',1} = R + \begin{cases} 1 \cdot \delta_{j+1-h\to j}\gamma_{j'\to j+1-h} & \text{if } j = h \\ \lambda_{0,j,h}^{j',1} \cdot \delta_{j+1-h\to j}\gamma_{j'\to j+1-h} & \text{if } j > h \end{cases}$$

where by induction and lemma 3.1 R is a linear combination of products of monomials with a smaller *deg*. Since by lemma 3.12 the coefficient of $\delta_{j+1-h\to j}\gamma_{j'\to j+1-h}$ is different from zero we obtain that for any $1 \le h \le j$ the element of $\mathcal{P}(v,d)$, $\delta_{j+1-h\to j}\gamma_{j'\to j+1-h}$, can be obtained as claimed. But now we observe that this element has $deg = d$ and that all the elements in $\mathcal{P}$ of deg equal to d can be obtained in this way for a good choice of h between 1 and j. □

Proof of theorem 3.6. By the lemma above and the fact that μ_d and π are projective we see that $\widetilde{\varphi}$ is proper. By lemmas 3.14 and 3.17, since by a result of Nakajima ([**21**] [**22**]) all the orbits in $\Lambda_0^+(v,d)$ and $\Lambda_0^+(\widetilde{d}, \widetilde{v})$ are closed we see that $\widetilde{\varphi}$ is also injective. Since by lemma 3.8 $M(v,d)$ and $\widetilde{\mathcal{S}}_{a,x}$ are smooth varieties of the same dimension and $\widetilde{\mathcal{S}}_{a,x}$ is connected we have proved that it is an isomorphism of holomorphic varieties and by consequence is also an algebraic isomorphism. In particular $\widetilde{\varphi}$ is surjective and μ_d is also surjective, so also φ_1 is surjective, but since it is a closed immersion of reduced varieties over $\mathbb{C}$ it must be an isomorphism of algebraic varieties. Finally $\varphi_0(0) = x \in \mathcal{S}_{a,x}$, so by the previous lemma $0 \in M^1(v,d)$ and $\varphi_1(0) = x$. *QED*

REMARK 3.20. The map $\widetilde{\varphi}$ restricted to $L(v,d)$ take a more explicit and simple form. Indeed it is easy to see that in this case δ vanishes so we have that all the polynomials T and S vanish also, and we have an explicit formula for $\widetilde{\varphi}$.

REMARK 3.21. In [**21**] is observed that the conjecture does not generalize to diagrams of type E and D. But it is an interesting and more general fact (see for example the stratification of quiver varieties constructed by Nakajima [**21**], [**22**] or the remark above) that some subvarieties can be described as an another quiver variety. From this point of vied we want to point out that it is possible to give an explicit pairs of injective maps $\widetilde{\psi}$ and ψ from $M(v,d)$ to $M(\widetilde{d},\widetilde{v})$ and from $M_0(v,w)$ to $M_0(\widetilde{d},\widetilde{v})$ respectively such that the diagram (29) commute and $\psi(0) = x$. As we said they have an explicit formula and so they look more simple than $\widetilde{\varphi}$ and φ_1 but their image is not contained in $\widetilde{\mathcal{S}}_{a,x}$ and $\mathcal{S}_{a,x}$ respectively, they "describe" different subvarieties.

3.1. Spaltenstein map. To understand the geometry of the generalized Springer fiber of type A_{n-1} $\mathcal{F}^x_a$ is very usefull (see for example [**26**]) to consider the following map $\alpha : \mathcal{F}^x_a \to Gr_{a_1}(\ker x)$ defined by

$$\alpha(0 \subset F_1 \subset \cdots F_{n-1} \subset D) = F_1.$$

If $H \in Gr_{a_1}(\ker x)$ it is very easy to see that $\alpha^{-1}(H)$ is isomorphic to a Springer fiber $\mathcal{F}^{x'}_{a'}$ of type A_{n-2} where $a' = (a_2,\dots,a_n)$ and x' depends on H. This fact suggest a way to study $\mathcal{F}^x_a$ by induction. For example it is a well known fact that we can use this map to prove that the number of irreducible component of $\mathcal{F}^x_a$ is equal to the number of a-semistandard Young tableaux of shape x. To avoid misleading interpretaion of this sentence we only say that if x is of type $1^2 \cdot 2^1 \cdot 3^2$ and $a = (3,4,0,3)$ then

$$\begin{matrix} 1 & 1 & & & \\ 2 & 2 & 2 & & \\ 4 & 4 & 4 & 2 & 1 \end{matrix}$$

is a a-semistandard Young tableaux of shape x.

It is intersting to observe that the map α can be described easily in the language of quiver varieties. First of all observe that $Gr_{a_1}(\ker x) \simeq Gr(a_1, a_1 + \cdots + a_n)$ is isomorphic to $\{$map of maximal rank in $Hom(D_1 \oplus \cdots \oplus D_n, V_1)\}/GL(V_1)$. The define

$$\alpha : L(d,v) \longrightarrow \{\text{map of maximal rank in } Hom(D_1 \oplus \cdots \oplus D_n, V_1)\}/GL(V_1)$$

$$\alpha(A,B,\gamma,0) = (\gamma_1 \oplus \cdots \oplus \gamma_{n-1\to 1}).$$

The fibers of this map can be described easily as varieties $L(d', v')$ for a graph of type A_{n-2} without using the theorem proved inthis chapter. Now using the computation above of the number of irreducible component of Springer fibers and Nakajima's theorem 1.37 we obtain the following well known formula for the character of the irreducible sl_n module of highest weight d:

$$ch_d = \sum_{\substack{T \text{ semistandard Young tableaux} \\ \text{of shape } 1_1^{d} \cdots (n-1)^{d_{n-1}}}} e^{weight(T)}$$

where the weight of a tableaux of type a is $\sum_{i=1}^{n-1}(a_i - a_{i+1})\omega_i$.

We don't give the (easy) details becouse they don't give nothing new but we hope that this explain the remark at the of the introduction.

3.2. sl_n-equivariance and equivalence with Ginzburg's construction. We showed that the varieties constructed by Nakajima appears as special subvarieties of partial flag varieties, but we didn't check if also the costruction of representation is equivalent. By the very definition and Lemma 3.9 in the case $d = (N, 0, \ldots, 0)$ is clear that Nakajima's construction coincide with Ginzburg's construction. For general d, v we saw that $L(d, v) = M(\widetilde{(d)}, \widetilde{(v)})_x$ for an appropriate choice of x, hence we have an isomorphism:

$$H_*(L(d, v)) \simeq H_*(M(\widetilde{d}, \widetilde{v})_x).$$

We would like to prove that this is an isomorphism of sl_n modules. More in general the problem of sl_n-equivariance can be stated as follows. Consider the embedding

$$\psi_Z = \varphi \times \varphi \colon Z(d, v', v) \longrightarrow Z(\widetilde{d}, \widetilde{v'}, \widetilde{v})$$

we have homorphism of algebra:

$$\mathbf{U}(sl_n) \longrightarrow H_{top}\Big(Z(d)\Big),$$
$$\mathbf{U}(sl_n) \longrightarrow H_{top}\Big(Z(\widetilde{d})\Big).$$

then problem of equivariance can be stated as follows: the embedding ψ induces an homomorphism of algebra $H(\psi_Z)$ such that the following diagram commute:

$$\begin{array}{ccc} \mathbf{U}(sl_n) & \longrightarrow & H_{top}\Big(Z(\widetilde{d})\Big) \\ & \searrow & \big\downarrow {\scriptstyle H(\psi_Z)} \\ & & H_{top}\Big((Z(d)\Big) \end{array}$$

I believe that this fact should follow from some regularity condition plus the following Lemma:

LEMMA 3.22. *Let* $\psi = \varphi \times \varphi \colon M(d, v - \alpha_i) \times M(d, v) \longrightarrow M(\widetilde{d}, \widetilde{v} - \alpha_i) \times M(\widetilde{d}, \widetilde{v})$ *then*

$$P_i(\widetilde{d}, \widetilde{v}) \cap \operatorname{Im} \psi = \psi(P_i(d, v)).$$

PROOF. The proof is equal to the proof of Lemma 3.17. $\square$

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