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Asymptotic Behaviour for the Kirchhoff Equation (*).

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Abstract. – *In this paper we study the asymptotic behaviour of the solution u of the Kirchhoff equation with small data. More precisely we show that*

$$\lim_{t\to\infty} \|\nabla^k(u-v)\|_2 + \|\nabla^k(u-v)_t\|_2 = 0 \quad \text{for every } k \in N$$

where v is a suitable solution of an appropriate wave equation. Moreover we give some estimates on $\lim_{t\to\infty} \|\nabla u\|_2$.

1. – Introduction.

Given a function m of class C^1 satisfying:

$$m(r) \geqslant \nu > 0\,, \qquad \forall r \geqslant 0$$

we consider the Cauchy problem on R^j:

$$u_{tt} - m\left(\int_{R^j} |\nabla u|^2\right) \Delta u = 0\,, \tag{1}$$

$$u(0, x) = u_0 \qquad u_t(0, x) = u_1\,, \tag{2}$$

where $u_0, u_1 \in C_0^\infty(R^j)$.

In the case $j = 1$ and $m(s) = 1 + s$ equation (1) has been proposed by G. KIRCHHOFF [9] as a model equation for the transversal motion of a stretched string. After the pioneering paper of S. BERNSTEIN [4] who proved local existence for initial data in suitable Sobolev spaces and the global existence for real analytic data, several authors have studied this (ore related) problem. We refer to A. AROSIO [1] and S. SPAGNOLO [10] for a complete bibliography.

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We remark that for the equation (1) there exists a conserved energy:

$$E(t, u) = \frac{1}{2}\left(\int_{R^j} |u_t|^2 + M\left(\int_{R^j} |\nabla u|^2\right)\right) = E(0, u) \tag{3}$$

with

$$M(s) = \int_0^s m(r)\, dr .$$

The global existence for (1)-(2) with (non analytic) small data has been proved by J. M. GREENBERG and S. H. HU [7] in the case $j = 1$ and extended to the case $j \geq 2$ in the following sense:

THEOREM 1 (P. D'ANCONA and S. SPAGNOLO [5]-[6]). – *For all $j \geq 1$ there exists some $\varepsilon_j > 0$ (depending on the function m) for which (1), (2) has a unique, global, smooth solution $u(t, x)$ as soon as:*

- *if $j = 1$*

$$\sum_{|\alpha| \leq 1} \int_R (1 + |x|^6)(|D^\alpha u_0|^2 + |D^\alpha u_1|^2)\, dx \leq \varepsilon_1 . \tag{4}$$

- *if $j > 1$*

$$\sum_{|\alpha| \leq 1} \int_{R^j} (1 + |x|^{2(j+1)})(|D^\alpha u_0|^2 + |D^\alpha u_1|^2)\, dx \leq \varepsilon_j . \tag{5}$$

Moreover the function $c(t)$ defined as

$$c(t) = \left(m\left(\int_{R^j} |\nabla u|^2\right)\right)^{1/2}$$

satisfies the estimate

- *if $j = 1$*

$$|c'(t)| \leq K(1 + t)^{-3} \tag{6}$$

for some $K = K(m, u_0, u_1)$;

- *if $j > 1$*

$$|c'(t)| \leq K(1 + t)^{-(j+1)} \tag{7}$$

for some $K = K(m, u_0, u_1)$.

Actually in [6] the estimate (7) was proved only for $k \leq j$, however the extension to the case $k \leq j+1$ can be easly proved by a simple modification of Lemma A in [6].

The purpose of the present paper is to complete this existence result by studying the asymptotic behaviour of the solution $u(t, x)$ for $t \to \infty$.

In order to state our result, we define, for any (u_0, u_1) satisfying (4)-(5) the constant $c_\infty = c_\infty(u_0, u_1)$ given by:

$$c_\infty^2 := \lim_{t \to \infty} c^2(t)$$

(the existence of a finite limit being assured by (6)-(7)) and we consider the corresponding wave equation:

$$v_{tt} - c_\infty^2 \Delta v = 0 . \tag{8}$$

We have then:

THEOREM 2. – *Let ε_j be as in Theorem 1. Then, for all (u_0, u_1) satisfying (4)-(5) there exists a solution v of the equation (8) such that*[1]

(i)

$$\lim_{t \to \infty} \|\nabla^k (u - v)\|_2 + \|\nabla^k (u - v)_t\|_2 = 0 \quad \textit{for every } k \in N; \tag{9}$$

(ii) *the limits*

$$\lim_{t \to \infty} \|\nabla u\|_2^2 = a_\infty \qquad \lim_{t \to \infty} \|u_t\|_2^2 = b_\infty$$

do exist in $\mathfrak{R}$ and are related with c_∞ by the equalities:

- $b_\infty = c_\infty^2 a_\infty , \quad c_\infty^2 = m(a_\infty),$
-

$$\frac{1}{2}(c_\infty^2 a_\infty + M(a_\infty)) = E(0, u); \tag{10}$$

(iii) *let us define:*

$$F(r) = \frac{1}{2}(m(r) r + M(r)) .$$

Let us suppose that F is nondecreasing or F satisfies:

- (HP) *there exists $0 < \bar{r}$ such that F is nondecreasing in $[0, \bar{r}]$ and nonincreasing in a right neighbourhood of $\bar{r}$.*

Then

$$a_\infty = \min\{r \colon F(r) = E(0, u)\} .$$

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(1) In the present paper we shall use the following notation:

$$\|f\|_p = \|f\|_{L^p(R^j)} .$$

REMARK 3. – *The equalities* (10) *and* (3) *allow us to calculate* a_∞ *and hence* c_∞ *in terms of the initial energy* $E(0, u)$.

Moreover we can rewrite (10) *as* $F(a_\infty) = E(0, u)$. *In the special case when* F *is one to one, or equivalently* (*since* $F'(r) = m(r) + (1/2)rm'(r) = (1/2r)(r^2 m(r))'$) *provided that* $r^2 m(r)$ *is strictly increasing, we have*:

$$a_\infty = F^{-1}(E(0, u)).$$

We note that F *is always strictly increasing in a right neighbourhood of* $r = 0$. *For instance in the Kirchhoff originary case* $m(r) = 1 + r$ *we have*:

$$c_\infty^2 = \frac{1}{3} + \frac{2}{3}(1 + 3E(0, u))^{1/2}.$$

REMARK 4 (see after the proof of Theorem 2). – *Under the assumptions of Theorem 2 we have continuous dependence from the initial data in the following sense*: *if* $(u_{0n}(x), u_{1n}(x)) \to (u_0(x), u_1(x))$ *in* H^∞, *and* $u_n(t, x)$ *and* $u(t, x)$ *are the corresponding solutions of* (1) *we have*:

$$\lim_{t \to \infty} (\|\nabla^k(u_n - u)\|_2^2 + \|\nabla^k(u_n - u)_t\|_2^2) = 0 \qquad k \in N.$$

Moreover denoting

$$a_{\infty, n} = \lim_{t \to \infty} \|\nabla u_n\|_2^2, \qquad a_\infty = \lim_{t \to \infty} \|\nabla u\|_2^2 \tag{11}$$

we obtain

$$a_{\infty, n} \to a_\infty \qquad \text{for } n \to \infty. \tag{12}$$

REMARK 5 (see after the proof of Theorem 2). – *Let us assume* $F(r)$ *satisfies* (*HP*).

Then it is possible to find $\bar{a} \le F(\bar{r})$ *in such a way that for every initial data* u_0 *and* u_1 *with energy* $E(0, u_0, u_1) < \bar{a}$ *the following properties are verified*:

i) *the problem* (1)-(2) *has a global solution,*

ii) *the estimate* (10) *is verified,*

iii) *we have* (12) *for* $u_{0,n} = \lambda_n u_0$ *and* $u_{1,n} = \lambda_n u_1$, $\lambda_n \to \lambda \in [0, 1]$.

Furthermore until $E(0, u_0, u_1) < \bar{a}$, *in* (10) *we have*

$$a_\infty = \min\{r\colon F(r) = E(0, u)\}. \tag{13}$$

Moreover i), ii), iii) *can not be verified simultaneously for every initial data* u_0 *and* u_1 *with energy* $E(0, u_0, u_1) \ge \bar{a}$.

REMARK 6. – *We can treat the case when* $t \to -\infty$ *like the case* $t \to +\infty$. *Indeed we have* $u(t, x) = w(-t, x)$, *where* u *is the solution of* (1)-(2) *and* w *is the solution of* (1) *with data* u_0 *and* $-u_1$.

Moreover let us denote by F the map:

$$F\colon C_s \to (H^\infty(R^j) \times H^\infty(R^j))$$

where

$$C_s = \{(u_0, u_1) \in (C_0^\infty(R^j) \times C_0^\infty(R^j)) \quad \text{satisfying (4) and (5)}\}$$

and $F((u_0, u_1)) = (v_0, v_1)$ where (v_0, v_1) are the data of the solution v of the wave equation in Theorem 2. Now we have the following result:

THEOREM 7. – *The map* F *is well defined and continuous in the* H^∞ *topology.*

The proof of Theorem 2 is based on a representation of the Fourier's transform of the solutions of the equation (1).

In the proof of Theorem 7 we use an «explicit» representation of the Fourier's transforms of (v_0, v_1).

2. – Proofs.

PROOF OF THEOREM 2.

1) *We prove the limit* (9). We adapt the argument in [2], to the case of unbounded domains.

Let us denote by $y(t, \xi)$ the Fourier transform of the solution u of (1)-(2). We have:

$$\begin{aligned} &y'' + c^2(t)|\xi|^2 y = 0, \\ &y(0) = y_0; \quad y'(0) = y_1. \end{aligned} \tag{14}$$

Let us fix ξ, there exist $a(\xi)$ and $\varphi(\xi)$ such that (see [3]):

$$\lim_{t \to \infty} y - a \sin\left(|\xi| \int_0^t c(t)\, dt + \varphi\right) = 0, \tag{15}$$

$$\lim_{t \to \infty} y' - a|\xi| c_\infty \cos\left(|\xi| \int_0^t c(t)\, dt + \varphi\right) = 0. \tag{16}$$

The constants $a(\xi)$ and $\varphi(\xi)$ are not unique, nevertheless, since y and y' are continuous functions of the parameter ξ, we can choose them in such a way that, as functions of ξ they are continuous.

Thanks to (6)-(7) we have:

$$\beta = \int_0^\infty (c(t) - c_\infty)\, dt < +\infty. \tag{17}$$

Let us define $\psi = \varphi + |\xi| \int_0^\infty (c(t) - c_\infty)\,dt$, then

$$\lim_{t \to \infty} y - a \sin(|\xi| t c_\infty + \psi) = 0\,,$$

$$\lim_{t \to \infty} y' - a|\xi| c_\infty \cos(|\xi| t c_\infty + \psi) = 0\,.$$

Let us define $z(t, \xi) = a \sin(|\xi| t c_\infty + \psi)$. We obtain

$$\lim_{t \to \infty} y(t, \xi) - z(t, \xi) = 0 \quad \text{and} \quad \lim_{t \to \infty} y'(t, \xi) - z'(t, \xi) = 0\,. \tag{18}$$

Moreover z is a solution of the Fourier transform of the equation (8). We need to prove that z is the Fourier transform of a solution v of (8) that verifies (9).

(*a*) Now let us given some estimates for the function y. We remark that, since $\int_0^\infty |(c^2(t))'|\,dt < +\infty$, it is easy to prove, by multiplying (14) for $|\xi|^{2k}$, that there exists constant a_k, b_k $(k \in N)$ such that:

- $$E_{c,k}(t, \xi) = c^2(t)|\xi|^{2k+2} y^2(t, \xi) + |\xi|^{2k} y'^2(t, \xi) \le a_k E_{c,k}(0, \xi)\,,$$

- $$E_k(t, \xi) = |\xi|^{2k+2} y^2(t, \xi) + |\xi|^{2k} y'^2(t, \xi) \le a_k E_k(0, \xi)\,. \tag{19}$$

Moreover

$$(y^2)'' = 2(y')^2 - 2c^2(t)|\xi|^2 y^2 \le 2b_0 E_{c,0}(0, \xi)$$

then

$$y^2(t, \xi) \le b_0 E_{c,0}(0, \xi) t^2 + 2y_0 y_1 t + y_0^2\,. \tag{20}$$

(*b*) Now let us find an «explicit» representation of z. Let us denote:

- $h(t) = \int_0^t c(\tau)\,d\tau$,

- $r(t, \xi) = \int_t^{+\infty} y(\tau, \xi)|\xi| \cos(|\xi| h(\tau))\, c'(\tau)\,d\tau$,

- $s(t, \xi) = \int_t^\infty y(\tau, \xi)|\xi| \sin(|\xi| h(\tau))\, c'(\tau)\,d\tau$.

We remark that $r(t, \xi)$ and $s(t, \xi)$ are finite for every (t, ξ). Indeed it is enough to apply Holder's inequality to

$$|c'(\tau)|^{1/2}(|c'(\tau)|^{1/2} |\xi|\, |y(\tau, \xi)|)\,,$$

and use (19).

By [3] page 137 we have

$$y(t, \xi) = z_1(t, \xi) + \sigma_1 \cos(|\xi| h(t)) + \sigma_2 \sin(|\xi| h(t)) = z_1(t, \xi) + \alpha(t, \xi)\,,$$

where

$$\sigma_1 = \left(\frac{c_\infty - c(t)}{c_\infty}\gamma - r(t, \xi)\right)(c(t)|\xi|)^{-1};$$

$$\sigma_2 = \left(\frac{c_\infty - c(t)}{c_\infty}\delta - s(t, \xi)\right)(c(t)|\xi|)^{-1};$$

and

$$y = (c(t)|\xi|)^{-1}((\gamma - r(t, \xi)) \cos(|\xi| h(t)) + (\delta - s(t, \xi)) \sin(|\xi| h(t)))\,,$$

for $\gamma(\xi)$ and $\delta(\xi)$ suitable functions of ξ. By the previous equalities we have:

$$\gamma(\xi) = y_0 c(0)|\xi| + r(0, \xi) \quad \text{and} \quad \delta(\xi) = y_1 + s(0, \xi)\,.$$

Now it is easy to calculate $\sigma_1(0, \xi)$ and $\sigma_2(0, \xi)$:

$$\sigma_1(0, \xi) = (c_\infty - c(0))\, c_\infty^{-1} y_0 - (c_\infty |\xi|)^{-1} r(0, \xi)\,,$$

$$\sigma_2(0, \xi) = ((c_\infty - c(0))(c(0)|\xi| c_\infty)^{-1} y_1 - (|\xi| c_\infty)^{-1} s(0, \xi))\,.$$

Then:

$$\begin{aligned} z_1(0, \xi) &= (|\xi| c_\infty)^{-1}(y_0 |\xi| c(0) + r(0, \xi))\,, \\ z_1'(0, \xi) &= c_\infty^{-1} c(0)(y_1 + s(0, \xi))\,. \end{aligned} \tag{21}$$

Moreover

$$\begin{aligned} z_1(t, \xi) &= (|\xi| c_\infty)^{-1} \gamma \cos(|\xi| h(t)) + (|\xi| c_\infty)^{-1} \delta \sin(|\xi| h(t)) = \\ &= z_1(0, \xi) \cos(|\xi| h(t)) + (|\xi| c(0))^{-1} z_1'(0, \xi) \sin(|\xi| h(t))\,. \end{aligned}$$

Now let us define

$$z(t, \xi) = z_1(0, \xi) \cos(c_\infty |\xi| t + |\xi| \beta) + \frac{z_1'(0, \xi)}{|\xi| c(0)} \sin(c_\infty |\xi| t + |\xi| \beta)\,. \tag{22}$$

Then z is a solution of the Fourier transform of (8).

By (21) and (22) we have:

$$c_\infty^2 \, |z(t,\xi)|^2 \leq 4\left(c^2(0)|y_0|^2 + \left(\int_0^{+\infty} |y(s,\xi)\,c'(s)|\,ds\right)^2 + \right.$$

$$\left. + |c_\infty t + \beta|^2 \left(\frac{\sin(c_\infty |\xi| t + |\xi|\beta)}{c_\infty |\xi| t + |\xi|\beta}\right)^2 (|y_1|^2 + |s(0,\xi)|^2)\right).$$

Therefore:

$$c_\infty^2 \|z(t,\cdot)\|_2^2 \leq 4\left(c^2(0)\|y_0\|^2 + \int_{R^j}\left(\int_0^{+\infty} |y(s,\xi)\,c'(s)|\,ds\right)^2 d\xi + \right.$$

$$\left. + (tc_\infty + \beta)^2 \left(\|y_1\|_2^2 + \int_{R^j} |s(0,\xi)|^2 d\xi\right)\right).$$

Now let us set

$$\eta = \int_{R^j}\left(\int_0^{+\infty} |y(s,\xi)\,c'(s)|\,ds\right)^2 d\xi,$$

by Holder's inequality we have:

$$\eta \leq \int_0^{+\infty} |c'(s)|^{2/5} ds \int_{R^j}\int_0^{+\infty} |c'(s)|^{8/5} |y(s,\xi)|^2 ds\, d\xi .$$

Therefore by (20) there exists a constant $\bar{b}_1$ such that

$$\eta \leq \bar{b}_1 (\| |\xi| y_0 \|_2^2 + \|y_1\|_2^2 + \|y_0\|_2^2).$$

In the same way we can obtain

$$\int_{R^j} |s(0,\xi)|^2 d\xi \leq \bar{b}_2 (\| |\xi| y_0 \|_2^2 + \|y_1\|_2^2).$$

Hence there exists a constant $b = b(u_0, u_1)$ such that

$$\|z(t,\cdot)\|_2^2 \leq b(1+t)^2 .$$

In the same way one can prove that

$$\| |\xi|^k z(t,\cdot)\|_2^2 < +\infty \quad \text{for every } t \text{ and } k \in N.$$

Moreover

$$z'(t,\xi) = -z_1(0,\xi)|\xi| c_\infty \sin(|\xi| c_\infty t + |\xi|\beta) + z_1'(0,\xi)\frac{c_\infty}{c(0)}\cos(|\xi| c_\infty t + |\xi|\beta).$$

Therefore we can easily prove that

$$\| |\xi|^k z'(t,\cdot)\|_2^2 < +\infty \quad \text{for every } t \text{ and } k \in N.$$

By this fact $z(t,\cdot)$ is the Fourier transform of a solution of (8) with data in H^∞.

(c) Let us show (9). Let v the reverse transform of z. Since u, and $v \in L^2(R^j)$ one has:

$$\|(u-v)\|_2^2 \leq \|(y - z_1)\|_2^2 + \|z_1 - z\|_2^2 = \|\alpha\|_2^2 + \|z_1 - z\|_2^2 .$$

Moreover we have

$$\nu|\alpha(t,\xi)| \leq K(2c_\infty(1+t)^2)^{-1}\left(c(0)|y_0| + \int_0^{+\infty} |y(s,\xi)|\,|c'(s)|\,ds\right) +$$

$$+ \int_t^{+\infty} |y(s,\xi)|\,|c'(s)|\,ds + K|h(t)|(2c_\infty(1+t)^2)^{-1}(|y_1| + |s(0,\xi)|) +$$

$$+ |h(t)| \int_t^{+\infty} |y(s,\xi)\|\xi\| c'(s)|\,ds .$$

Hence, by Holder's inequality we have for some constants c_1 and c_2:

$$\nu\|\alpha(t,\cdot)\|_2^2 < c_1(\|y_0\|_2^2 + \| |\xi| y_0\|_2^2 + \|y_1\|_2^2) \times$$

$$\times\left((1+t)^{-4} + \int_t^{+\infty} |c'(s)|^{2/5} ds \int_t^{\infty} (1+s)^{-14/5} ds\right) +$$

$$+ c_2 t(\| |\xi| y_0\|_2^2 + \|y_1\|_2^2)\left((1+t)^{-4} + \left(\int_t^{+\infty} |c'(s)|\,ds\right)^2\right).$$

Therefore

$$\lim_{t\to\infty} \|\alpha\|_2^2 = 0 .$$

Moreover

$$|z_1(t,\xi) - z(t,\xi)| = |z_1(0,\xi)|\,|\cos(|\xi| h(t)) - \cos(c_\infty |\xi| t + |\xi|\beta)| +$$

$$+ \frac{|z_1'(0,\xi)|}{|\xi| c(0)} |\sin(|\xi| h(t)) - \sin(c_\infty |\xi| t + |\xi|\beta)| \leq$$

$$\leq (|z_1(0,\xi)| + |z_1'(0,\xi)|)\left(|\xi| + \frac{1}{c(0)}\right) \int_t^{+\infty} |c(s) - c_\infty|\,ds .$$

Hence

$\|z_1 - z\|_2^2 \leq$

$$\leq 2\left(\left\|\left(|\xi| + \frac{1}{c(0)}\right)z_1(0,\cdot)\right\|_2^2 + \left\|\left(|\xi| + \frac{1}{c(0)}\right)z_1'(0,\cdot)\right\|_2^2\right) \times \left(\int_t^{+\infty}\frac{K}{2}(1+s)^{-2}ds\right)^2.$$

By this we obtain

$$\lim_{t\to\infty}\|z_1 - z\|_2^2 = 0.$$

In the same way we can prove that

$$\lim_{t\to\infty}\|\nabla^k(u-v)\|_2^2 = \lim_{t\to\infty}\||\xi|^k a\|_2^2 = 0 \quad \text{for every } k \in N.$$

Now we can prove in the same way that

$$\lim_{t\to\infty}\|\nabla^k(u-v)\|_2^2 = 0 \quad \text{for every } k \in N,$$

or, otherwise, it is enough to prove that in

$$\int_{R^j}|\xi|^{2k}(y'(t,\xi) - z'(t,\xi))^2 d\xi$$

we can pass to the limit under the integral. Indeed it is easy to see that for every ξ we have

$$\lim_{t\to\infty}(y'(t,\xi) - z_1'(t,\xi)) = 0 \quad \text{and} \quad \lim_{t\to\infty}(z_1'(t,\xi) - z'(0,\xi)) = 0.$$

Furthermore we can use the Lebesgue's theorem for the dominated convergence, thanks to estimates on energy of (14) (see (19)) and the conservation of energies for the Fourier transform of equation (8).

2) *Now let us prove* (ii). By (38), (39), (43) of [6], we have:

(23) $$|(\|\nabla u\|_2^2)'| \leq M(1+t)^{-2}$$

for some constant M. Therefore there exists

$$\lim_{t\to\infty}\|\nabla u\|_2^2 = a_\infty.$$

Thanks to [8] if v is a solution of (8) we have:

$$c_\infty^2 \lim_{t\to\infty}\|\nabla v\|_2^2 = \lim_{t\to\infty}\|v_t\|_2^2 < +\infty.$$

Thanks to (9) of Theorem 2 we obtain that there exists:

$$\lim_{t\to\infty}\|u_t\|_2^2 = b_\infty$$

and that

$$c_\infty^2 a_\infty = b_\infty.$$

3) *Now let us prove* (iii). We need some preliminaries.

(*a*) Let us define:

$$E_3(t,u) = \frac{1}{2}(\|\nabla u_t\|_2^2 + c^2(t)\|\Delta u\|_2^2).$$

Thanks to (6)-(7) there exists a constant H such that:

(24) $$E_3'(t,u) \leq H(1+t)^{-3}E_3(t,u);$$

therefore

(25) $$E_3'(t,u) \leq c_3 E_3(0,u).$$

(*b*) Let $(u_{0n}, u_{1n}) \to (u_0, u_1)$ in H^∞ (with (u_{0n}, u_{1n}) and (u_0, u_1) satisfying (4), (5)) and let $u_n(t,x)$ and $u(t,x)$ be the corresponding solutions of (1); we have:

$$\lim_{t\to\infty}\|\nabla(u_n - u)\|_2 + \|(u_n - u)_t\|_2 = 0.$$

Indeed, let us define $w^n = u - u_n$. We have:

$$w_{tt}^n - m(\|\nabla u\|_2^2)\Delta w^n = (m(\|\nabla u\|_2^2) - m(\|\nabla u_n\|_2^2))\Delta u_n.$$

Let us define

$$S(t) = \|w_t^n(t)\|_2^2 + m(\|\nabla u(t)\|_2^2)\|\nabla w^n(t)\|_2^2.$$

We remark that in (6)-(7) the constant K depends continuously by the initial data. Then the constant H in (24) depends continuously by the data. Therefore it is easy to see that $S'(t) \leq CS(t)$, since $m(s)$ is a locally Lipschitz function, where C is a constant independent of n. By this we obtain $\lim_{n\to\infty} S(t) = 0$ for each t, then

(26) $$\lim_{t\to\infty}\|w_t^n(t)\|_2^2 + \|\nabla w^n(t)\|_2^2 = 0.$$

(*c*) Let us define

$$a_{\infty,n} = \lim_{t\to\infty}\|\nabla u_n\|_2^2, \qquad a_\infty = \lim_{t\to\infty}\|\nabla u\|_2^2$$

we obtain $a_{\infty,n} \to a_\infty$ for $n \to \infty$.

Indeed let us fix $\varepsilon > 0$, by (23) there exists t_ε independent of n (since M in (23) depends continuously on the data), such that:

$$a_{1,n} := |a_{\infty,n} - \|\nabla u_n(t_\varepsilon)\|_2^2| \leq \varepsilon$$

$$a_1 := |a_\infty - \|\nabla u(t_\varepsilon)\|_2^2| \leq \varepsilon.$$

Moreover thanks to (26) there is n_1 such that, for $n \geq n_1$, we have:

$$a_{2,n} := \left| \|\nabla u_n(t_\varepsilon)\|_2^2 - \|\nabla u(t_\varepsilon)\|_2^2 \right| \leq \varepsilon .$$

Hence

$$|a_{\infty,n} - a_\infty| \leq a_{1,n} + a_{2,n} + a_1 \leq 3\varepsilon .$$

Now we show (iii). Let us remark that, since F is an increasing function near the origin, $F(r) \neq 0$ for $r > 0$ and $\lim_{r \to +\infty} F(r) = +\infty$, for small data there exists only a solution of $F(r) = E(0, u)$. Let u_0 and u_1 be as in (4) and (5). Let us denote by u_λ the solution of (1) with data $u_{0\lambda} = \lambda u_0$ and $u_{1\lambda} = \lambda u_1$, and energy $E(0, u_\lambda)$. Let us define

$$a_\lambda = \lim_{t \to \infty} \|\nabla u_\lambda\|_2^2$$

and let us indicate

$$\lambda_1 = \sup \{\lambda \leq 1 \colon a_\lambda = \min\{r \colon F(r) = E(0, u_\lambda)\}\} .$$

We show that $\lambda_1 = 1$. Let us remark that if z_1 and z_2 ($z_1 \neq z_2$) are such that $F(z_1) = F(z_2) = E(0, u_{\lambda_1})$ then $|z_1 - z_2| > h_0 > 0$. By this and 3c) it is easy to prove that λ_1 is a maximum.

In a similar way, if F is nondecreasing we can prove that, if we suppose $\lambda_1 < 1$,

$$a_\lambda = \min\{r \colon F(r) = E(0, u_\lambda)\}$$

in a neighbourhood of λ_1, and then it is not a maximum.

If F satisfies HP we remark that $E(0, u) \leq F(\bar{r})$. Otherwise, let $E(0, u_\lambda) = F(\bar{r})$, then, as in the previous case, $\lambda_1 \geq \lambda$ and if $\lambda_n > \lambda$, $\lambda_n \to \lambda$, $F(r_n) = E(0, u_{\lambda_n})$ then $r_n - \bar{r} > h > 0$, in contrast to 3c). Therefore as in previous case $\lambda_1 = 1$.

PROOF OF REMARK 4. – This proof for the cases of

$$\lim_{n \to +\infty} \|(\nabla(u_n - u)\|_2^2 + \|(u_n - u)_t\|_2^2 = 0 , \qquad \text{and} \qquad \lim_{n \to +\infty} a_{\infty,n} = a_\infty$$

is contained in part 3*b*) and 3*c*) of the proof of Theorem 2. In the other cases, that is:

$$\lim_{n \to +\infty} \|(\nabla^k(u_n - u)\|_2^2 + \|\nabla^k(u_n - u)_t\|_2^2 = 0 ,$$

the proof is the same. We remark that, since:

$$(\|u_n - u\|_2^2)' \leq 2\|u_n - u\|_2 \|(u_n - u)_t\|_2 ,$$

then for some constant C_1 we have

$$\|u_n - u\|_2 \leq \|u_{n,0} - u_0\|_2 + C_1 e^t (\|\nabla(u_{n,0} - u_0)\|_2 + \|u_{n,1} - u_1\|_2) .$$

PROOF OF REMARK 5. – By Theorem 2 part 3 there exists $\bar{a} > 0$, eventually $\bar{a} = +\infty$, such that for $E(0, u_0, u_1) < \bar{a}$, i), ii), iii) of Remark 5 are verified. Moreover let u_0 and u_1 by such that $E(0, u_0, u_1) < \bar{a}$, and let u be the corresponding solution of (1). As in the proof of 3 of Theorem 2 we can show that u satisfies (13).

If $\bar{a} \geq F(\bar{r})$, (13) is in contrast with iii), because $E(0, u) = F(\bar{r})$ is a point of discontinuity, indeed if $F(r) = F(\bar{r})$, $r > \bar{r}$, and there exists $r^n \to r$ such that $F(r^n) > F(r)$ then $|r - \bar{r}| > h_1 > 0$.

PROOF OF THEOREM 7. – First, we remark that the map F is well defined, that is, for any $(u_0, u_1) \in C_s$ there exists only one solution of (8) asymptotic to u in the sense of (9). Indeed if v and $\bar{v}$ are two such functions, then they are asymptotic, but $w = v - \bar{v}$ is a solution of (8), for which we have:

$$E_0(t, w) = \|w_t\|_2 + c_\infty^2 \|\nabla w\|_2^2 = E_0(0, w)$$

Therefore $E_0(0, w) = 0$, hence $w = 0$.

Now let us prove that the map F is continuous in H^∞.

Let us suppose that

$$(u_0^n, u_1^n) \in C_s \to (u_0, u_1) \in C_s \qquad \text{for } n \to +\infty \text{ in } H^\infty .$$

We use the same notations of the proof of Theorem 2 and if the considered quantities are related to (u_0^n, u_1^n) we add the index n.

Then, by Remark 4,

$$c_\infty^n \to c_\infty \qquad \text{for } n \to +\infty .$$

Moreover we have:

$$\|v_0^n - v_0\|_2^2 = \|z^n(0, \cdot) - z(0, \cdot)\|_2^2 = \|\varrho\|_2^2 .$$

Furthermore, by (21) and (22) we obtain

$$|\varrho(\xi) \leq c_\infty^{-1} |\cos(|\xi|\beta) - \cos(|\xi|\beta^n)| \left(c(0)|y_0| + \int_0^{+\infty} |y(s, \xi)| \, |c'(s)| \, ds \right) + \tag{27}$$

$$+ \left| \frac{c_\infty^n - c_\infty}{c_\infty^n c_\infty} \right| \left(c(0)|y_0| + \int_0^{+\infty} |y(s, \xi) c'(s)| \, ds \right)$$

$$+ (c_\infty^n)^{-1} (c(0)|y_0 - y_0^n| + |y_0^n| \, |c(0) - c^n(0)|) +$$

$$+(c_\infty^n)^{-1}\int_0^{+\infty}|y(s,\xi)-y^n(s,\xi)||c'(s)|ds+$$

$$+(c_\infty^n)^{-1}\int_0^{\infty}|y^n(s,\xi)||c'(s)||\cos(|\xi|h(s))-\cos(|\xi|h^n(s))|ds+$$

$$+(c_\infty^n)^{-1}\int_0^{\infty}|y^n(s,\xi)||c'(s)-(c^n(s))'|ds+$$

$$+(c_\infty|\xi|)^{-1}|\sin(|\xi|\beta)-\sin(|\xi|\beta^n)|\left(|y_1|+\int_0^{+\infty}|y(s,\xi)||\xi||c'(s)|ds\right)$$

$$+\left|\frac{\sin(|\xi|\beta^n)}{|\xi|\beta^n}\right||\beta^n|\left|\frac{c_\infty^n-c_\infty}{c_\infty^n c_\infty}\right|\left(|y_1|+\int_0^{+\infty}|y(s,\xi)||\xi||c'(s)|ds\right)+$$

$$+|\beta^n|(c_\infty^n)^{-1}|y_1-y_1^n|+|\beta^n|(c_\infty^n)^{-1}\int_0^{+\infty}|\xi||y(s,\xi)-y^n(s,\xi)||c'(s)|ds+$$

$$+|\beta^n|(c_\infty^n)^{-1}\int_0^{\infty}|y^n(s,\xi)||c'(s)||\xi||\sin(|h(s))-\sin((|\xi|h^n(s))|ds+$$

$$+|\beta^n|(c_\infty^n)^{-1}\int_0^{\infty}|\xi||y^n(s,\xi)||c'(s)-(c^n(s))'|ds.$$

Let us remark that there exists a constant c_4 such that:

$$\|y_0\|_2^2,\ \||\xi|y_0\|_2^2,\ \||\xi|^2y_0\|_2^2,\ \|y_1|\xi|\|_2^2,\ c(0),\ c_\infty,$$
$$\|y_0^n\|_2^2,\ \||\xi|y_0^n\|_2^2,\ \|y_1^n\|_2^2,\ \||\xi|^2y_0^n\|_2^2,\ c^n(0),\ c_\infty^n\le c_4.$$

Now let us give some estimates (as in proof of 1*b*), 1*c*) of Theorem 2) on the terms of (27).

1) Let us set

$$a_0=c_\infty^{-2}\int_{R^j}|\cos(|\xi|\beta)-\cos(|\xi|\beta^n)|^2\left(c(0)|y_0|+\int_0^{+\infty}|y(s,\xi)||c'(s)|ds\right)^2d\xi.$$

We have for some constant c_5:

(28) $$a_0\le c_5|\beta-\beta^n|^2(\||\xi|y_0\|_2^2+\|y_1\|_2^2).$$

Let us remark that

$$\lim_{n\to+\infty}(c(t)-c^n(t))=0\quad\text{for every } t.$$

Indeed, since

$$c(t)=(m(\|\nabla u\|_2^2))^{1/2},\qquad c^n(t)=(m(\|\nabla u^n\|_2^2)_{1/2},$$

it is enough to prove that

$$\lim_{n\to+\infty}|\|\nabla u\|_2^2-\|\nabla u^n\|_2^2|=0\quad\text{for every } t.$$

Now let us remark that $u-u_n$ satisfies the following equation:

(29) $$w_{tt}-c^2(t)\Delta w=(c^2(t)-(c^n)^2(t))\Delta u_n.$$

Let us set

$$E_w(t)=\|w_t\|_2^2+c^2(t)\|\nabla w\|_2^2.$$

We have, for some constant d_1, independent of n:

$$E_w'(t)\le d_1E_w(t).$$

Hence

$$E_w(t)\le E_w(0)e^{d_1t}.$$

Therefore

$$\lim_{n\to+\infty}|\|\nabla u\|_2^2-\|\nabla u^n\|_2^2|=0\quad\text{for every } t.$$

Moreover in

$$|\beta-\beta^n|=\left|\int_0^{+\infty}(c(t)-c_\infty-c^n(t)+c_\infty^n)dt\right|$$

we can pass to the limit under the integral, since:

$$|c(t)-c_\infty-c^n(t)+c_\infty^n|\le|c(t)-c_\infty|+|c^n(t)-c_\infty^n|\le 2K(1+t)^{-2}.$$

Hence

$$\lim_{n\to+\infty}a_0=0.$$

2) Let us set

$$a_1=\left|\frac{c_\infty^n-c_\infty}{c_\infty^n c_\infty}\right|^2\int_{R^j}\left(c(0)|y_0|+\int_0^{+\infty}|y(s,\xi)||c'(s)|ds\right)^2d\xi.$$

There exists a constant c_6 such that:

$$\alpha_1 \leq c_6 \left| \frac{c_\infty^n - c_\infty}{c_\infty^n c_\infty} \right|^2 (\|y_0\|_2^2 + \| |\xi| y_0 \|_2^2 + \|y_1\|_2^2).$$

Then

$$\lim_{n \to +\infty} \alpha_1 = 0.$$

3) Let us denote

$$\alpha_2 = \int_{R^j} ((c_\infty^n)^{-1} (c(0)|y_0 - y_0^n| + |y_0^n| |c(0) - c^n(0)|))^2 d\xi.$$

There exists a constant c_7 such that:

$$\alpha_2 \leq c_7 (\|y_0 - y_0^n\|_2^2 + \|y_0^n\|_2^2 |c(0) - c^n(0)|^2).$$

Moreover since $(m(s))^{1/2}$ is a locally Lipschitz function, there exists a constant c_8 such that:

$$|c(0) - c^n(0)| \leq c_8 | \|\nabla u_0\|_2^2 - \|\nabla u_0^n\|_2^2 | \leq 2c_4 c_8 \|\nabla u_0 - \nabla u_0^n\|_2.$$

Hence

$$\alpha_2 \leq c_7 (\|u_0 - u_0^n\|_2^2 + 4c_4^3 c_8^2 \|\nabla u_0 - \nabla u_0^n\|_2^2).$$

Therefore

$$\lim_{n \to +\infty} \alpha_2 = 0.$$

4) Now let us evaluate:

$$\alpha_3 = (c_\infty^n)^{-2} \int_{R^j} \left(\int_0^{+\infty} |y(s, \xi) - y^n(s, \xi)| |c'(s)| ds \right)^2 d\xi.$$

Let w be a solution of (29), then we obtain for some constant d_2:

$$E_w'(t) \leq d_2 ((1+t)^{-3} E_w(t) + |c(t) - c^n(t)|).$$

Hence for some constant $\bar{d}_2$ we obtain

$$E_w(t) \leq \bar{d}_2 \left(E_w(0) + \int_0^t |c(s) - c^n(s)| ds \right). \tag{30}$$

Furthermore:

$$(\|w\|_2^2)' \leq 2\|w\|_2 \|w_t\|_2.$$

Therefore, for some constant d_3 we have:

$$\|w\|_2 \leq d_3 \left(\|u_0 - u_0^n\|_2 + t \left(E_w(0) + \int_0^t |c(s) - c^n(s)| ds \right) \right)^{1/2}. \tag{31}$$

Then we have

$$\alpha_3 \leq d_4 (c_\infty^n)^{-2} \int_0^{+\infty} |c'(s)|^{8/5} \|w(s)\|_2^2 ds \int_0^\infty (1+t)^{-6/5} dt. \tag{32}$$

Now let us remark that, by (31) in (32) we can pass to the limit for $n \to +\infty$ by using the Lebesgue's theorem for the dominate convergence, hence:

$$\lim_{n \to +\infty} \alpha_3 = 0.$$

5) Furthermore let us denote:

$$\alpha_4 = (c_\infty^n)^{-2} \int_{R^j} \left(\int_0^\infty |y^n(s, \xi)| |c'(s)| \,|\cos(|\xi| h(s)) - \cos((|\xi| h^n(s))| ds \right)^2 d\xi.$$

We have:

$$\alpha_4 \leq (c_4)^{-2} \int_{R^j} \left(\int_0^\infty |y^n(s, \xi)| |c'(s)| |\xi| |h(s) - h^n(s)| ds \right)^2 d\xi. \tag{33}$$

Hence for some constant d_5:

$$\alpha_4 \leq d_5 \int_0^\infty |c'(s)|^{8/5} |h(s) - h^n(s)|^2 ds \int_0^\infty |c'(s)|^{2/5} (\| |\xi| y_0^n \|_2^2 + \|y_1^n\|_2^2). \tag{34}$$

Moreover we have:

$$|h(s) - h^n(s)| \leq \int_0^s |c(r) - c^n(r)| dr.$$

Therefore in (34) we can pass to the limit for $n \to +\infty$ by using the Lebesgue's theorem, and then

$$\lim_{n \to +\infty} \alpha_4 = 0.$$

6) Let us set:

$$\alpha_5 = \int_{R^j} (c_\infty)^2 \left(\int_0^\infty |y^n(s, \xi)| |c'(s) - (c^n(s))'| ds \right)^2 d\xi.$$

We have for some constant d_6:

$$\alpha_5 \leq (c_\infty)^{-2} \int_{R^j} \int_0^\infty |y^n(s,\xi)|^2 |c'(s) - (c^n(s))'|^{8/5} ds\, d\xi \int_0^\infty |c'(s) - (c^n(s))'|^{2/5} ds \leq$$

$$\leq d_6 (\|y_0^n\|_2^2 + \||\xi| y_0^n\|_2^2 + \|y_1^n\|_2^2) \int_0^{+\infty} (1+t)^{-14/5} dt \int_0^\infty |c'(s) - (c^n(s))'|^{2/5} ds .$$

Now we need to estimate $\int_0^\infty |c'(s) - (c^n(s))'|^{2/5} ds$. We remark that, as in the previous cases it is possible to pass to the limit by using the Lebesgue's theorem.

Moreover we have:

$$|c'(s) - (c^n(s))'| \leq d_8 |m'(\|\nabla u\|_2^2)| |\langle \nabla u, \nabla u_t\rangle - \langle \nabla u^n, \nabla u_t^n\rangle| +$$
$$+ d_8 |\langle \nabla u^n, \nabla u_t^n\rangle| |m'(\|\nabla u\|_2^2) - m'(\|\nabla u^n\|_2^2)| +$$
$$+ d_9 |m(\|\nabla u\|_2^2) - m(\|\nabla u\|_2^2)|^{1/2} \leq$$
$$\leq \bar{d}_8 |\langle \nabla(u - u^n), \nabla u_t\rangle + \langle \nabla u^n, \nabla(u - u^n)_t\rangle| +$$
$$+ \bar{d}_9 |m'(\|\nabla u\|_2^2) - m'(\|\nabla u^n\|_2^2)| + e_0 |\|\nabla u\|_2^2 - \|\nabla u^n\|_2^2|^{1/2} \leq$$
$$\leq e_1 (\|\nabla(u - u^n)\|_2 \|\nabla u_t\|_2 + \|\nabla u^n\|_2 \|\nabla(u - u^n)_t\|_2 +$$
$$+ |m'(\|\nabla u\|_2^2) - m'(\|\nabla u^n\|_2^2) + \|\nabla(u - u^n)\|_2^{1/2})$$

and (as in 5) of this theorem) it is easy to see that:

$$\lim_{n\to+\infty} \|\nabla(u - u^n)_t\|_2 + \|\nabla(u - u^n)\|_2 = 0 \quad \text{for every } t;$$

then

$$\lim_{n\to+\infty} \alpha_5 = 0 .$$

7) Let us set

$$\alpha_6 = \int_{R^j} \frac{|\sin(|\xi|\beta) - \sin(|\xi|\beta^n)|^2}{(c_\infty |\xi|)^2} \left(|y_1| + \int_0^{+\infty} |y(s,\xi)| |\xi| |c'(s)| ds \right)^2 d\xi .$$

We obtain, for some constant e_2

$$\alpha_6 \leq e_2 |\beta - \beta^n|^2 (\|y_1\|_2^2 + \||\xi| y_0\|_2^2)$$

and, as in 1) (of this theorem):

$$\lim_{n\to+\infty} \alpha_6 = 0 .$$

8) Let us set

$$\alpha_7 = \int_{R^j} \left| \frac{\sin(|\xi|\beta^n)}{|\xi|\beta^n} \right|^2 |\beta^n|^2 \left| \frac{c_\infty^n - c_\infty}{c_\infty^n c_\infty} \right|^2 \left(|y_1| + \int_0^{+\infty} |y(s,\xi)| |\xi| |c'(s)| ds \right)^2 d\xi .$$

Then we have for some constant e_3:

$$\alpha_7 \leq e_3 |c_\infty - c_\infty^n|^2 (\|y_1\|_2^2 + \||\xi| y_0\|_2^2) .$$

Hence

$$\lim_{n\to+\infty} \alpha_7 = 0 .$$

9) Let us define

$$\alpha_8 = |\beta^n|^2 (c_\infty^n)^{-2} \|y_1 - y_1^n\|_2^2 .$$

Then we have:

$$\lim_{n\to+\infty} \alpha_8 = 0 .$$

10) Let us consider

$$\alpha_9 = \int_{R^j} |\beta^n|^2 (c_\infty^n)^{-2} \left(\int_0^{+\infty} |\xi| |y(s,\xi) - y^n(s,\xi)| |c'(s)| ds \right)^2 d\xi .$$

As in 4) (of this theorem) one can prove (by using (30)) that:

$$\lim_{n\to+\infty} \alpha_9 = 0 .$$

11) Let us set:

$$\alpha_{10} = \int_{R^j} \frac{|\beta^n|^2}{(c_\infty^n)^2} \left(\int_0^\infty |y^n(s,\xi) c'(s)| |\xi| |\sin(|\xi| h(s)) - \sin((|\xi| h^n(s))| ds \right)^2 d\xi .$$

We obtain for some constant e_4:

$$\alpha_{10} \leq e_4 \int_0^\infty |c'(s)|^{8/5} |h(s) - h^n(s)|^2 ds \int_0^\infty |c'(s)|^{2/5} (\||\xi|^2 y_0^n\|_2^2 + \||\xi| y_1^n\|_2^2) ds .$$

Therefore as in 5):

$$\lim_{n\to+\infty} \alpha_{10} = 0 .$$

12) Let us define:

$$\alpha_{11} = \int_{R^j} |\beta^n|^2 (c_\infty^n)^{-2} \left(\int_0^\infty |\xi| |y^n(s,\xi)| |c'(s) - (c^n(s))'| ds \right)^2 d\xi .$$

There exists a constant e_5 such that:

$$\alpha_{11} \leq e_5(\| |\xi|^2 y_0^n \|_2^2 + \| |\xi| y_1^n \|_2^2) \int_0^{+\infty} |c'(s) - (c^n)'(s)| \, ds .$$

Therefore as in 6) (of this theorem) we have:

$$\lim_{n \to +\infty} \alpha_{11} = 0 .$$

By 1)-12) we have:

$$\lim_{n \to +\infty} \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} = 0 .$$

Therefore

$$\lim_{n \to +\infty} \|v_0^n - v_0\|_2^2 = 0 .$$

In the same way we can prove that, for every $k \in N$:

$$\lim_{n \to +\infty} \|\nabla^k (v_0^n - v_0)\|_2^2 = \lim_{n \to +\infty} \| |\xi|^k (z(0, \cdot) - z^n(0, \cdot))\|_2^2 = 0$$

$$\lim_{n \to +\infty} \|\nabla^k (v_1^n - v_1)\|_2^2 = \lim_{n \to +\infty} \| |\xi|^k (z'(0, \cdot) - z^n(0, \cdot))' \|_2^2 = 0 .$$

Therefore $(v_0^n, v_1^n) \to (v_0, v_1)$ in H^∞.

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