Hyperbolic-parabolic singular perturbation for Kirchhoff equations with weak dissipation

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Abstract

We consider Kirchhoff equations with a small parameter ε in front of the second-order time-derivative, and a dissipative term whose coefficient may tend to 0 as $t \to +\infty$ (weak dissipation).

In this note we present some recent results concerning existence of global solutions, and their asymptotic behavior both as $t \to +\infty$ and as $\varepsilon \to 0^+$. Since the limit equation is of parabolic type, this is usually referred to as a hyperbolic-parabolic singular perturbation problem.

We show in particular that the equation exhibits hyperbolic or parabolic behavior depending on the values of the parameters.

1 Introduction

Let H be a separable real Hilbert space. For every x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x\rangle \geq 0$ for every $x \in D(A)$, so that for every $\alpha \geq 0$ the power $A^\alpha x$ is defined provided that x lies in a suitable domain $D(A^\alpha)$.

Let $b : [0, +\infty) \to (0, +\infty)$ and $m : [0, +\infty) \to [0, +\infty)$ be two given functions. For every $\varepsilon > 0$ we consider the Cauchy problem

$$\varepsilon u_\varepsilon''(t) + b(t)u_\varepsilon'(t) + m(|A^{1/2}u_\varepsilon(t)|^2)Au_\varepsilon(t) = 0, \tag{1.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{1.2}$$

This is the dissipative version of the celebrated equation introduced by G. Kirchhoff in [19] as a simplified model for transversal vibrations of elastic strings. We refer to the survey [14] for the non-dissipative case where $\varepsilon = 1$ and $b(t) \equiv 0$. Let us set

$$\mu := \inf_{\sigma \geq 0} m(\sigma), \qquad \delta := \inf_{t \geq 0} b(t), \qquad \nu := \inf\left\{\frac{\langle Ax, x\rangle}{|x|^2} : x \in D(A),\ x \neq 0\right\}.$$

Several features of (1.1) depend on the values of μ, δ, ν. Let us recall some standard terminology.

- *Non-degenerate vs degenerate equations* These terms refer to the nonlinearity. Equation (1.1) is called *nondegenerate* or *strictly hyperbolic* if $\mu > 0$, and *degenerate* or *weakly hyperbolic* if $\mu \geq 0$. The Cauchy problem (1.1), (1.2) is called *mildly degenerate* if $\mu \geq 0$ but

$$m(|A^{1/2}u_0|^2) \neq 0. \tag{1.3}$$

  Whenever we consider degenerate equations, we always limit ourselves to the mildly degenerate case. The *really degenerate* case where $m(|A^{1/2}u_0|^2) = 0$ seems to be still quite unexplored.

- *Constant vs weak dissipation* We have *constant dissipation* when $b(t) = \delta > 0$ for every $t \geq 0$, and *weak dissipation* when $b(t) \to 0$ as $t \to +\infty$. Almost all known results for the constant dissipation case can be easily extended to non-constant dissipation coefficients provided that $\delta > 0$ and $b'(t)$ is bounded. For simplicity we often limit ourselves to the model case where $b(t) = (1+t)^{-p}$ for some $p \geq 0$, the case $p = 0$ corresponding to constant dissipation.

  In this note we don't consider equations with *strong dissipation*, which usually refers to dissipative terms of the form $A^\alpha u_\varepsilon'(t)$ with $\alpha > 0$, or better $\alpha \geq 1/2$.

- *Coercive vs non-coercive operators* The operator A is called *coercive* when $\nu > 0$, and it is called *noncoercive* when $\nu \geq 0$. This property of the operator has a great influence on the asymptotic behavior of solutions.

The singular perturbation problem in its generality consists in proving the convergence of solutions of (1.1), (1.2) to solutions of the first order problem

$$b(t)u'_\varepsilon(t) + m(|A^{1/2}u_\varepsilon(t)|^2)Au_\varepsilon(t) = 0, \qquad u(0) = u_0, \tag{1.4}$$

obtained setting formally $\varepsilon = 0$ in (1.1), and omitting the second initial condition in (1.2). Following the approach introduced by J. L. LIONS [20] in the linear case, one defines the corrector $\theta_\varepsilon(t)$ as the solution of the second order *linear* problem

$$\varepsilon\theta''_\varepsilon(t) + b(t)\theta'_\varepsilon(t) = 0 \qquad \forall t \geq 0,$$

$$\theta_\varepsilon(0) = 0, \qquad \theta'_\varepsilon(0) = u_1 + \frac{1}{b(0)}m(|A^{1/2}u_0|^2)Au_0 =: w_0. \tag{1.5}$$

It is easy to see that $\theta'_\varepsilon(0) = u'_\varepsilon(0) - u'(0)$, hence this corrector keeps into account the boundary layer due to the loss of one initial condition. Finally one defines $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ in such a way that

$$u_\varepsilon(t) = u(t) + \theta_\varepsilon(t) + r_\varepsilon(t) = u(t) + \rho_\varepsilon(t) \qquad \forall t \geq 0. \tag{1.6}$$

With these notations, the singular perturbation problem consists in proving that $r_\varepsilon(t) \to 0$ or $\rho_\varepsilon(t) \to 0$ in some sense as $\varepsilon \to 0^+$. The general problem can be split into at least six subproblems.

(1) *Parabolic problem: global existence and decay estimates* This is the first and usually easiest step in the theory. It consists in proving that (1.4) admits a unique global solution $u(t)$, and then in estimating its decay rate as $t \to +\infty$. This decay rate is afterwards used as a benchmark for the decay rate of solutions of the hyperbolic problem.

(2) *Local existence for the hyperbolic problem and local-in-time error estimates* Let $T > 0$ be fixed. This subproblem consists in proving that, for every $\varepsilon > 0$ small enough, the solution $u_\varepsilon(t)$ of the hyperbolic problem (1.1), (1.2) is defined (at least) on the interval $[0, T]$, and in this interval $u_\varepsilon(t)$ converges to the solution $u(t)$ of the limit problem. In this case the smallness of ε, as well as the convergence rates, may depend on T.

(3) *Hyperbolic problem: global existence* This subproblem consists in proving that problem (1.1), (1.2) admits a global-in-time solution provided that $\varepsilon > 0$ is small enough. From the point of view of existence, this is a strengthening of the previous step, and of course in general it requires stronger assumptions.

Existence of global solutions without the smallness assumption on ε is a widely open question, which seems to be as difficult as the non-dissipative case (see Section 4).

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(4) *Hyperbolic problem: decay estimates* Once we know that the hyperbolic problem admits a global-in-time solution $u_\varepsilon(t)$, a natural question concerns its behavior as $t \to +\infty$ (ε is now small and fixed). What one expects in reasonable situations is that $u_\varepsilon(t)$ decays as the solution $u(t)$ of the corresponding parabolic equation. This is what has been actually proved in many cases.

(5) *Singular perturbation problem: global-in-time error estimates* This subproblem is just the global-in-time version of subproblem (2). The goal is therefore to give *time-independent* estimates on $\rho_\varepsilon(t)$ or $r_\varepsilon(t)$ as $\varepsilon \to 0^+$.

(6) *Singular perturbation problem: decay-error estimates* This is the meeting point of subproblems (4) and (5), and it is the ultimate goal of the theory. It consists in estimating in the same time the behavior of $u_\varepsilon(t)$ as $t \to +\infty$ and as $\varepsilon \to 0^+$. The general form of a decay-error estimate is something like

$$|A^\alpha \rho_\varepsilon(t)| \leq \omega(\varepsilon)\gamma(t) \quad \text{or} \quad |A^\alpha r_\varepsilon(t)| \leq \omega(\varepsilon)\gamma(t).$$

Of course one expects $\gamma(t)$ to be the decay rate of solutions of the parabolic problem (or even better), and $\omega(\varepsilon)$ to be the convergence rate which appears in the local-in-time error estimates.

This program has generated a considerable literature in the last thirty years, for which we refer to the introductions of the following sections. In this note we sum up the state of the art and the main open questions. A rough overview is provided by Table 1, where we show, under different assumptions, which subproblems have received a reasonable or partial answer up to now. We focus in particular on the model dissipation coefficient of the form $b(t) := (1+t)^{-p}$ with $p \geq 0$, and on nonlinear terms which are either non-degenerate or of the form $m(\sigma) = \sigma^\gamma$ for some $\gamma > 0$ (note that we allow also the non-Lipschitz case $\gamma \in (0,1)$).

	$p = 0$	$p \in (0,1], \nu > 0$	$p \in (0,1], \nu \geq 0$
$\mu > 0$	1-2-3-4-5-6	1-2-3-4p-5-6p	1-2-3-4-5-6
$m(\sigma) = \sigma^\gamma, \gamma \geq 1$	1-2-3-4-5p	1-2-3-4	1-2-3p-4p
$m(\sigma) = \sigma^\gamma, \gamma \in (0,1)$	1-2-3-4	1-2-3-4	1-2-3p-4p
$m(\sigma) \geq 0$ (Lip. cont.)	1-2-3-4	1-2	1-2

Table 1: state of the art on subproblems (1) through (6). Numbers refer subproblems, a "p" means that in that case only partial (non-optimal) results have been obtained

Looking at Table 1, one can guess that $p = 1$ plays a special role in the theory. This is true also in the linear case. Let us indeed consider equation

$$au''(t) + \frac{b}{(1+t)^p}u'(t) + cAu(t) = 0, \tag{1.7}$$

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where a, b, c are positive parameters, and $p \geq 0$. This equation was investigated by T. Yamazaki [32] and J. Wirth [30]. They proved that (1.7) has both parabolic and hyperbolic features, and which nature prevails depends on p. When $p < 1$ the equation has parabolic behavior, in the sense that all its solutions decay to 0 as $t \to +\infty$ as solutions of the parabolic equation with $a = 0$. When $p > 1$ the same equation has hyperbolic behavior, meaning that every solution is asymptotic to a suitable solution of the non-dissipative equation with $b = 0$ (and in particular all non-zero solutions do not decay to zero). In the critical case $p = 1$ the nature of the problem depends on b/a, with the parabolic behavior prevailing as soon as the ratio is large enough.

Our results for Kirchhoff equations are consistent with the linear theory. Indeed we have always hyperbolic behavior when $p > 1$, meaning that non-zero global solutions (provided that they exist) cannot decay to 0. When $p \leq 1$ we were able to prove that the behavior is of parabolic type in many cases. In all such situations the critical exponent $p = 1$ falls in the parabolic regime, but this is simply due to the fact that in our equation we have that $b = 1$ and $a = \varepsilon$ is small enough, hence the ratio b/a is always big enough.

For shortness's sake we don't include proofs in this note. Nevertheless, we conclude this introduction by mentioning the useful energies and the technical reasons why the problem becomes harder and harder when the equation is degenerate, the dissipation is weak, and the operator is non-coercive. In the parabolic case all estimates follow from the monotonicity of the classical energies

$$
E_k(t) := |A^{k/2} u(t)|^2, \qquad P(t) := \frac{|Au(t)|^2}{|A^{1/2} u(t)|^2}.
$$

In the hyperbolic case, all known techniques for proving global existence for (1.1) require an a priori estimate such as

$$
\varepsilon \cdot \frac{\left| m'(|A^{1/2} u_\varepsilon(t)|^2) \right|}{m(|A^{1/2} u_\varepsilon(t)|^2)} \cdot |Au_\varepsilon(t)| \cdot |u_\varepsilon'(t)| \leq b(t). \tag{1.8}
$$

If $\mu > 0$ and $b(t)$ is a positive constant, an a priori bound on $Au_\varepsilon(t)$ and $u_\varepsilon'(t)$, together with the smallness of ε, is enough to establish (1.8). When $b(t) \to 0$ as $t \to +\infty$, the boundedness is no more enough, and we need some a priori informations on the decay of $Au_\varepsilon(t)$ and $u_\varepsilon'(t)$. This means that global existence and decay estimates become intimately tied, and they must be treated together. The main energies involved in these estimates are

$$
E_{\varepsilon,k}(t) := \varepsilon \frac{|A^{k/2} u_\varepsilon'(t)|^2}{c_\varepsilon(t)} + |A^{(k+1)/2} u_\varepsilon(t)|^2, \qquad G_\varepsilon(t) := \frac{|u_\varepsilon'(t)|^2}{c_\varepsilon^2(t)}, \tag{1.9}
$$

where $c_\varepsilon(t) := m(|A^{1/2} u_\varepsilon(t)|^2)$. They are both extensions of the first energy of the parabolic case. The use of $E_{\varepsilon,k}(t)$ is quite classical, and dates back to [3, 31], while $G_\varepsilon(t)$ was introduced by the authors in [8].

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The degenerate case is more complex. Let us assume for example that $m(\sigma) = 0$ if and only if $\sigma = 0$. Then the decay of the solution implies that the denominator in the left-hand side of (1.8) tends to 0. When $m(\sigma) = \sigma^\gamma$ with $\gamma \in (0,1)$, then also the term with m' diverges to $+\infty$ as the solution approaches 0. This complicates proofs both in the case of constant, and in the case of weak dissipation.

The basic idea to deal with degenerate nonlinear terms of the form $m(\sigma) = \sigma^\gamma$ is to exploit that $\sigma m'(\sigma)/m(\sigma)$ has a finite limit as $\sigma \to 0^+$. This reduces (1.8) to

$$\varepsilon \cdot \frac{|Au_\varepsilon(t)| \cdot |u_\varepsilon'(t)|}{|A^{1/2}u_\varepsilon(t)|^2} \leq b(t). \tag{1.10}$$

This inequality has been approached using (1.9) and the further energies

$$P_\varepsilon := \frac{\varepsilon}{c_\varepsilon} \frac{|A^{1/2}u_\varepsilon|^2 |A^{1/2}u_\varepsilon'|^2 - \langle Au_\varepsilon, u_\varepsilon'\rangle^2}{|A^{1/2}u_\varepsilon|^4} + \frac{|Au_\varepsilon|^2}{|A^{1/2}u_\varepsilon|^2}, \qquad Q_\varepsilon := \frac{|u_\varepsilon'|^2}{c_\varepsilon^2 |A^{1/2}u_\varepsilon|^2}.$$

These energies have been introduced by the first author in [7] as a hyperbolic version of the second energy of the parabolic case.

If $m(\sigma)$ is a general nonnegative (even Lipschitz continuous) function, then it may happen that $\sigma m'(\sigma)/m(\sigma)$ is unbounded in a neighborhood of $\sigma = 0$. In this case (1.10) does not imply (1.8), and each step seems to require new ideas. For this reason we are quite skeptic about a future relevant progress in the last line of Table 1.

Concerning the coerciveness of the operator, it is well known that small eigenvalues deteriorate the decay estimates on solutions, even in the parabolic case, and we have seen that decay estimates are fundamental also for existence issues.

This note is organized as follows. Section 2 is devoted to subproblem (1), namely existence and decay estimates for the parabolic problem. Section 3 is devoted to subproblem (2), namely local existence results for the hyperbolic equation and local-in-time error estimates for the singular perturbation problem. In Section 4 we show that (1.1) has hyperbolic behavior whenever $p > 1$. Section 5 is devoted to global existence and decay-error estimates in the nondegenerate case. Section 6 is devoted to the degenerate case. Finally, Section 7 is a collection of open problems.

2   The parabolic problem

The theory of parabolic equations of Kirchhoff type is quite well established. This equation appeared for the first time in the pioneering paper [2] by S. Bernstein. He considered the concrete equation in the interval $(0,1)$, with a nondegenerate nonlinearity and constant dissipation, and he proved that for every initial condition in the Sobolev space $H^1((0,1))$ the equation admits a unique solution, which is actually analytic in the space variable for every $t > 0$ (the classical regularizing effect of parabolic equations). This result was afterwards extended by many authors (see [1, 21]).

The more general version is probably stated in [15]. The basic fact observed in [15] is that any solution $u(t)$ of (1.4) can be written in the form

$$u(t) = v(\alpha(t)),$$

where $v(t)$ is the solution of the linear Cauchy problem with constant coefficients

$$v'(t) + Av(t) = 0, \qquad v(0) = u_0, \tag{2.1}$$

and $\alpha : [0, +\infty) \to [0, +\infty)$ is the solution of the ordinary differential equation

$$b(t)\alpha'(t) = m(|A^{1/2}v(\alpha(t))|^2), \qquad \alpha(0) = 0.$$

In other words, the solution of (1.4) is always a time reparametrization of the solution of the heat-like equation (2.1). At this point it is quite easy to prove the following existence result.

Theorem 2.1 (Global existence for the parabolic problem) *Let H be a Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator on H with dense domain. Let $m : [0, +\infty) \to [0, +\infty)$ be a locally Lipschitz continuous function, and let $u_0 \in D(A)$.*
Then problem (1.4) has a unique global solution

$$u \in C^1\left([0, +\infty); H\right) \cap C^0\left([0, +\infty); D(A)\right).$$

If in addition $m(|A^{1/2}u_0|^2)Au_0 \neq 0$, hence $u'(0) \neq 0$, then the solution is non-stationary, and $u \in C^1\left((0, +\infty); D(A^\alpha)\right)$ for every $\alpha \geq 0$.

Decay estimates for $u(t)$ can be deduced from decay estimates for (2.1) and the asymptotic behavior of the parametrization $\alpha(t)$. Concerning (2.1), it is well known that the asymptotic behavior of solutions depends on the coerciveness of the operator A. If A is coercive with some constant $\nu > 0$, then solutions decay exponentially to 0, with a rate depending on ν. In this case we have indeed that

$$|A^{1/2}u_0|^2 \exp\left(-2\frac{|Au_0|^2}{|A^{1/2}u_0|^2}t\right) \leq |A^{1/2}v(t)|^2 \leq |A^{1/2}u_0|^2 \exp(-2\nu t).$$

If A is non-coercive ($\nu \geq 0$), then decay rates are slower. We have indeed that

$$|A^{1/2}u_0|^2 \exp\left(-2\frac{|Au_0|^2}{|A^{1/2}u_0|^2}t\right) \leq |A^{1/2}v(t)|^2 \leq \frac{|u_0|^2}{2t}, \qquad |Av(t)|^2 \leq \frac{|u_0|^2}{2t^2}.$$

Note in particular that the estimates from below and from above for $|A^{1/2}v(t)|^2$ involve different rates. This range of rates cannot be improved because, when the

$m(\sigma) \geq \mu > 0$	$\nu > 0$	$c_1 e^{-\alpha_1(1+t)^{p+1}} \leq \|A^{1/2}u(t)\|^2 \leq c_2 e^{-\alpha_2(1+t)^{p+1}}$ $c_1 e^{-\alpha_1(1+t)^{p+1}} \leq \|Au(t)\|^2 \leq c_2 e^{-\alpha_2(1+t)^{p+1}}$ $c_1(1+t)^{2p} e^{-\alpha_1(1+t)^{p+1}} \leq \|u'(t)\|^2 \leq c_2(1+t)^{2p} e^{-\alpha_2(1+t)^{p+1}}$
	$\nu \geq 0$	$c_1 e^{-\alpha_1(1+t)^{p+1}} \leq \|A^{1/2}u(t)\|^2 \leq \dfrac{c_2}{(1+t)^{p+1}}$ $\|Au(t)\|^2 \leq \dfrac{c}{(1+t)^{2(p+1)}}$ $\|u'(t)\|^2 \leq \dfrac{c}{(1+t)^2}$
$m(\sigma) = \sigma^\gamma$	$\nu > 0$	$\dfrac{c_1}{(1+t)^{(p+1)/\gamma}} \leq \|A^{1/2}u(t)\|^2 \leq \dfrac{c_2}{(1+t)^{(p+1)/\gamma}}$ $\dfrac{c_1}{(1+t)^{(p+1)/\gamma}} \leq \|Au(t)\|^2 \leq \dfrac{c_2}{(1+t)^{(p+1)/\gamma}}$ $\dfrac{c_1}{(1+t)^{2+(p+1)/\gamma}} \leq \|u'(t)\|^2 \leq \dfrac{c_1}{(1+t)^{2+(p+1)/\gamma}}$
	$\nu \geq 0$	$\dfrac{c_1}{(1+t)^{(p+1)/\gamma}} \leq \|A^{1/2}u(t)\|^2 \leq \dfrac{c_2}{(1+t)^{(p+1)/(\gamma+1)}}$ $\|Au(t)\|^2 \leq \dfrac{c}{(1+t)^{(p+1)/\gamma}}$ $\|u'(t)\|^2 \leq \dfrac{c}{(1+t)^{[2\gamma^2+(1-p)\gamma+p+1]/(\gamma^2+\gamma)}}$

Table 2: Decay estimates for the parabolic problem

operator has a sequence of eigenvalues converging to 0, any intermediate rate is realized by a suitable solution.

Once we know the decay of $v(t)$, we can easily deduce the asymptotic behavior of $\alpha(t)$, hence also the asymptotic behavior of $u(t)$. In Table 2 we sum up the decay estimates which can be obtained in this way, limiting ourselves for simplicity to dissipation coefficients of the form $b(t) = (1+t)^{-p}$ with $p \geq 0$, and to nonlinear terms which are either non-degenerate or of the form $m(\sigma) = \sigma^\gamma$ with $\gamma > 0$.

We stress that in all these cases solutions decay to zero, and the decay rate becomes stronger and stronger as p grows. This contrasts with the hyperbolic case, where solutions cannot decay when $p > 1$ (see Section 4).

3 Local-in-time error estimates

All the local existence results for the non-dissipative equation (see [14, Theorem 2.1]) can be easily extended to the dissipative case. This provides a continuum of local existence results, with the regularity requirements on the initial data depending on the continuity modulus of m. In this note we limit ourselves to Lipschitz continuous nonlinear terms, or to the non-Lipschitz case $m(\sigma) = \sigma^\gamma$ with $\gamma \in (0,1)$, where the nondegeneracy assumption (1.3) makes the problem just mildly non-Lipschitz. In all these cases the equation is locally well posed for initial data in Sobolev spaces.

In this section we focus on a property which is slightly stronger than local existence, and which could be called *almost global existence*. The first result is indeed that the life span of $u_\varepsilon(t)$ tends to $+\infty$ as $\varepsilon \to 0^+$.

Theorem 3.1 (Hyperbolic problem: almost global existence) *Let H be a Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator on H with dense domain. Let $m : [0, +\infty) \to [0, +\infty)$ and $b : [0, +\infty) \to (0, +\infty)$ be two locally Lipschitz continuous functions. Let us assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy the non-degeneracy assumption (1.3), and let $T > 0$.*

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (1.1), (1.2) has a unique solution

$$u_\varepsilon \in C^2\left([0,T];H\right) \cap C^1\left([0,T];D(A^{1/2})\right) \cap C^0\left([0,T];D(A)\right).$$

Then we study the convergence of $u_\varepsilon(t)$ to the solution $u(t)$ of the limit problem.

Theorem 3.2 (Singular perturbation: local-in-time error estimates) *Let H, A, $m(\sigma)$, $b(t)$, u_0, u_1, T, ε_0 be as in Theorem 3.1. Let $u(t)$ be the solution of the corresponding parabolic problem (1.4), and let $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ be defined by (1.6).*

Then we have the following conclusions.

(1) *Without further assumptions on initial data, hence $(u_0, u_1) \in D(A) \times D(A^{1/2})$, we have that*

$$|\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 + |r_\varepsilon'(t)|^2 \to 0 \quad \text{uniformly in } [0,T],$$

$$\int_0^T |A^{1/2}r_\varepsilon'(t)|^2\, dt \to 0.$$

(2) *If in addition we assume that $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, then there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that*

$$|\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + \varepsilon|r_\varepsilon'(t)|^2 \leq C\varepsilon^2 \qquad \forall t \in [0,T],$$

$$\int_0^T |r_\varepsilon'(t)|^2\, dt \leq C\varepsilon^2.$$

(3) If in addition we assume that $(u_0, u_1) \in D(A^2) \times D(A)$, then there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that

$$|A\rho_\varepsilon(t)|^2 + |r'_\varepsilon(t)|^2 + \varepsilon|A^{1/2}r'_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall t \in [0, T],$$

$$\int_0^T |A^{1/2}r'_\varepsilon(t)|^2\, dt \leq C\varepsilon^2.$$

We point out that the remainder $r_\varepsilon(t)$ is well suited for estimates involving derivatives, because it doesn't feel the effects of the boundary layer due to the loss of one initial condition. On the contrary, the remainder $\rho_\varepsilon(t)$ is better suited for estimates without derivatives. This is because, for example, $A\rho_\varepsilon(0)$ is defined whenever $u_0 \in D(A)$, while $Ar_\varepsilon(0)$ requires $u_0 \in D(A^2)$ (see definition (1.5) of w_0).

Both the existence and the convergence result are local-in-time, namely constants, error estimates, and the smallness of ε do depend on the interval $[0, T]$ chosen at the beginning. On the other hand, the assumptions required on $b(t)$ and $m(\sigma)$ are quite weak. The dichotomy between hyperbolic and parabolic behavior mentioned in the introduction appears only as $t \to +\infty$, hence it plays no role on a fixed time interval. In particular we don't need to assume that $p \leq 1$ in the case where $b(t) = (1+t)^{-p}$.

From Theorem 3.2 it is clear that convergence rates for the singular perturbation problem depend on the regularity of initial data. This situation is consistent with the linear case. Indeed in [9] we considered the linear equation

$$\varepsilon u''_\varepsilon(t) + u'_\varepsilon(t) + Au_\varepsilon(t) = 0$$

and the corresponding limit parabolic problem, and we proved similar results. We also proved that an error estimate such as $|A^{1/2}\rho_\varepsilon(t)|^2 \leq C\varepsilon^2$ is possible only when $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$.

We just remark that uniform convergence, without any rate, requires initial data in spaces such as $D(A) \times D(A^{1/2})$, hence with "gap 1/2" between the regularity of u'_ε and u_ε, a typical feature of hyperbolic problems. On the contrary, if we want some convergence rate, we have to work in spaces such as $D(A^{3/2}) \times D(A^{1/2})$ or $D(A^2) \times D(A)$, hence with "gap 1", a typical feature of parabolic problems. In our opinion, this gives further evidence that the parabolic nature dominates in the limit.

A formal proof of Theorem 3.1 and Theorem 3.2, as they are stated, has never been put into writing. Error estimates have been considered in at least three papers, always with constant dissipation. B. F. Esham, and R. J. Weinacht [5] proved error estimates in the nondegenerate case with initial data in $D(A^{3/2}) \times D(A)$. The second author [16] considered the degenerate case, proving uniform convergence for data in $D(A) \times D(A^{1/2})$, and error estimates for more regular data. Finally, the authors [11] proved error estimates in the degenerate case with the optimal requirement that initial data are in $D(A^{3/2}) \times D(A^{1/2})$ (see [11, Proposition A.1]). It should be quite standard to extend those proofs to the case of weak dissipation, just because on a fixed time interval the function $b(t)$ is always strictly positive.

4   The hyperbolic regime

In this section we show that when the dissipation is too weak, namely $b(t) \to 0$ too fast, then equation (1.1) behaves in a hyperbolic way, in the sense that its non-zero global solutions (provided that they exist) do not decay to 0 as $t \to +\infty$. Of course this doesn't prevent such solutions from existing (which remains an open problem), but it shows that the problem cannot be approached using the standard methods based on estimates such as (1.8) or (1.10). Since solutions of the parabolic problem always decay to 0, this shows also that no decay-error estimate can be true in this case. Note that condition (4.1) is equivalent to $p > 1$ when $b(t) = (1+t)^{-p}$.

Theorem 4.1 (Hyperbolic regime) *Let H be a Hilbert space, and let A be a non-negative self-adjoint (unbounded) operator on H with dense domain.*

Let $m : [0, +\infty) \to [0, +\infty)$ be a continuous function. Let $b : [0, +\infty) \to (0, +\infty)$ be a continuous function such that

$$\int_0^{+\infty} b(s)\, ds < +\infty. \tag{4.1}$$

Let $(u_0, u_1) \in D(A) \times D(A^{1/2})$ be such that

$$|u_1|^2 + \int_0^{|A^{1/2}u_0|^2} m(\sigma)\, d\sigma > 0. \tag{4.2}$$

Let us assume that for some $\varepsilon > 0$ problem (1.1), (1.2) has a global solution

$$u_\varepsilon \in C^2([0, +\infty); H) \cap C^1([0, +\infty); D(A^{1/2})) \cap C^0([0, +\infty); D(A)). \tag{4.3}$$

Then we have that

$$\liminf_{t \to +\infty} \left(|u_\varepsilon'(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2 \right) > 0. \tag{4.4}$$

The proof of this result is very simple, and relies on the usual Hamiltonian

$$\mathcal{H}_\varepsilon(t) := \varepsilon |u_\varepsilon'(t)|^2 + \int_0^{|A^{1/2}u_\varepsilon(t)|^2} m(\sigma)\, d\sigma.$$

Assumption (4.2) is equivalent to say that $\mathcal{H}_\varepsilon(0) > 0$. Moreover we have that

$$\mathcal{H}_\varepsilon'(t) = -2b(t)|u_\varepsilon'(t)|^2 \geq -\frac{2}{\varepsilon}b(t)\mathcal{H}_\varepsilon(t) \qquad \forall t \geq 0,$$

hence

$$\mathcal{H}_\varepsilon(t) \geq \mathcal{H}_\varepsilon(0) \exp\left(-\frac{2}{\varepsilon} \int_0^t b(s)\, ds \right) \qquad \forall t \geq 0.$$

For a fixed $\varepsilon > 0$, the right-hand side is greater than a positive constant independent on t because of (4.1) and the fact that $\mathcal{H}_\varepsilon(0) > 0$. This implies (4.4).

5 The nondegenerate case

In this section we focus on the hyperbolic equation (1.1) under the non-degeneracy assumption $\mu > 0$.

The case with constant dissipation was considered independently by E. H. de Brito [3] and by Y. Yamada [31]. They proved existence of a global solution provided that ε is small enough. Decay estimates for these solutions were proved by Y. Yamada [31] in the non-coercive case, and by E. H. de Brito [4] and by M. Hosoya and Y. Yamada [18] in the coercive case. All these estimates were afterwards reobtained as a particular case of the theory developed in [10].

More recently, H. Hashimoto and T. Yamazaki [17] proved that for initial data $(u_0, u_1) \in D(A^{3/2}) \times D(A)$ one has that

$$|\rho_\varepsilon(t)|^2 + (1+t)|A^{1/2}\rho_\varepsilon(t)|^2 + \varepsilon(1+t)^2|r'_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall t \geq 0,$$

where of course C doesn't depend on t and ε. When $(u_0, u_1) \in D(A^2) \times D(A)$, the coefficient ε in the left-hand side may be dropped, thus providing a better convergence rate on $r'_\varepsilon(t)$. This is a first example of *decay-error estimate*.

The weakly dissipative case was considered only in last years. Apart from a result obtained in a special situation by M. Nakao and J. Bae [24], the problem in its full generality was solved by T. Yamazaki [33] in the subcritical case $p < 1$ with some technical requirements on initial data, and then by the authors [12] (see also [34]) in the general case $p \leq 1$ with minimal requirements on initial data.

The results are the following.

Theorem 5.1 (Hyperbolic problem: global existence) *Let H be a Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator on H with dense domain. Let $\mu > 0$, and let $m : [0, +\infty) \to [\mu, +\infty)$ be a locally Lipschitz continuous function. Let $b(t) := (1+t)^{-p}$ with $p \in [0, 1]$, and let $(u_0, u_1) \in D(A) \times D(A^{1/2})$.*

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (1.1), (1.2) has a unique global solution u_ε satisfying (4.3).

Theorem 5.2 (Hyperbolic problem: decay estimates) *Under the same assumptions of Theorem 5.1 there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that*

$$|u_\varepsilon(t)|^2 + (1+t)^{p+1}|A^{1/2}u_\varepsilon(t)|^2 + (1+t)^2|u'_\varepsilon(t)|^2 \leq C \qquad \forall t \geq 0,$$

$$\varepsilon|A^{1/2}u'_\varepsilon(t)|^2 + |Au_\varepsilon(t)|^2 \leq \frac{C}{(1+t)^{2(p+1)}} \qquad \forall t \geq 0,$$

$$\int_0^{+\infty} (1+t)^p \left(|u'_\varepsilon(t)|^2 + |A^{1/2}u_\varepsilon(t)|^2\right) dt \leq C,$$

$$\int_0^{+\infty} (1+t)^{2p+1} \left(|A^{1/2}u'_\varepsilon(t)|^2 + |Au_\varepsilon(t)|^2\right) dt \leq C.$$

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Theorem 5.3 (Singular perturbation: decay-error estimates)   *Let H, A, μ, $m(\sigma)$, $b(t)$, p, u_0, u_1, ε_0 be as in Theorem 5.1. Let $u(t)$ be the solution of the corresponding parabolic problem (1.4), and let $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ be defined by (1.6).*
    Then we have the following conclusions.

(1) *Without further assumptions on initial data, namely $(u_0, u_1) \in D(A) \times D(A^{1/2})$, we have that*

$$|\rho_\varepsilon(t)|^2 + (1+t)^{p+1}|A^{1/2}\rho_\varepsilon(t)|^2 + (1+t)^{2(p+1)}|A\rho_\varepsilon(t)|^2 + (1+t)^2|r'_\varepsilon(t)|^2 \to 0$$

uniformly in $[0, +\infty)$, and

$$\int_0^{+\infty} (1+t)^p \left(|r'_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2\right) dt \to 0,$$

$$\int_0^{+\infty} (1+t)^{2p+1} \left(|A^{1/2}r'_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2\right) dt \to 0.$$

(2) *If in addition we assume that $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, then there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that*

$$|\rho_\varepsilon(t)|^2 + (1+t)^{p+1}|A^{1/2}\rho_\varepsilon(t)|^2 + \varepsilon(1+t)^{p+1}|r'_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall t \geq 0,$$

$$\int_0^{+\infty} (1+t)^p \left(|r'_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2\right) dt \leq C\varepsilon^2.$$

(3) *If in addition we assume that $(u_0, u_1) \in D(A^2) \times D(A)$, then there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that*

$$(1+t)^{2(p+1)}|A\rho_\varepsilon(t)|^2 + (1+t)^2|r'_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall t \geq 0,$$

$$\int_0^{+\infty} (1+t)^{2p+1} \left(|A^{1/2}r'_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2\right) dt \leq C\varepsilon^2.$$

We point out that in the previous three theorems the operator A is never assumed to be coercive.

The global-in-time convergence rates (with respect to ε) appearing in Theorem 5.3 are optimal because they coincide with the local-in-time convergence rates of Theorem 3.2, which in turn are the same of the linear case.

The decay rates (with respect to time) appearing in Theorem 5.2 and Theorem 5.3 are optimal for non-coercive operators. In this case indeed they coincide with the decay rates of solutions of the corresponding parabolic equation, as shown in Table 2.

In the coercive case these decay rates are not optimal. In the case $p = 0$ indeed we know that solutions exponentially decay to zero as solutions of the parabolic problem (see [4, 18, 10]). We strongly suspect that the same is true also for every $p \in [0, 1]$, namely that solutions decay as shown in the first three rows of Table 2. Of course also the decay rates in Theorem 5.3 should be changed accordingly. We give no precise statement or reference because this part of the theory has never been put into writing.

6 The degenerate case

Several papers have been devoted to global existence and decay estimates for equation (1.1) in the degenerate case $\mu \geq 0$. Let us begin with constant dissipation. In this case global existence results (provided that the problem is mildly degenerate and ε is small enough) were proved by K. Nishihara and Y. Yamada [25] in the case where $m(\sigma) = \sigma^\gamma$ (with $\gamma \geq 1$), by the authors [8] in the case where $m(\sigma) \geq 0$ is any Lipschitz continuous function, and by the first author [6, 7] in the non-Lipschitz case where $m(\sigma) = \sigma^\gamma$ with $\gamma \in (0, 1)$.

Decay estimates have long been studied for equations with constant dissipation. In the case $m(\sigma) = \sigma^\gamma$ with $\gamma \geq 1$, the first decay estimates were obtained by K. Nishihara and Y. Yamada [25] in the coercive case, and by K. Ono [29] in the non-coercive case. The case $m(\sigma) = \sigma^\gamma$ with $\gamma \in (0, 1)$ was considered in [6]. In the special case $m(\sigma) = \sigma$, T. Mizumachi [22, 23] and K. Ono [26, 27] proved better decay estimates, namely estimates with decay rates which are faster than those obtained by putting $\gamma = 1$ in the previous ones. This in particular showed that the previous results were not optimal.

A complete answer was given by the authors in [10], where the case of a general nonlinearity $m(\sigma) \geq 0$ is considered. The decay rates obtained in [10] coincide with the decay rates of solutions of the parabolic problem.

Let us consider now the equation with weak dissipation, focussing on the model case

$$\varepsilon u_\varepsilon''(t) + \frac{1}{(1+t)^p} u_\varepsilon'(t) + |A^{1/2} u_\varepsilon(t)|^{2\gamma} A u_\varepsilon(t) = 0, \tag{6.1}$$

of course with the mild non-degeneracy assumption (1.3). The only previous result we are aware of was obtained by K. Ono [28]. In the special case $\gamma = 1$ he proved that a global solution exists provided that ε is small and $p \in [0, 1/3]$. The reason of the slow progress in this field is hardly surprising. In the weakly dissipative case existence and decay estimates must be proved in the same time. The better are the decay estimates, the stronger is the existence result.

Ten years ago decay estimates for degenerate equations were far from being optimal, but for the special case $\gamma = 1$. In [10] a new method for obtaining optimal decay estimates was introduced. This allowed a substantial progress on equation (6.1).

Let us begin with our existence and decay results proved in [13]. The first one concerns the coercive case.

Theorem 6.1 (Coercive case: global existence and decay estimates) *Let H be a Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator with dense domain. Let us assume that A is coercive ($\nu > 0$). Let $\gamma > 0$, and let $p \in [0, 1]$. Let us assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy (1.3).*

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (6.1), (1.2) has a unique global solution satisfying (4.3).

Moreover there exist positive constants C_1 and C_2 such that

$$\frac{C_1}{(1+t)^{(p+1)/\gamma}} \leq |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)/\gamma}} \qquad \forall t \geq 0,$$

$$\frac{C_1}{(1+t)^{(p+1)/\gamma}} \leq |Au_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)/\gamma}} \qquad \forall t \geq 0,$$

$$|u_\varepsilon'(t)|^2 \leq \frac{C_2}{(1+t)^{2+(p+1)/\gamma}} \qquad \forall t \geq 0.$$

We point out that Theorem 6.1 is optimal both in the sense that all $p \in [0,1]$ are considered, and in the sense that solutions decay as in the parabolic case (see Table 2).

In the non-coercive case we have the following result.

Theorem 6.2 (Non-coercive case: global existence and decay estimates) *Let H be a Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator with dense domain. Let $\gamma \geq 1$, and let*

$$0 \leq p \leq \frac{\gamma^2 + 1}{\gamma^2 + 2\gamma - 1}. \tag{6.2}$$

Let us assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfy (1.3).

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (6.1), (1.2) has a unique global solution satisfying (4.3).

Moreover there exist constants C_1 and C_2 such that

$$\frac{C_1}{(1+t)^{(p+1)/\gamma}} \leq |A^{1/2}u_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)/(\gamma+1)}} \qquad \forall t \geq 0,$$

$$|Au_\varepsilon(t)|^2 \leq \frac{C_2}{(1+t)^{(p+1)/\gamma}} \qquad \forall t \geq 0,$$

$$|u_\varepsilon'(t)|^2 \leq \frac{C_2}{(1+t)^{[2\gamma^2+(1-p)\gamma+p+1]/(\gamma^2+\gamma)}} \qquad \forall t \geq 0.$$

Theorem 6.2 doesn't represent a final answer in the non-coercive case. Let indeed p_γ denote the right-hand side of (6.2). It is easy to see that $p_\gamma \leq 1$ for every $\gamma \geq 1$, with equality only when $\gamma = 1$, and asymptotically as $\gamma \to +\infty$. Since we have hyperbolic behavior when $p > 1$ (see Section 4), and parabolic behavior for $p \in [0, p_\gamma]$, this means that there is a non-man's land between p_γ and 1 where things are not clear yet.

The only case where this region is empty is when $\gamma = 1$. In this case all exponents $p \in [0,1]$ fall in the parabolic regime, and this improves the result obtained in [28] ($p \in [0, 1/3]$) also in the case $m(\sigma) = \sigma$.

We stated Theorem 6.2 assuming $\gamma \geq 1$. In the case $\gamma \in (0,1)$ we have a weaker result, namely global existence for $p \in [0, \gamma/(\gamma + 2)]$ (see [13, Remark 2.6]). Figure 1 represents hyperbolic and parabolic regimes, and the no-man's land in between.

The singular perturbation problem is still quite open in the degenerate case. We have indeed only the following partial result for the constant dissipation case (see [11]).
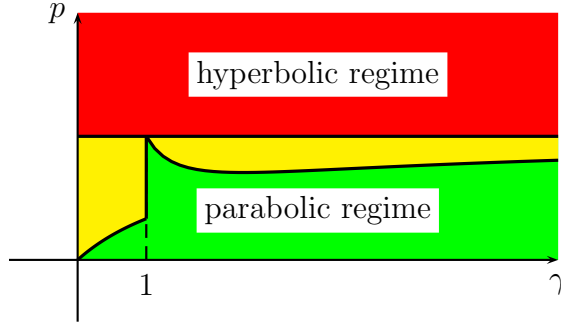
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Figure 1: parabolic and hyperbolic regimes in the degenerate non-coercive case

Theorem 6.3 (Constant dissipation: global-in-time error estimates) *Let H be a Hilbert space, and let A be a nonnegative self-adjoint (unbounded) operator with dense domain. Let $u_\varepsilon(t)$ be the solution of equation (6.1) with $\gamma > 0$, $p = 0$, and initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ satisfying (1.3). Let $u(t)$ be the solution of the corresponding parabolic problem, and let $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ be defined by (1.6).*

Then we have the following conclusions.

(1) *Without further assumptions on initial data, namely $(u_0, u_1) \in D(A) \times D(A^{1/2})$, we have that*

$$|\rho_\varepsilon(t)|^2 + |A^{1/2}\rho_\varepsilon(t)|^2 + |A\rho_\varepsilon(t)|^2 + |r'_\varepsilon(t)|^2 \to 0$$

uniformly in $[0, +\infty)$, and

$$\int_0^{+\infty} \left(|r'_\varepsilon(t)|^2 + |A^{1/2}r'_\varepsilon(t)|^2 \right) dt \to 0.$$

(2) *If in addition we assume that $\gamma \geq 1$ and $(u_0, u_1) \in D(A^{3/2}) \times D(A^{1/2})$, then there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that*

$$|\rho_\varepsilon(t)|^2 + \varepsilon|A^{1/2}\rho_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall t \geq 0,$$

$$\int_0^{+\infty} |r'_\varepsilon(t)|^2 \, dt \leq C\varepsilon.$$

(3) *If in addition we assume that $\gamma \geq 1$ and $(u_0, u_1) \in D(A^{3/2}) \times D(A)$, then there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that*

$$|\rho_\varepsilon(t)|^2 + \varepsilon^{2/3}|A^{1/2}\rho_\varepsilon(t)|^2 + \varepsilon^{4/3}|A\rho_\varepsilon(t)|^2 + \varepsilon^{4/3}|r'_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall t \geq 0.$$

Theorem 6.3 is far from being optimal. First of all most of the convergence rates in the second and third statement are weaker than the corresponding rates in Theorem 3.2. Moreover, all statements present just error estimates, and not decay-error estimates as in Theorem 5.3. It is possible to add some decays with some extra work, but in any case they are so far from those appearing in Theorem 6.1 and Theorem 6.2 that we decided not to include them. Last but not least, Theorem 6.3 is limited to equations with constant dissipation.

7 Open problems

The main open problem in the theory of Kirchhoff equations is existence of global solutions. We have seen that in the dissipative case an affirmative answer can be given provided that ε is small enough. So the first question is whether this condition is necessary or not.

Open problem 1 *Let us consider equation (1.1) with $m : [0, +\infty) \to [1, +\infty)$ of class C^∞, and constant dissipation $b(t) \equiv 1$. Let us assume that $(u_0, u_1) \in D(A^\infty) \times D(A^\infty)$, where $D(A^\infty)$ is the intersection of all spaces $D(A^\alpha)$ with $\alpha \geq 0$.*
Does the Cauchy problem (1.1), (1.2) admit a global solution for every $\varepsilon > 0$?

We stated the question with generous assumptions both on the nonlinearity (smoothness and strict hyperbolicity), and on initial data (regularity). In any case there are no counterexamples, even with less regular terms and data, or with $b(t) \equiv 0$.

Even assuming the smallness of ε, one may ask if a global solution exists under assumptions weaker than those required in the previous sections. This leads to the following question.

Open problem 2 *Let us consider the Cauchy problem (1.1), (1.2) in each of the following situations.*

- *In the hyperbolic regime where $b(t) = (1 + t)^{-p}$ with $p > 1$.*

- *In the case where assumption $b(t) = (1 + t)^{-p}$ with $p \leq 1$ is replaced by the weaker condition (4.1).*

- *In the really degenerate case $m(|A^{1/2}u_0|^2) = 0$.*

Is it possible to prove global existence provided that ε is small enough?

A third question related to global existence issues concerns the regularity of initial data. All existence results stated in the previous sections assume that $(u_0, u_1) \in D(A) \times D(A^{1/2})$. On the other hand, the classical local existence results for the non-dissipative equation require the weaker assumption $(u_0, u_1) \in D(A^{3/4}) \times D(A^{1/4})$. Therefore a natural question is whether the global existence results for dissipative equations can be extended to this weaker class of data.

In the constant dissipation case, it is not difficult to give an affirmative answer when $\mu > 0$ or when $m(\sigma) = \sigma^\gamma$ with $\gamma \geq 2$. On the contrary, the proof given in [8] for a general locally Lipschitz continuous non-linearity $m(\sigma) \geq 0$ seems to require in an essential way that $(u_0, u_1) \in D(A) \times D(A^{1/2})$. So the problem is the following.

Open problem 3 *Let us consider equation (1.1) with constant dissipation $b(t) \equiv 1$, and with any locally Lipschitz continuous nonlinearity $m(\sigma) \geq 0$. Let us assume that $(u_0, u_1) \in D(A^{3/4}) \times D(A^{1/4})$ satisfy the non-degeneracy condition (1.3).*
Does problem (1.1), (1.2) admit a global solution for every small enough ε?

The last open question concerning existence is how to fill the no-man's zone left by Theorem 6.2 and described in Figure 1.

Open problem 4 *Let H, A, u_0, u_1 be as in Theorem 6.2. Let us assume that either $\gamma \in (0,1)$ and $p \in (\gamma/(\gamma+2), 1]$, or that $\gamma > 1$ and $p \in (p_\gamma, 1]$, where p_γ is the right-hand side of (6.2).*

Does problem (6.1), (1.2) admit a global solution whenever ε is small enough?

All previous examples suggest that the answer should be affirmative, but a proof seems to require some new ideas.

The singular perturbation problem is arguably the new frontier in this research field. This problem has been quite well understood only in the nondegenerate case, in which case, however, the decay rates are optimal only for non-coercive operators. A first open question is therefore the following.

Open problem 5 *Let the assumptions of Theorem 5.3 be satisfied. Let us assume also that the operator A is coercive ($\nu > 0$).*

Prove the same conclusions of Theorem 5.3 with all polynomial decay rates such as $(1+t)^\beta$ replaced by exponential decay rates of the form $\exp(\alpha(1+t)^{p+1})$, where α is a suitable constant.

The singular perturbation problem is quite open in the degenerate case. One should try to extend Theorem 6.3 in order to allow weak dissipations, and involve better decay and convergence rates. An example of open question is the following.

Open problem 6 *Let H, A, γ, p, u_0, u_1, ε_0 be as in Theorem 6.1. Let $u(t)$ be the solution of the corresponding parabolic problem, and let $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ be defined by (1.6).*

Under the appropriate conditions on initial data, prove that there exists a constant C such that for every $\varepsilon \in (0, \varepsilon_0)$ we have that

$$(1+t)^{(p+1)/\gamma}|A^{1/2}\rho_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall t \geq 0,$$

$$(1+t)^{2+(p+1)/\gamma}|r'_\varepsilon(t)|^2 \leq C\varepsilon^2 \qquad \forall t \geq 0.$$

In this estimates we require on $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ the same decay rates (as $t \to +\infty$) of $u_\varepsilon(t)$ and $u(t)$ separately, and we require the same convergence rates (as $\varepsilon \to 0^+$) of the local-in-time error estimates. We actually suspect that in the degenerate case the remainders $r_\varepsilon(t)$ and $\rho_\varepsilon(t)$ decay faster than $u_\varepsilon(t)$ and $u(t)$.

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