Divisors normally generated on reduced curves *

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Abstract

In this paper we show that a divisor $\mathcal{H}$ on a reduced 1- connected curve C (possibly singular or reducible) is normally generated if $\deg \mathcal{H}_{|B} \geq 2p_a(B) + 1$ for all subcurve $B \subseteq C$.

Introduction

Let C be a curve (a pure projective scheme of dimension 1) over an algebraically closed field of characteristic 0 and let $\mathcal{H}$ be an invertible sheaf on C.

A general problem is to study the behaviour of the rational map $\varphi_{|\mathcal{H}|}$ associated to the linear system $|\mathcal{H}|$ of divisors of sections of $H^0(C, \mathcal{H})$.

In [C-F-H-R] it is developed a criterion (valid in all characteristics) which gives sufficient numerical conditions for the very ampleness of $\mathcal{H}$ on a curve C. It states that if the degree of $\mathcal{H}$ on each generically Gorenstein subcurve B of C is at least $2p_a(B) + 1$, then necessarily $\mathcal{H}$ is very ample on C.

Then one can study some property of the embedded scheme e.g., if it has no "*trisecants*", and some property of the homogeneous ring $R = \bigoplus_{k \geq 0} H^0(C, \mathcal{H}^{\otimes k})$. To this aim we recall the following definition.

Definition. *Let X be a pure projective scheme of dimension n and $\mathcal{H}$ an invertible sheaf on X. $\mathcal{H}$ is said to be* k-normal *if the multiplication map*

$$\rho_k : (H^0(X, \mathcal{H}))^{\otimes k} \longrightarrow H^0(X, \mathcal{H}^{\otimes k})$$

is surjective.

$\mathcal{H}$ is said to be normally generated *if the maps ρ_k are surjective for all $k \in \mathbb{N}$.*

This means that the homogeneous ideal R is generated in degree 1.

It is easy to prove that $\mathcal{H}$ is normally generated iff it is very ample and the embedding $\varphi_{|\mathcal{H}|} : X \hookrightarrow \mathbb{P}^N$ is such that $H^0(\mathbb{P}^N, (n)) \twoheadrightarrow H^0(\varphi(X), \mathcal{O}_{\varphi(X)}(n))$ $\forall n \in \mathbb{N}$.

The idea is to study this property for projective scheme of dimension 1, trying to extend the classical results of Noether, Petri, Mumford, St. Donat and of Green and Lazarsfeld on the normal generation of divisors on smooth curves under some conditions of numerical connectedness.

Our first partial result is the following (for the notation we refer to §1)

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Theorem A *Let C be a 1-connected reduced curve and let $\mathcal{H}$ be an invertible sheaf on C such that*

$$\deg \mathcal{H}_{|B} \geq 2p_a(B) + 1 \quad \textit{for all subcurve } B \subseteq C.$$

Then $\mathcal{H}$ is normally generated on C.

This is a generalization of a classical result due to Castelnuovo and Mumford in the case where C is smooth and irreducible.

We will prove this theorem using simple methods. We hope to extend with different methods this result to the general case.

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1 Notation and preliminary results

Notation

For all the paper we will assume C to be a reduced curve (a pure projective scheme of pure dimension 1 such that for every point $P \in C$ the local ring $\mathcal{O}_{C,P}$ has no nilpotent elements) over an algebraically closed field of characteristic 0.

$\mathcal{H}$ An invertible sheaf on C.

$|\mathcal{H}|$ Linear system of divisors of sections of $H^0(C, \mathcal{H})$.

$\mathcal{H}.C$ The degree of $\mathcal{H}$ on C; it can be defined for every torsion free sheaf of rank 1 by

$$\mathcal{H}.C = \deg \mathcal{H}_{|C} = \chi(\mathcal{H}) - \chi(\mathcal{O}_C).$$

$p_a(C)$ The arithmetic genus of C, $p_a(C) = 1 - \chi(\mathcal{O}_C)$.

ω_C Dualising sheaf of C (see [Ha], Chap. III, §7).

If $C = C_1 \cup C_2$ scheme theoretically with $\dim C_1 \cap C_2 = 0$ and $x \in C_1 \cap C_2$, we can define (cf. [Ca], p. 54)

$$(C_1.C_2)_x = \operatorname{length} \mathcal{O}_{C_1 \cap C_2, x}; \quad C_1.C_2 = \sum_{x \in C_1 \cap C_2} \operatorname{length} \mathcal{O}_{C_1 \cap C_2, x}$$

Note that if $C = C_1 \cup C_2$, with $\dim C_1 \cap C_2 = 0$, then we recover the classical formula

$$p_a(C) = p_a(C_1) + p_a(C_2) + C_1.C_2 - 1$$

Sometimes, with abuse of notation, we will denote the curve C_2 as $C - C_1$.

Definition 1.1 *A (reduced) curve C is* numerically m-connected *if* $C_1.C_2 \geq m$ *for every decomposition* $C = C_1 \cup C_2$

We recall that in [C-F-H-R] it is introduced a notion of m-connectedness fof C Gorenstein, possibly non reduced, in terms of the degree of the dualising sheaf ω_C on each subcurve B.

A *cluster* Z of *degree* $\deg Z = r$ is simply a 0-dimensional subscheme with length $\mathcal{O}_Z = \dim_k \mathcal{O}_Z = r$.

If $C = C_1 \cup C_2$ with $\dim C_1 \cap C_2 = 0$ we will denote by $\mathcal{O}_{C_1}(C_2)$ the cluster on C_1 defined by the ideal $\mathcal{I}_{C_2} \otimes \mathcal{O}_{C_1}$.

We recall the notion of k-very ampleness as introduced by M. Beltrametti P. Francia and A. J. Sommese in [B-F-S] which turns out to be useful to understand the embedding related to $|\mathcal{H}|$.

Definition 1.2 *Let X be a pure projective scheme of dimension m and $\mathcal{H}$ an invertible sheaf on X. $\mathcal{H}$ is said to be* k-very ample *($k \geq 0$) if for any cluster $(Z, \mathcal{O}_Z)$ of X of degree $d \leq k+1$ the map*

$$r_Z : H^0(\mathcal{H}) \longrightarrow H^0(\mathcal{H} \otimes \mathcal{O}_Z)$$

is surjective.

The starting point of our work is the following theorem, proved in [C-F-H-R], which guarantees the very ampleness of a Cartier divisor $\mathcal{H}$ on C under the numerical conditions of theorem A.

A curve $B \subsetneq C$ is *generically Gorenstein* if, outside a finite set, ω_B is locally isomorphic to $\mathcal{O}_B$.

Note that if C is reduced or if $C \subset S$ a smooth surface then this condition is automatically verified.

Theorem 1.3 (Curve embedding theorem) *$\mathcal{H}$ is very ample on C if for every generically Gorenstein subcurve $B \subset C$, either*

1. *$\mathcal{H}.B \geq 2p_a(B) + 1$, or*
2. *$\mathcal{H}.B \geq 2p_a(B)$, and there does not exist a 0-dimensional scheme $Z \subset B$ of degree 2 such that $\mathcal{I}_Z \otimes \mathcal{H} \cong \omega_B$.*

More generally, suppose that $Z \subset C$ is a 0-dimensional scheme (of any degree) such that the restriction

$$H^0(C, \mathcal{H}) \to H^0(\mathcal{H} \otimes \mathcal{O}_Z)$$

is not onto. Then there exists a generically Gorenstein subcurve B of C and an inclusion $\varphi\colon \mathcal{I}_Z\mathcal{O}_B(\mathcal{H}) \hookrightarrow \omega_B$ not induced by a map $\mathcal{H} \to \omega_B$.

In particular if for each generically Gorenstein subcurve B of C we have

$$sH.B \geq (2p_a(B) + k)$$

then $\mathcal{H}$ is k-very ample on C.

(cf. Theorem 1.1 in [C-F-H-R]).

Since in this case $H^1(C, \mathcal{H}) = 0$ (cf. [Ca-Fr], lemma 2.1), we get an embedding $\varphi_{|\mathcal{H}|} : C \to \mathbb{P}^N$, where $N = \mathcal{H}.C - p_a(C)$.

2 Proof of Theorem A

The proof of theorem A, as in the classical case, is essentially an application of a lemma of Castelnuovo on base point free systems. For the reader's benefit we reproduce here the essential structure of Mumford's original proof (cf. [Mu], thm. 2).

Lemma 2.1 (Generalized lemma of Castelnuovo) *Let $\mathcal{F}$ and $\mathcal{H}$ be invertible sheaves on C such that*

1. $H^1(\mathcal{H} \otimes \mathcal{F}^{-1}) = 0$;
2. $|\mathcal{F}|$ *is a base point free system on* C.

Then the multiplication map

$$H^0(\mathcal{H}) \otimes H^0(\mathcal{F}) \to H^0(\mathcal{H} \otimes \mathcal{F})$$

is surjective.

Proof.

Let $s \in H^0(C, \mathcal{F})$ be a section. $\forall\, x \in C$ on an open subset U with $U \ni x$, if we consider an isomorphism $\mathcal{F}_{|U} \cong \mathcal{O}_{C|U}$, s can be considered as a function.

Since $|\mathcal{F}|$ is base point free we can choose s so that $\forall\, x \in C$ $s(x)$ is not a zero-divisor of the stalk $\mathcal{H}_x$.

Then s induces the following exact sequences

$$0 \to \mathcal{F} \xrightarrow{\sigma} \mathcal{H} \otimes \mathcal{F} \to \Delta \to 0 \tag{1}$$

$$0 \to \mathcal{H} \otimes \mathcal{F}^{-1} \xrightarrow{\sigma'} \mathcal{H} \to \Delta' \to 0 \tag{2}$$

where, on an open set U, $\sigma : H^0(U, \mathcal{F}) \to H^0(U, \mathcal{H} \otimes \mathcal{F})$ and $\sigma' : H^0(U, \mathcal{H} \otimes \mathcal{F}^{-1}) \to H^0(U, \mathcal{H})$ are defined by $\alpha \mapsto \alpha \otimes s$, while $\Delta \cong \Delta'$ is a skyscraper sheaf of finite length.

In particular we get $H^1(\mathcal{H}) = 0$ and then the following diagram has exact rows and columns:

$$\begin{array}{ccccccccc}
0 \to & H^0(\mathcal{F}) \otimes H^0(\mathcal{H} \otimes \mathcal{F}^{-1}) & \xrightarrow{\mathrm{id}\otimes\sigma'} & H^0(\mathcal{F}) \otimes H^0(\mathcal{H}) & \to & H^0(\mathcal{F}) \otimes H^0(\Delta) & \to 0 \\
 & {\scriptstyle q_1}\downarrow & & {\scriptstyle q_2}\downarrow & & {\scriptstyle q_3}\downarrow & \\
0 \to & H^0(\mathcal{H}) & \xrightarrow{\sigma} & H^0(\mathcal{F} \otimes \mathcal{H}) & \to & H^0(\Delta) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & S_1 & \xrightarrow{\tau} & S_2 & \to & S_3 & \to 0
\end{array}$$

where for $i = 1,2,3$ q_i is the natural multiplication map and $S_i = \mathrm{coker}(q_i)$.

We can define a map $j_1 : H^0(\mathcal{H}) \to H^0(\mathcal{F}) \otimes H^0(\mathcal{H})$ by $j_1(\alpha) = \alpha \otimes s$. Then the following diagram

$$\begin{tikzcd} & H^0(\mathcal{F}) \otimes H^0(\mathcal{H}) \arrow[d, "q_2"] \\ H^0(\mathcal{H}) \arrow[ur, "j_1"] \arrow[r, "\sigma"] & H^0(\mathcal{F} \otimes \mathcal{H}) \end{tikzcd}$$

is commutative since $q_2(\alpha \otimes s) = \alpha \cdot s$. This implies $\tau \equiv 0$ and then $S_2 \cong S_3$. Now, since $(\Delta, \mathcal{O}_\Delta)$ is a skyscraper sheaf of finite length and $|\mathcal{F}|$ is base point free system we can pick a section $t \in H^0(C, \mathcal{F})$ such that $t(x) \neq 0$ for all $x \in \mathrm{Supp}(\Delta)$. Then

$$\langle t \rangle \otimes_{\mathbb{K}} H^0(\Delta) \xrightarrow{\sim} H^0(\Delta),$$

that is $S_3 = S_2 = 0$. □

The fundamental step in the proof of Theorem A is then to find a base point free system on C satisfying the above conditions.

We will find such invertible sheaf by an inductive argument.

Observe that the condition $\mathcal{H}.C \geq (2p_a(C) + 1)$ is equivalent to the inequality $(\omega_C - \mathcal{H}).C \leq -1$, whence there exists an irreducible $C_1 \subset C$ such that $(\omega_C - \mathcal{H}).C_1 < 0$, (that is $\mathcal{H}.C_1 \geq 2p_a(C_1) + C_1.C_2 - 1$).

Following the paper [Ca-Fr] we define such a C_1 to be *$\mathcal{H}$-positive* and we denote by C_2 the curve $C - C_1$.

Furthermore either C_1 is unique and then $\mathcal{H}.C_1 \geq 2p_a(C_1) + C_1.C_2 + 1$ or there exists at least one other irreducible *$\mathcal{H}$-positive* curve.

Such a curve C_1 will yield an important role in case II of the next proposition:

Lemma 2.2 *Let C and $\mathcal{H}$ be as in theorem A and let C_1 be an irreducible $\mathcal{H}$-positive subcurve of C. Then the exact sequence*

$$0 \to \mathcal{O}_{C_1}(\mathcal{H}) \otimes \mathcal{I}_{C_2} \to \mathcal{O}_C(\mathcal{H}) \to \mathcal{O}_{C_2}(\mathcal{H}) \to 0$$

is exact on global sections, that is $|\mathcal{H}|_{|C_2} = |\mathcal{H}_{|C_2}|$.

Proof. $\mathcal{O}_{C_1} \otimes \mathcal{I}_{C_2}$ defines on C_1 a cluster of length $C_1.C_2$ and by thm. 1.3 $\mathcal{H}$ is $(C_1.C_2 - 1)$-*very ample* on C_1. □

Proposition 2.3 *Let C and $\mathcal{H}$ be as in theorem A. Then there exists an invertible subsheaf $\mathcal{F}$ of $\mathcal{H}$ such that*

1. $\deg \mathcal{F}_{|B} \geq p_a(B) + 1$ *for all B subcurve of C.*
2. $H^1(C, \mathcal{F}) = 0$.
3. *$|\mathcal{F}|$ is a base point free system on C.*
4. $H^1(C, \mathcal{H} \otimes \mathcal{F}^{-1}) = 0$.

PROOF. Let $C = \bigcup_{i=1}^{\nu} \Gamma_i$, with Γ_i irreducible components. The proof will be made by induction on the number ν of components of C.

If $\nu = 1$ it is a classical result; essentially it is a consequence of the *General Position Theorem* (cf. e.g. [ACGH], pp. 109-112).

Taking Δ a cluster of δ smooth distinct points on C in sufficiently general position, with $d - p_a(C) - 1 \geq \delta \geq p_a(C)$, we define $\mathcal{F}$ by the following sequence

$$0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{O}_\Delta \otimes \mathcal{H} \cong \mathcal{O}_\Delta \to 0.$$

We will consider the case C 2-connected and C 2-disconnected separately.

CASE I: There exists a decomposition $C = B_1 \cup B_2$, with $B_1.B_2 \leq 1$.

In this case we may assume by induction that there exists a sheaf $\mathcal{F}$ such that $\mathcal{F}_{|B_1}$ and $\mathcal{F}_{|B_2}$ satisfy conditions 1,...,4 . Then, for $\{i, j\} = \{1, 2\}$, we have the following exact sequences

$$0 \to \mathcal{O}_{B_i}(\mathcal{F} - B_j) \to \mathcal{O}_C(\mathcal{F}) \to \mathcal{O}_{B_j}(\mathcal{F}) \to 0$$

By our hypotheses the above sequences are exact on global section and furthermore we argue that $|\mathcal{O}_C(\mathcal{F})|$ is a base point free system on C.

It remains to show that $H^1(C, \mathcal{H} \otimes \mathcal{F}^{-1}) = 0$.

Take x a smooth point on B_1 such that $|\mathcal{H} \otimes \mathcal{F}^{-1} \otimes \mathcal{O}_{B_1} \otimes \mathcal{M}_x^{-1}|$ is a base point free system on B_1 and $x, \mathcal{O}_{B_1}(B_2)$ impose independent conditions to this system. Then we have $H^1(B_1, \mathcal{H} \otimes \mathcal{F}^{-1} \otimes \mathcal{O}_{B_1} \otimes \mathcal{I}_{B_2}) = 0$ and we conclude since we can assume $H^1(B_2, \mathcal{H} \otimes \mathcal{F}^{-1} \otimes \mathcal{O}_{B_2})) = 0$ by induction.

CASE II: For all decompositions $C = C_1 \cup C_2$, $C_1.C_2 \geq 2$.

Let $C = C_1 \cup C_2$, with C_1 $\mathcal{H}-positive$. For simplicity we let

$$\begin{cases} \mathcal{H}.C = d & p_a(C) = p \\ \mathcal{H}.C_1 = d_1 & p_a(C_1) = p_1 \\ \mathcal{H}.C_2 = d_2 & p_a(C_2) = p_2 \\ C_1.C_2 = m & \end{cases}$$

Take Δ in the following way:

$$\begin{cases} \Delta_{|C_1} = \Delta_1 & \text{consists of } d_1 - p_1 - m + 1 \text{ smooth distinct points on } C_1 \\ \Delta_{|C_2} = \Delta_2 & \text{consists of } \delta_2 \text{ smooth distinct points on } C_2 \end{cases}$$

so that $p_2 \leq \delta_2 \leq d_2 - (p_2 + 1)$ and

$$\deg \mathcal{F}_{|C_1} = p_1 + m - 1, \quad \deg \mathcal{H} \otimes \mathcal{F}_{|C_1}^{-1} \geq p_a(C_1)$$

$$\deg \mathcal{F}_{|B} \geq p_a(B) + 1, \quad \deg \mathcal{H} \otimes \mathcal{F}_{|B}^{-1} \geq p_a(B) \quad \forall B \subseteq C_2.$$

(More precisely if there exists a decomposition $C = \bigcup_{i=1}^{\nu} \Gamma_i$, with Γ_i irreducible components, s.t. $\forall h = 1, ..., \nu - 1$ if we let $A_h = \bigcup_{i=h}^{\nu} \Gamma_i$ we have

$$H^1(\Gamma_h, \mathcal{H} \otimes \mathcal{O}_{\Gamma_h} \otimes \mathcal{I}_{A_{h+1}}) = 0 \quad \text{and} \quad \Gamma_h.A_{h+1} \geq 2 \qquad (*)$$

then we can choose $\mathcal{F}$ s.t. $\deg \mathcal{F}_{|C_2} = p_a(C_2) + 1$, $h^0(C_2, \mathcal{F}) = 2$.)

Furthermore we may assume $p_1+m-1+d_2-\delta_2 \leq N$, $d_1-p_1-m+1+\delta_2 \leq N$.

Now we consider the splitting $C = C_1 \cup C_2$

On C_1 :

Since C_1 is irreducible by classical results on irreducible curve we may assume that $\mathcal{O}_{C_1}(\mathcal{F})$ satisfies conditions 1,..., 4. Furthermore we may assume

Claim 2.4 *For a suitable choice of Δ_1 we have*

$$H^0(C_1, \mathcal{F} \otimes \mathcal{O}_{C_1} \otimes \mathcal{I}_{C_2}) = H^1(C_1, \mathcal{F} \otimes \mathcal{O}_{C_1} \otimes \mathcal{I}_{C_2}) = 0$$

Proof. Let us consider the image of the embedding associated to the complete system $|\mathcal{H}_{|C_1}|$, $\phi_{\mathcal{H}} : C_1 \to \mathbb{P}^{N_1}$, where $N_1 = d_1 - p_1$.

By our choice of C_1 we have $H^1(C_1, \mathcal{H} \otimes \mathcal{O}_{C_1} \otimes \mathcal{I}_{C_2}) = 0$, that is (the image of) the cluster $\mathcal{O}_{C_1}(C_2)$ behaves like $m = C_1.C_2$ independent points in $\mathbb{P}^{N_1}$. But then, since C_1 is irreducible and $\phi_{\mathcal{H}}(C_1)$ is non-degenerate, we can choose other $N_1 - m + 1 = d_1 - p_1 - m + 1$ independent points on C_1 , i.e., for a general Δ_1, $H^1(C_1, \mathcal{H} \otimes \mathcal{I}_{\Delta_1} \otimes \mathcal{O}_{C_1}(-C_2)) = 0$. $\square$

On C_2 :

$\mathcal{H} \otimes \mathcal{I}_{\Delta_2} \otimes \mathcal{O}_{C_2} = \mathcal{O}_{C_2}(\mathcal{F})$ is an invertible subsheaf of $\mathcal{O}_{C_2}(\mathcal{H})$ of degree $\geq p_a(B) + 1$ on all $B \subseteq C_2$.

By induction on the number ν of components we may assume that for a general choice of Δ_2 the conditions 1,..., 4 are satisfied.

Furthermore we have the following

Claim 2.5 *For a general choice of Δ_2, $h^0(C_2, \mathcal{F} \otimes \mathcal{O}_{C_2}(-C_1)) \leq h^0(C_2, \mathcal{F}) - 2$.*

Furthermore if C_1 is the unique $\mathcal{H}-$positive subcurve then we can take $\mathcal{F}$ s.t. $h^0(C_2, \mathcal{F} \otimes \mathcal{O}_{C_2} \otimes \mathcal{I}_{C_1}) = 0$.

Proof. As before let us consider the embedding $\varphi_{|\mathcal{H}|_{|C_2}} : C_2 \hookrightarrow \mathbb{P}^{N_2}$.

Note that we can take $\delta_2 = N_2 + 1 - h^0(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2}(-C_1))$ linearly independent smooth points on C_2 and such that their linear span is independent from the linear subspace generated by $\mathcal{O}_{C_2}(C_1)$.

($N_2 + 1 - h^0(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2} \mathcal{O}_{C_2} \otimes \mathcal{I}_{C_1}) \leq d_2 - p_2 - 1$ implies that Δ_2 as required does exist).

Since $\mathcal{H}$ is very ample and $\deg(\mathcal{O}_{C_2}(C_1)) \geq 2$ we may choose Δ_2 so that the cluster $\mathcal{O}_{C_2}(C_1)$ imposes at least 2 independent conditions to $|\mathcal{F}_{C_2}|$. The first assertion then follows immediately.

For the second assertion if C_2 satisfies condition $(*)$ then we are done since then $h^0(C_2, \mathcal{F}) = 2$, that is we can assume $\mathcal{O}_{C_2}(C_1)$ to be a cluster which imposes independent conditions on $|\mathcal{F}|$.

If not, let $C = \bigcup_{i=1}^{\nu} \Gamma_i$ and let Γ_h be an irreducible subcurve of C which does not satisfies condition $*$, i.e. such that

$$H^1(\Gamma_h, \mathcal{H} \otimes \mathcal{O}_{\Gamma_h} \otimes \mathcal{I}_{A_{h+1}}) = 0, \quad \Gamma_h.A_{h+1} \leq 1.$$

Note that by construction we may assume $\deg \mathcal{F}_{|\Gamma_h} = p_a(\Gamma_h) + 1$. Moreover since by our hypotheses $2p_a(\Gamma_h) + 1 \leq \mathcal{H}.\Gamma_h < 2p_a(\Gamma_h) + \Gamma_h.(C - \Gamma_h) - 1$ then $\Gamma_h.C_1 \geq 2$ that is we should choose $\mathcal{F}$ so that $h^0(\Gamma_h, \mathcal{F} \otimes \mathcal{O}_{\Gamma_h}(-C_1)) = 0$.

Let $B \subset C_2$ be the union of all such Γ_h's. We obviously have $h^0(B, \mathcal{F} \otimes \mathcal{O}_B \otimes \mathcal{I}_{C_1}) = 0$. Now each connected component E of $C_2 - B$ satisfies condition $(*)$ and by our hypotheses $C_1.E \geq 2$, which implies $h^0(E, \mathcal{F} \otimes \mathcal{O}_E(-C_1)) = 0$ and then $h^0(C_2, \mathcal{F} \otimes \mathcal{O}_{C_2} \otimes \mathcal{I}_{C_1}) = 0$. □

Now we can finish the proof of proposition 2.3.

By the following exact sequence

$$H^1(C_1, \mathcal{O}_{C_1}\mathcal{F} \otimes \mathcal{O}_{C_1} \otimes \mathcal{I}_{C_2}) \to H^1(C, \mathcal{F}) \to H^1(C_2, \mathcal{F})$$

we argue that $H^1(C, \mathcal{F}) = 0$ since the first term is zero by claim 2.4 and the third is zero by induction hypothesis.

Note again by claim 2.4, that we have $H^0(C, \mathcal{F}) \xrightarrow{\sim} H^0(C_2, \mathcal{F})$.

If $x \in C_2$, we conclude since $H^0(C_2, \mathcal{F})$ is b.p.f. by induction.

Let $x \in C_1 \setminus C_2$.

If there exists another $\mathcal{H}$*-positive* irreducible subcurve D_1 then we can consider the decomposition $C = D_1 \cup (C - D_1)$ to conclude since then $x \in C - D_1$.

If C_1 is the unique $\mathcal{H}$*-positive* subcurve of C then by claim 2.5 we have an injective map $H^0(C, \mathcal{F}) \to H^0(C_1, \mathcal{F})$ and for $\mathcal{F}$ general we may assume that the image is a base point free subsystem of dimension r.

Indeed, let us consider a divisor Λ, with support on C_2, such that $\mathcal{H}' := \mathcal{H} \otimes \mathcal{O}_C(\Lambda)$ satisfy the condition

$$\deg \mathcal{H}'_{|B} \geq 2p_a(B) + B.C_1 - 1 \quad \text{for all subcurve } B \subseteq C_2$$

and let $\mathcal{F}' := \mathcal{H}' \otimes \mathcal{I}_{\Delta_1}$.Note that $\mathcal{F}'_{|C_1} \cong \mathcal{F}_{|C_1}$. Now, the restriction map $H^0(C, \mathcal{F}') \to H^0(C_1, \mathcal{F})$ is onto and a general subspace $W \subset H^0(C_1, \mathcal{F})$ of dimension r is base point free. Choose $\{s_1, \dots, s_r\} \subset H^0(C, \mathcal{F}')$ such that $\langle s_1, \dots, s_r \rangle \xrightarrow{\sim} W$. Then, for Δ_2, Λ sufficiently general, we may assume

$$\text{im}\{H^0(C, \mathcal{F}) \to H^0(C, \mathcal{F}')\} = \langle s_1, \dots, s_r \rangle$$

which means $\text{im}\{H^0(C, \mathcal{F}) \to H^0(C_1, \mathcal{F})\} = W$ is a base point free system.

It remains to show that $H^1(C, \mathcal{H} \otimes \mathcal{F}^{-1}) = 0$.

Let $\mathcal{F} \cong \mathcal{O}_C(\Sigma)$, where Σ is a cluster of $d - \delta$ smooth points s.t.

$$\begin{cases} \Sigma_{|C_1} = \Sigma_1 & \text{consists of } p_1 + m - 1 \text{ smooth general points on } C_1 \\ \Sigma_{|C_2} = \Sigma_2 & \text{consists of } d_2 - \delta_2 \text{ smooth general points on } C_2 \end{cases}$$

We are done if we show that the points of Σ should be taken in such a way that they impose independent conditions on $|\mathcal{H}|$.

If we consider the embedding $\varphi_{|\mathcal{H}|} : C \to \mathbb{P}^N$, where $N = H.C - p_aC$ and we identify C and its subcurve with their images in $\mathbb{P}^N$ it is enough to prove that the points of Σ may be taken projectively independent.

To simplify the computations we let $N_1 = h^0(C_1, \mathcal{H}) - 1$; $N_2 = h^0(C_2, \mathcal{H}) - 1$; $l = h^1(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2} \otimes \mathcal{I}_{C_1})$; (recall that $C_1.C_2 = m$). Thus we have

$C_1 \subset V_1$ where V_1 is a linear subspace of dimension $= N_1 - l$
$C_2 \subset V_2$ where V_2 is a linear subspace of dimension $= N_2$
$\dim(V_1 \cap V_2) = (m - l - 1)$

The last equality follows from the exact sequence

$$H^0(C_2, \mathcal{H}) \to H^0(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2}(C_1)) \to H^1(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2} \otimes \mathcal{I}_{C_1})$$

Since we can choose δ_2 such that $d_2 - \delta_2 \le \dim(V_2)$ by induction and by our choice $p_1 + m - 1 + d_2 - \delta_2 \le N$ we are done if we show that $p_1 + m - 1 \le \dim(V_1)$.

Now

$$p_1 + m - 1 \le \dim(V_1) \iff d_1 - 2p_1 \ge m + l - 1.$$

We will show this inequality using the fact that $\mathcal{H}.B \ge 2p_a(B) + 1 \;\; \forall\, B \subseteq C$.

If C_2 is irreducible we are done because $\mathcal{H}$ is $(d_2 - 2p_2)$-very ample on C_2 and $d_2 > \max\{2p_2,\ 2p_1 + 2p_2 + 2m - d_1\}$, that is, $l \le \max\{m - d_2 - 2p_2, 0\} \le d_1 - 2p_1 - m$.

If C_2 is reducible, let us consider a decomposition $C_2 = A_2 \cup B_2$ with A_2 irreducible s.t. $h^1(A_2, \mathcal{H} \otimes \mathcal{O}_{A_2}\mathcal{I}_{B_2}) = 0$.

By the following exact sequence

$$H^1(A_2, \mathcal{H} \otimes \mathcal{O}_{A_2} \otimes \mathcal{I}_{B_2 \cup C_1}) \to H^1(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2} \otimes \mathcal{I}_{C_1}) \to H^1(B_2, \mathcal{H} \otimes \mathcal{O}_{B_2} \otimes \mathcal{I}_{C_1}),$$

since by induction

$$h^1(B_2, \mathcal{O}_{B_2}(\mathcal{H} - C_1)) < B_2.C_1 - (H.B_2 - 2p_a(B_2))$$

and by the k-very-ampleness of $\mathcal{H}$ on A_2

$$h^1(A_2, \mathcal{O}_{A_2}(\mathcal{H} - B_2 - C_1)) < C_1.A_2 - (H.A_2 - A_2.B_2 - 2p_a(A_2))$$

we argue that

$$\begin{aligned} l &\le C_1.(A_2 + B_2) - H.(A_2 + B_2) + (2p_a(A_2) + 2p_a(B_2) + A_2.B_2) - 2 \le \\ &\le C_1.C_2 - (H.C_2 - 2p_a(C_2)) = d_1 - 2p_1 - m. \end{aligned}$$

Q.E.D. for proposition 2.3

Note that in the above proposition no connectedness hypotheses are required.

Proof of Theorem A. For $n = 0, 1$ it is obvious since C 1-connected implies

$h^0(C,\mathcal{O}) = 1$ (cf. [CFHR], thm. 3.3 for the general case) and we are considering the complete linear system $|\mathcal{H}|$.

To show that $\mathcal{H}$ is 2-normal on C we apply the generalized lemma of Castelnuovo. Consider the following commutative diagram:

$$\begin{array}{ccccc} H^0(C,\mathcal{F}) \otimes H^0(C,\mathcal{H}) & \hookrightarrow & H^0(C,\mathcal{H}) \otimes H^0(C,\mathcal{H}) & \twoheadrightarrow & H^0(C,\mathcal{O}_\Delta \otimes \mathcal{H}) \otimes H^0(C,\mathcal{H}) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(C,\mathcal{F} \otimes \mathcal{H}) & \hookrightarrow & H^0(C,\mathcal{H}^{\otimes 2}) & \twoheadrightarrow & H^0(C,\mathcal{O}_\Delta \otimes \mathcal{H}^{\otimes 2}) \end{array}$$

$\mathcal{H}$ is very ample on C (in particular it is base point free).

$\mathcal{O}_\Delta \otimes \mathcal{H}^{\otimes 2} \cong \mathcal{O}_\Delta \otimes \mathcal{H} \cong \mathcal{O}_\Delta$ is a skyscraper sheaf of finite length.

We can pick a section $s \in H^0(C,\mathcal{H})$ such that for all $x \in \mathrm{Supp}(\Delta)$ $s(x) \neq 0$. Then

$$H^0(\Delta \otimes \mathcal{H}) \otimes_{\mathbb{K}} \langle s \rangle \xrightarrow{\sim} H^0(\Delta \otimes \mathcal{H}^{\otimes 2}),$$

that is, the third map is onto.

To show that $\mathcal{H}$ is 2-normal it suffices to see that the map $H^0(C,\mathcal{F}) \otimes H^0(C,\mathcal{H}) \to H^0(C,\mathcal{F} \otimes \mathcal{H})$ is surjective and this follows by lemma 2.1 and proposition 2.3.

For $n \geq 3$ we use induction applying the generalized lemma of Castelnuovo to the sheaves $\mathcal{H}^{\otimes(n-1)}$ and $\mathcal{H}$, since by lemma 2.1 of [Ca-Fr] if $\deg \mathcal{H}_{|B} \geq 2p_a(B) - 1$ for all subcurve $B \subseteq C$ then $H^1(C,\mathcal{H}) = 0$.

Q.E.D. for Theorem A

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