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MILD SOLUTIONS OF EVOLUTION EQUATIONS AND MEASURES OF NONCOMPACTNESS

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1. INTRODUCTION

Let X be a Banach space. We consider the Cauchy problem

$$u' = Au + f(t,u) \qquad (t \in I, u(t) \in X)$$
$$u(0) = u_0; \tag{1.1}$$

where $I := [0,a]$, and A is the infinitesimal generator of a C°-semigroup.

It is well known that if f is only continuous (even if $A = 0$) the problem (1.1) does not always have solutions [1]. Like in the finite dimensional case, under the Lipschitz condition we have existence and uniqueness of mild solution of problem (1.1) (for the definition of mild solution see definition 2.3). If f is compact we have at least a mild solution (we can alternatively assume the compactness of the semigroup generated by A, see [2–5]. Also if we assume that f is weakly compact and weakly continuous we have at least a mild solution [6].

Many authors have considered the Cauchy problem (1.1), in particular in the case $A = 0$, under hypotheses based on noncompactness measures (see definition 2.1 and [7–9]), which include Lipschitz and compactness conditions.

CASE A = 0

$$u' = f(t,u) \qquad (t \in I, u(t) \in X);$$
$$u(0) = u_0. \tag{1.2}$$

The first result has been obtained by [10]. They supposed f continuous and $f = f_1 + f_2$, where f_1 is a Lipschitz continuous operator and f_2 is a compact one (for a generalization see [11]). Later on [12] supposed that f is uniformly continuous in (t,u) and α-Lipschitz continuous, that is for every bounded subset W of X, and for $t \in I$

$$\alpha(f(\{t\} \times W)) \le K\alpha(W)$$

where α is the Hausdorff noncompactness measure (see definition 2.1). Ambrosetti [12] uses in his proof the Darbo fixed point theorem [13]. Szufla [14] generalized this result by supposing f continuous and such that:

(i) there exists a constant K such that, for every bounded subset W of X, $\alpha(f(I \times W)) \leq K\alpha(W)$.

A stronger formulation of this result is due to [15]. They supposed $f(t, x) := F(t, x, x)$; where $F(\cdot, \cdot, y)$ is compact, $F(t, x, \cdot)$ is Lipschitz continuous, and $F(\cdot, \cdot, \cdot)$ is continuous. Later on, [16] treated the case when f is a Caratheodory operator (that is continuous in u and measurable in t). Later on, [17] assumed that $f(t, x) := F(t, x, x)$, with F uniformly continuous in (t, x, y) and $\alpha(F\{t\} \times V \times W) \leq h(t)\alpha(V)$, for V, W bounded subsets of X, $t \in I$, and $h(t)$ integrable real function.

We remark that many other authors [7, 16, 18–31] weakened hypothesis (i), by assuming a hypothesis of this type;

(ii) there exists a function g: $R \times R \to [0, +\infty[$ such that, for every bounded subset W of X, and for every $t \in I$, $\alpha(f(\{t\} \times W)) \leq g(t, \alpha(W))$, where g is a Kamke function (that is the Cauchy problem $y' = g(t, y), y(0) = 0$ has only the solution $y = 0$) (see [32]).

Furthermore [32] showed that when f is uniformly continuous in (t, u) the properties of type (ii) used by above considered authors are equivalent. We can *not* treat this case by our technique.

Some other authors considered the weak noncompactness measure (see definition 2.2), instead of noncompactness measures. Mitchen and Smith [33] supposed that f is weakly continuous and satisfies a hypothesis of type (i) with respect to weak noncompactness measure (see also [34]). Cramer *et al.* [35] generalized this result by weakening hypothesis of type (i) of [33]. Later on [36] showed that the Cauchy problem (2) has a solution by supposing that f is a Caratheodory operator (for the weak topology) and satisfies (i) for the weak noncompactness measure.

CASE A $\neq$ 0

Reference [44] showed that the Cauchy problem (1) has at least a mild solution when f is continuous and $f = f_1 + f_2$, where f_1 is a Lipschitz continuous operator and f_2 is a compact one. Later on, [37] treated problem (1.1) under α-Lipschitz hypotheses. They supposed that f is continuous and satisfies hypothesis (i), and that A is the generator of a contraction semigroup (theorem 3.3 and theorem 3.6). Their proof seems to be uncompleted (see Mathematical Reviews, MR 91h: 34099). Later on [38] showed the problem $u' \in -Au + f(t, u), u(0) = u_0 \in$ closure $(D(A))$ has an integral solution if A is m-accretive linear operator (i.e. for $x, z \in D(A)$, $y \in Ax, w \in Az$) one has $[x - x, y - w]_{-} \geq 0$ and the range of $(I + tA)$ is the whole X (for each $t > 0$)) which generates an equicontinuous semigroup and f is a locally uniformly continuous, locally bounded operator and α-Lipschitz continuous operator with constant $K < 1/2a$.

In fact, the Cauchy problem (1.1) has at least a mild solution under two sets of hypotheses:

(a) f is a Caratheodory operator and satisfies property i) (theorem 3.1);

(b) f is a Caratheodory operator with respect to weak topology, in particular $f(t, \cdot)$ is weakly sequentially continuous, and satisfies property i) for the weak noncompactness measure. Furthermore the semigroup generated by A is sequentially weakly continuous in (t, u) on the bounded subsets (theorem 3.2).

Our technique is based on the approximating delayed problems introduced by Tonelli [39] already used by Pianigiani [16], Cramer *et al.* [35] and Song [26].

This paper is organized as follows:
in Section 2 we give some definitions;
in Section 3 we state some lemmata, the main result and remarks;
in Section 4 we give the proofs.

2. PRELIMINARIES

Let us recall the following definitions.

Definition 2.1. Let B be a bounded subset of X. The Hausdorff noncompactness measure of B is defined as

$$\alpha(B) := \inf\{\epsilon > 0\colon B \text{ can be covered by a finite number of balls of radius } \epsilon\}$$

Definition 2.2. Let B be a bounded subset of X. The weak noncompactness measure of B is defined as

$$\alpha_W(B) := \inf\{\epsilon > 0\colon \text{there exists a weakly compact set } K \subseteq X \text{ such that } \mathbf{B} \subseteq K + \epsilon\mathbf{B}\},$$

where $\mathbf{B}$ is the ball of center 0 and radius 1 in X.

Given a subset A of X, we denote by **cl(A)** its closure, **co(A)** its convex hull.

Let us recall some properties of α (for the proofs see [7–9]). Let A and B be bounded subsets of X, then

(1) $\alpha(\mathrm{co}(B)) = \alpha(B)$;
(2) $\alpha(\mathrm{cl}(B)) = \alpha(B)$;
(3) $\alpha(A \cup B) \leq \max\{\alpha(A), \alpha(B)\}$;
(4) $\alpha(A) = 0$ if and only if A is relatively compact;
(5) $\alpha(\lambda B) = |\lambda|\alpha(B)$ for every $\lambda \in \mathbf{R}$;
(6) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;
(7) $\alpha(B) \leq \alpha(A)$ if $B \subseteq A$.

The same properties hold true, with respect to weak topology, for α_W (see [40]); we denote them by (1_W)–(7_W).

We recall that a C°-semigroup on X is an operator $S\colon [0, +\infty[\times X \to X$ such that:

(1) for every $t \geq 0$, $S(t)$ is a linear and continuous operator on X;
(2) for every $t \geq 0$, $s \geq 0$, $S(t+s) = S(t)\,S(s)$;
(3) $S(0) = \text{Identity}$;
(4) for every x in X, $S(t)x \to x$ for $t \to 0^+$.

We recall that A is the infinitesimal generator of a C°-semigroup if

$$D(A) = \left\{x\colon \text{there exists finite } \lim_{h\to 0} h^{-1}(S(h)x - x)\right\}$$

and

$$Ax = \lim_{h\to 0} h^{-1}(S(h)x - x).$$

If S is a C°-semigroup and A is its infinitesimal generator, we set

$$e^{-tA} := S(t).$$

We recall that, if e^{-tA} is a C°-semigroup, then there exist two constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|e^{-tA}\| \leq M e^{\omega t} \tag{2.1}$$

(see for example [41]).

Definition 2.3. Let A be the generator of a C°-semigroup. We call u mild solution of the Cauchy problem (1.1) on $[0, a]$ if

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(s, u(s))\, ds \qquad (0 \leq t \leq a).$$

3. THE MAIN RESULT

Let $f:[0,a] \times X \to X$, and let e^{-tA} be a C°-semigroup. Our technique is based on the delayed problems

$$u_n(t) = \begin{cases} u_0 & \text{if } t \leq 0, \qquad (3.1.1) \\ e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(s, u_n(s - a/n))\, ds & \text{if } 0 \leq t \leq a. \qquad (3.1.2) \end{cases} \qquad (P_n)$$

Let us set

$$\Omega := \bigcup_{0 \leq t \leq a} \{u_n(t) : n \in N\}. \tag{3.2}$$

We need these lemmata:

LEMMA 3.1. We have

$$\alpha(\Omega) \leq a\, \alpha\left(\bigcup_{0 \leq t \leq a} e^{-tA} f([0,a] \times \Omega) \right), \tag{3.3}$$

and

$$\alpha_W(\Omega) \leq a\, \alpha_W\left(\bigcup_{0 \leq t \leq a} e^{-tA} f([0,a] \times \Omega) \right). \tag{3.4}$$

LEMMA 3.2. Let D be a bounded subset of X. Let M, ω be the constants introduced in (2.1). Then for every $a > 0$ we have

$$\alpha\left(\bigcup_{0 \leq t \leq a} e^{-tA}(D) \right) \leq M e^{\omega a} \alpha(D). \tag{3.5}$$

Furthermore, if the semigroup is sequentially weakly continuous on bounded subsets we have

$$\alpha_W\left(\bigcup_{0\le t\le a} e^{-tA}(D)\right)\le M e^{\omega a}\alpha_W(D). \tag{3.6}$$

Let B be the ball of center u_0 and radius R in X and let $I := [0, a_0]$. Let e^{-tA} be a C°-semigroup. Let $f: I\times B \to X$ be an operator that satisfies these properties:

(a) in the case of the strong topology

(3.7) there exists a constant K such that, for every bounded subset W of X

$$\alpha(f(I\times W))\le K\alpha(W);$$

(3.8) $f(\cdot, u)$ is strongly measurable for u in a dense subset of B;
(3.9) $f(t,\cdot)$ is continuous for almost every t in $[0, a_0]$;

(b) in the case of the weak topology

(3.7_w) there exists a constant K such that, for every bounded subset W of X

$$\alpha_W(f(I\times W))\le K\alpha_W(W).$$

(3.8_w) for every continuous function $u: I\to B$, the function $e^{-(\cdot)A}f(\cdot, u(\cdot))$ is weakly (or Pettis) integrable;
(3.9_w) $f(t,\cdot)$ is weakly sequentially continuous for almost every t in $[0, a_0]$.

Now we state the results.

THEOREM 3.1. (case a) Let D be a bounded subset of X, B the ball of center u_0 and radius R, and $I := [0, a_0]$.

Let A be the infinitesimal generator of a C°-semigroup. Let $f: I\times B\to D$ be an operator satisfying (3.7)–(3.9).

Then the Cauchy problem (1) has at least a mild solution $u\in C^\circ([0, a_1]; X)$, with

$$a_1 := \max\{b\le a_0 : \|e^{-hA}u_0 - u_0\| + h\, M e^{\omega h}\|D\|\le R \qquad 0\le h\le b\},$$

where M and ω are the constants introduced in (2.1) and $\|D\| := \sup_{d\in D}\|d\|$ (and $I = [0, a_0]$).

THEOREM 3.2. (case b) Let D be a bounded subset of X, B the ball of center u_0 and radius R, and $I := [0, a_0]$.

Let A be the infinitesimal generator of a C°-semigroup. Let us suppose that this semigroup is sequentially weakly continuous in (t, u) on the bounded subsets of $I\times X$. Let $f: I\times B\to D$ be an operator satisfying (3.7_w)–(3.9_w).

Then the Cauchy problem (1) has at least a mild solution $u\in C^\circ([0, a_1]; X)$, with

$$a_1 := \max\{b\le a_0 : \|e^{-hA}u_0 - u_0\| + h\, M e^{\omega h}\|D\|\le R \qquad 0\le h\le b\},$$

where M and ω are the constants introduced in (2.1) and $\|D\| := \sup_{d\in D}\|d\|$.

Now we give some examples of applications of theorem 3.1.

PROPOSITION 3.1. Hypothesis (3.7) is verified if:

(1) f is compact (that is the image of the bounded subsets of B is relatively compact);

(2) f is uniformly continuous in (t,u) and Lipschitz continuous in u;

(3) f is sum of an operator of type (1) and one of type (2) (see [10]);

(4) $f(u)=F(u,u)$, with $F:B\times B\to X$, $F(\cdot,w)$ Lipschitz continuous uniformly with respect to w (with constant $\leq L$), and $F(v,\cdot)$ compact for every v in B (see [15]).

COROLLARY 3.1. Let $m\in C(R^+;R)$. Then the problem

$$\begin{aligned} &u_{tt}-u_{xx}=u\cdot m\left(\int_0^1 |u_x|^2\,\mathrm{d}x\right) \qquad (t>0, 0\leq x\leq 1)\\ &u(0)=u_0\in H_0^1\\ &u_t(0)=u_1\in L^2 \end{aligned} \tag{3.10}$$

has at least a mild solution.

Finally, we give some applications of theorem 3.2.

PROPOSITION 3.2. Let us assume for simplicity that f is autonomous. Then the hypothesis (3.7_w) is verified if:

(1_w) f is weakly compact;

(2_w) f is Lipschitz and weakly continuous;

(3_w) f is sum of a operator of type (1_w) and of one of type (2_w);

(4_w) $f(u)=F(u,u)$, with $F:B\times B\to X$, $F(\cdot,w)$ Lipschitz continuous uniformly with respect to w (with constant $\leq L$), weakly continuous (uniformly with respect to w), and $F(v,\cdot)$ weakly compact for every v in B.

Remark 3.1. We reobtain [6], that is:

Let X be a reflexive space, let D be a bounded subset of X, B the ball of center u_0 and radius R, and $I:=[0,a_0]$. Let A be the infinitesimal generator of a C°-semigroup. Let $f:I\times B\to D$ be an operator that satisfies (3.8_w)||(3.9_w).

Then the Cauchy problem (1) has at least a mild solution.

Remark 3.2. The hypotheses of theorem 3.1 can be weakened. We can prove the existence of a mild solution of problem (1.1) also if we substitute (3.8) with (3.8_w), and (3.9) with:

(3.9′) $f(t,\cdot)$ is demicontinuous for almost every t (i.e. $f(t,\cdot)$ is continuous from X with the strong topology to X with the weak topology).

4. PROOFS

Proof of lemma 3.1. Let us observe that

$$\begin{aligned}\Omega\subseteq &\{\mathrm{e}^{-tA}u_0:0\leq t\leq a\}\\ &+\bigcup_{0\leq t\leq a}\bigcup_{n\in N} t\,\mathrm{cl}\big(\mathrm{co}\big(\{\mathrm{e}^{-(t-s)A}f(s,u_n(s-a/n)):0\leq s\leq t\}\big)\big)\end{aligned}$$

$$\subseteq \{e^{-tA}u_0 : 0 \le t \le a\} + \bigcup_{0 \le t \le a} t\, \mathrm{cl}(\mathrm{co}(\{e^{-(t-s)A} f([0,a] \times \Omega) : 0 \le s \le t\})).$$

From this we have

$$\Omega \subseteq \{e^{-tA}u_0 : 0 \le t \le a\} + [0,a]\, \mathrm{cl}(\mathrm{co}(\{e^{-(t-s)A} f([0,a] \times \Omega) : 0 \le s \le a\})).$$

Therefore, thanks to properties (1)–(2) and (4)–(6) of α

$$\alpha(\Omega) \le \alpha([0,a]\, \mathrm{cl}(\mathrm{co}(\{e^{-(t-s)A} f([0,a] \times \Omega) : 0 \le s \le a\})))$$
$$\le a\,\alpha(\{e^{-(t-s)A} f([0,a] \times \Omega) : 0 \le s \le a\}).$$

The proof of (3.4) is similar, by using (1_w)–(2_w) and (4_w)–(6_w). ∎

Proof of lemma 3.2.

Proof of (3.5). Let $\eta > \alpha(D)$, let $u_1, \ldots, u_n$ be the centers of balls of radius η which cover D. Let us fix $\epsilon > 0$. Let $\delta > 0$ be such that

$$[0 \le s \le t \le a, |t - s| \le \delta] \Rightarrow \left[\|e^{-tA}u_i - e^{-sA}u_i\| \le \epsilon \qquad (i = 1, \ldots, n)\right].$$

Let $t_1, \ldots, t_m$ be the centers of balls of radius δ which cover $[0,a]$. We will show that $(e^{-t_jA}u_i)_{i \in I_n, j \in I_m}$ are the centers of balls of radius $\eta M e^{\omega a} + \epsilon$ which cover $(\bigcup_{0 \le t \le a} e^{-tA}(D))$. Indeed, let $u \in D$. Let i, j be integers such that $|t - t_j| \le \delta$, $\|u - u_i\| \le \eta$. Then

$$\|e^{-tA}u - e^{-t_jA}u_i\| \le \|e^{-tA}(u - u_i)\| + \|e^{-tA}u_i - e^{-t_jA}u_i\|$$
$$\le M e^{\omega a}\|u - u_i\| + \epsilon \le \eta M e^{\omega a} + \epsilon.$$

Therefore

$$\alpha\left(\bigcup_{0 \le t \le a} e^{-tA}(D)\right) \le \eta M e^{\omega a} + \epsilon,$$

and, since η and ϵ are arbitrary

$$\alpha\left(\bigcup_{0 \le t \le a} e^{-tA}(D)\right) \le M e^{\omega a}\alpha(D).$$ ∎

Proof of (3.6). Let $\eta > \alpha(D)$, and let K be a weakly compact subset of X, such that

$$D \subseteq K + \eta \mathbf{B};$$

then

$$e^{-tA}(D) \subseteq e^{-tA}(K) + \eta M e^{\omega a}\mathbf{B};$$

and therefore

$$\bigcup_{0 \le t \le a} e^{-tA}(D) \subseteq \bigcup_{0 \le t \le a} e^{-tA}(K) + \eta M e^{\omega a}\mathbf{B}.$$

Since $[0,a] \times K$ is weakly compact (= sequentially weakly compact) and the semigroup is sequentially weakly continuous in (t,u) on the bounded subsets of $I \times X$, $\bigcup_{0 \le t \le a} e^{-tA}(K)$ is weakly compact. From this we have (3.6).

Proof theorem 3.1.

Step 1

Let a be a constant such that

$$\begin{cases} aM\mathrm{e}^{\omega a}K < 1; \\ 0 < a \leq a_1. \end{cases}$$

Let us consider for $n \in N$ the problems (P_n). Let us show that these problems have solutions $u_n \in C^{\circ}(]-\infty,\ a];\ B)$. It is easy show that if $f: I \to X$ is strongly measurable, then $\mathrm{e}^{-(\cdot)A}f(\cdot)$ is strongly measurable. Let us set $t_k = ka/n$ $(k = 0, 1, \ldots, n)$, and let us show by finite induction that the problems (P_n) have solutions on $]-\infty,\ t_k]$. If $k = 0$ it is trivial. Now let us suppose that u_n is a solution of (P_n) defined on $]-\infty,\ t_{k-1}]$. Since u_n is a continuous function on $]-\infty,\ t_{k-1}]$, it follows that $f(\cdot, u_n(\cdot - a/n))$ is strongly measurable on $]-\infty, t_k]$ (see [42]). Furthermore the function $\mathrm{e}^{-(t-\cdot)A}f(\cdot, u_n(\cdot - a/n))$ is bounded, and therefore integrable. Using (3.1.2) we can therefore continue u_n to a solution of (P_n) defined on $]-\infty,\ t_k]$. Furthermore u_n is continuous and, since $a \leq a_1$, u_n takes its values in B.

Step 2

Now let us show that

$$\Omega := \bigcup_{0 \leq t \leq a} \{u_n(t) : n \in \mathbf{N}\}$$

is relatively compact.

Thanks to property (4) of α it is enough to show that

$$\alpha(\Omega) = 0.$$

By applying (3.3) to the functions u_n, we have

$$\alpha(\Omega) \leq a\alpha\Big(\bigcup_{0 \leq t \leq a} \mathrm{e}^{-tA}f([0, a] \times \Omega)\Big).$$

From (3.5) and (3.7) we obtain

$$\alpha\Big(\bigcup_{0 \leq t \leq a} \mathrm{e}^{-tA}f([0, a] \times \Omega)\Big) \leq M\mathrm{e}^{\omega a}\alpha(f([0, a] \times \Omega))$$

$$\leq M\mathrm{e}^{\omega a}K\ \alpha(\Omega).$$

Therefore

$$(1 - aM\mathrm{e}^{\omega a}K)\ \alpha(\Omega) \leq 0.$$

Since $aM\mathrm{e}^{\omega a}K < 1$, this implies that

$$\alpha(\Omega) = 0.$$

Step 3

Now let us show that the functions u_n are equicontinuous. Let $0 \le r \le t \le a$.

$$\begin{aligned}\|u_n(t) - u_n(r)\| &\le \|e^{-tA}u_0 - e^{-rA}u_0\| + \left\| \int_r^t e^{-(t-s)A} f(s, u_n(s - a/n))\, ds \right\| \\ &\quad + \left\| \int_0^r (e^{-(t-r)A} - I) e^{-(r-s)A} f(s, u_n(s - 1/n))\, ds \right\| \\ &\le \|e^{-tA}u_0 - e^{-rA}u_0\| + |t - r|\, \|D\| M e^{\omega a} \\ &\quad + \left\| (e^{-(t-r)A} - I) \int_0^r e^{-(r-s)A} f(s, u_n(s - a/n))\, ds \right\|.\end{aligned}$$

Let us set

$$\Phi := \left\{ \int_0^r e^{-(r-s)A} f(s, u_n(s - a/n))\, ds : 0 \le r \le a, n \in N \right\}.$$

Since

$$\Phi \subseteq \Omega - \{e^{-rA}u_0 : 0 \le r \le a\}$$

is relatively compact, then

$$\lim_{\sigma \to 0} \sup_{x \in \Phi} \|(e^{-\sigma A} - I)x\| = 0.$$

Since the other two terms are small, independently from n, then step 3 is proved.

Step 4

Thanks to step 2, step 3, and to the Ascoli theorem for sequences (see Appendix 1) there exists a *subsequence* of (u_n) that converges uniformly to a continuous function u. This function, thanks to the Lebesgue theorem for the dominate convergence, is a mild solution of (1.1).

We have thus showed that problem (1.1) has a mild solution on $[0, a]$. If $a = a_1$ the proof is complete. If this is not the case, since $u(a) \in B$, we can repeat the previous argument on $[a, a']$, where $a' = \min\{2a, a_1\}$. We remark that in this case $(a' - a)KMe^{\omega(a' - a)} < 1$. We obtain in such a way a mild solution of the Cauchy problem (1.1) on $[0, a']$. If $a' < a_1$ we can repeat previous argument; at the end we obtain a mild solution of problem (1.1) defined on $[0, a_1]$. ∎

Proof of theorem 3.2. We can follow the outline of the proof of theorem 3.1, but it is necessary to specify some technical details.

The integrals in problems (P_n) are Pettis integrals. Thanks to (3.8_w) it is easy to show the existence of solutions of these problems.

We can show that Ω(Ω is the set in (3.2)) is relatively weakly compact like in step 2 of proof of theorem 3.1, by using (3.4) and (3.6) in place of (3.3) and (3.5).

We can prove that the functions u_n are weakly equicontinuous by using an argument similar to step 3 of proof of theorem 3.1.

The last part of the proof is almost as in theorem 3.1, by applying the Lebesgue theorem for the dominant convergence to

$$\langle \Psi, e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s, u_n(s - a/n))\,ds \rangle$$

for each Ψ in X'(= dual space of X). ■

Proof of proposition 3.1. Let us observe that (1)–(3) imply directly (3.7).

We show that (4) ⇒ (3.7).

Let V, W be bounded subsets of B, and let $\eta > \alpha(V)$. Let $v_1, \ldots, v_n$ be the centers of balls of radius η which cover V. Let $\epsilon > 0$, and let $w_{j,k}$ ($j = 1, \ldots m_k$; $k = 1 \ldots n$) be the centers of balls of radius ϵ which cover $F(v_k, W)$. Let us fix (v, w) in $V \times W$. Let v_k, w_h be such that

$$\|v - v_k\| \le \eta, \qquad \|F(v_k, w) - F(v_k, w_h)\| \le \epsilon.$$

Then we have

$$\begin{aligned} \|F(v,w) - F(v_k, w_h)\| &\le \|F(v,w) - F(v_k, w)\| + \|F(v_k, w) - F(v_k, w_h)\| \\ &\le L\eta + \epsilon. \end{aligned}$$

Since η, ϵ are arbitrary, we have

$$\alpha(F(V, W)) \le L\alpha(V),$$

and therefore

$$\alpha(f(V)) = \alpha(F(V, V)) \le L\alpha(V).$$ ■

Proof of corollary 3.1. Let us set $v := u_t$, and $U := (u, v)$, $U_0 := (u_0, u_1)$. The equation (3.10) is equivalent to

$$U' + A_1 U^T = F_1(U)$$
$$U(0) = U_0$$

where

$$A_1 := \begin{pmatrix} 0, & -I \\ A, & 0 \end{pmatrix} \qquad F_1(U) := \begin{pmatrix} 0 \\ u(k^2 + m)\left(\int_0^1 |u_x|^2\right) \end{pmatrix}$$

where $Au = -u_{xx} + k^2 u$. $-A_1$ is the generator of a C^0-semigroup (for a proof see [43]). Furthermore, if we suppose that F_1 is defined on a bounded subset of $X(:= H_0^1 \times L^2)$, F_1 satisfies 4) of remark 3.1 because $F_1(U) = F(U, U)$ where, for $V = (v_1, v_2)$ and $W = (w_1, w_2)$ $F(V, W) = (0, v_1(k^2 + m(\int_0^1 |w_{1_x}|^2)))^T$, $F(\cdot, W)$ is Lipschitz continuous, uniformly with respect to W(W lies in a bounded subset of X) and $F(V, \cdot)$ is compact since $k^2 + m(\int_0^1 |w_{1_x}|^2)$ lies in a bounded subset of R and v_1 is fixed), and F_1 is continuous. Therefore we can use theorem 3.1. ■

Proof of proposition 3.2. It is obvious that $(1_w), (2_w), (3_w) \Rightarrow (4_w)$.

Let us show that $(4_w) \Rightarrow (3.7_w)$.

Let V, W be bounded subsets of B. Let $\eta > \alpha_W(V)$. Let K be a weakly compact subset of X such that

$$V \subseteq K + \eta \mathbf{B}.$$

Then

$$F(V,W) \subseteq \bigcup_{w \in W} F(K,w) + L\eta\mathbf{B}$$
$$= F(K,W) + L\eta\mathbf{B}.$$

Let us show that $F(K,W)$ is relatively weakly compact. Let us show equivalently that it is relatively sequentially weakly compact. Let us set $x_n := f(k_n\, w_n)$, where $k_n \in K$, $w_n \in W$. There exists a subsequence $(k_{n_h})_{n_h}$ of $(k_n)_n$ that weakly converge to some k. Let $y_{n_h} := F(k, w_{n_h})$. This sequence has a subsequence $y_{n_{h_1}}$ that weakly converge to some y. Then for each $\Psi \in X'$ (the dual space of X), and for $n_{h_1} \to \infty$, we have

$$\langle \Psi, x_{n_{h_1}} - y \rangle = \langle \Psi, F\left(k_{n_{h_1}}, w_{n_{h_1}}\right) - F\left(k, w_{n_{h_1}}\right) \rangle + \langle \Psi, F\left(k, w_{n_{h_1}}\right) - y \rangle \to 0.$$

Therefore $\alpha_W(f(W)) \leq L\alpha_W(W)$. ∎

Proof of remark 3.1. Since X is a reflexive space and f is bounded then it follows that f is weakly compact, and therefore verifies (3.7_W). Furthermore the adjoint semigroup is a C°-semigroup (see proposition 2, Appendix 2), and then the semigroup is sequentially weakly continuous in (t,u) on the bounded subsets of $I \times X$ (see proposition 1, Appendix 2). Therefore we can use theorem 3.2.

Proof of remark 3.2. The proof is similar to the proof of theorem 3.2. The only difference is that we prove that Ω (Ω is the set in (3.2)) is relatively compact and (u_n) are equicontinuous as in theorem 3.1. ∎

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APPENDICES

Appendix 1

The following theorem is a variant of the classical Ascoli theorem. The proof can be performed by a technique very similar to the technique used by [6] in lemma 5.12.

THEOREM (Ascoli theorem for sequences)

Let X be a compact metric space, Y a Hausdorff topological vector space. Let $\mathsf{C} := C(X, Y)$ be endowed with the compact open topology. Let $\mathsf{F} \subseteq \mathsf{C}$ be such that:

(1) $\mathsf{F}(x) := \{f(x) : f \in \mathsf{F}\}$ is relatively sequentially compact for each $x \in X$;

(2) F is equicontinuous at each $x \in X$.

Then F is relatively sequentially compact.

Appendix 2

Let X be a Banach space, and let A be the infinitesimal generator of a C°-semigroup. One can easily show the following results.

PROPOSITION 1. If the adjoint semigroup is a C°-semigroup, then the semigroup is sequentially weakly continuous on the bounded subsets in (t, u).

PROPOSITION 2. If X is a reflexive Banach space, then the adjoint semigroup is a C°-semigroup.