Arithmetically Cohen–Macaulay algebraic curves *

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Abstract

Let C be a numerically connected curve lying on a smooth algebraic surface. We show that an invertible sheaf $\mathcal{H} \overset{\mathrm{num}}{\sim} \omega_C \otimes \mathcal{A}$ is normally generated on C if $\mathcal{A}$ is an ample invertible sheaf of degree ≥ 3.

As a corollary we show that on a smooth algebraic surface of general type the invertible sheaf $K_S^{\otimes 3}$ yields a projectively normal embedding of S assuming K_S ample, $(K_S)^2 \geq 3$, $p_g(S) \geq 2$ and $q(S) = 0$.

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1 Introduction

Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a curve (possibly non reduced or reducible) lying on a smooth algebraic surface S (Γ_i's are the irreducible components of C and n_i's are the multiplicities). Write $\mathcal{O}_C$ for the structure sheaf of C, ω_C for the dualizing sheaf of C and $p_a(C) = 1 - \chi(\mathcal{O}_C)$ for the arithmetic genus of C.

Let $\mathcal{H}$ be an invertible sheaf on C. We recall the notion of normal generation as introduced by Mumford in [9] .

Definition . $\mathcal{H}$ is said to be *k-normal* if the multiplication map

$$\rho_k : (H^0(C, \mathcal{H}))^{\otimes k} \longrightarrow H^0(C, \mathcal{H}^{\otimes k})$$

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is surjective. $\mathcal{H}$ is said to be *normally generated* if the maps ρ_k are surjective for all $k \in \mathbb{N}$.

This corresponds to say that the graded ring $R(C,\mathcal{H}) = \bigoplus_{k\geq 0} H^0(C,\mathcal{H}^{\otimes k})$ is generated in degree 1. We remark that if $\mathcal{H}$ is ample on C then $\mathcal{H}$ turns out to be normally generated if and only if $\mathcal{H}$ is very ample and the associated embedded scheme $\varphi_{|\mathcal{H}|}(C) \subset \mathbb{P}(H^0(C,\mathcal{H})^\vee)$ is arithmetically Cohen–Macaulay, i.e., its homogeneous coordinate ring is a Cohen–Macauly ring.

Our first result is the following

Theorem 1. *Let C be a numerically connected curve contained in a smooth algebraic surface and let $\mathcal{H} \overset{\text{num}}{\sim} \omega_C \otimes \mathcal{A}$, with $\mathcal{A}$ an ample invertible sheaf such that*

$$\deg \mathcal{H}_{|B} \geq \ 2p_a(B) + 1 \quad \forall \textit{ subcurve } \ B \subseteq C$$

Then $\mathcal{H}$ is normally generated on C

In [5] it was shown that the above theorem is true except in some exceptional cases. In the present paper we show that also in these particular cases $\mathcal{H}$ is normally generated. The proof of the theorem rely on the theory of Koszul cohomology developed by Green in [6] and the classical " Mumford's argument" for 1-dimension Cohen–Macaulay projective scheme (see Lemma 5), via a detailed analysis of numerically connected curves with multiple components.

The study of the ring $R(C,\mathcal{H})$ for a numerically connected curve has many applications in several aspects of the theory of algebraic surfaces. For instance if one consider the resolution of a normal surface singularity $\pi : S \to X$ or the *relative canonical algebra* of a fibration $f : S \to B$ (cf. [5]).

Another application of these kind of results can be found in the analysis of pluricanonical maps of algebraic surface of general type. Indeed, simply by restriction to divisors in an appropriate linear system we can get further information on the degree of the generators of the canonical rings or on the projective normality of a surface embedded by a pluricanonical system. In particular the above theorem 1 turns out to be particular useful in the special situations where it is impossible to consider theorems of Bertini type.

Our result on pluricanonical embeddings is the following

Theorem 2. *Suppose that S is a smooth surface of general type with K_S ample.*

Assume $(K_S)^2 \geq 3$, $p_g(S) = h^0(S,K_S) \geq 2$ and $q = h^1(S,\mathcal{O}_S) = 0$.

Then $K_S^{\otimes 3}$ is normally generated on S.

We remark that, with mild assumptions, the very ampleness of $K_S^{\otimes 3}$ has been proved in [2] (see [2, §5] for the proof). Here the hypotheses $p_g(S) \geq 2$

and $q = 0$ are needed to apply our restriction methods. The theorem then follows by the methods used in [3] simply applying theorem 1.

The paper is organized as follows: in §2 some useful background results are pointed out; in §3 we study Koszul groups for curves with many components; in §4 we analyze some properties of even curves and even divisors; in §5 we prove Thm. 1; in §6 we study the tricanonical embedding of a surface of general type S with $(K_S)^2 \geq 3$.

2 Notation and background results

2.1 Notation and conventions

We work over an algebraically closed field $\mathbb{K}$ of characteristic ≥ 0.

Throughout this paper by a curve C we will mean a curve $C = \sum_{i=1}^{s} n_i \Gamma_i$ lying on a smooth algebraic surface S, and we will denote by Γ_i each irreducible component of C and by n_i the multiplicity of Γ_i in C.

Let $\mathcal{F}$ be an invertible sheaf on C. For each i the natural inclusion map $\epsilon_i : \Gamma_i \to C$ induces a map $\epsilon_i^* : \mathcal{F} \to \mathcal{F}_{|\Gamma_i}$. Setting $d_i = \deg \mathcal{F}_{|\Gamma_i}$ we define the multidegree of $\mathcal{F}$ on C $\mathbf{d} := (d_1, ..., d_s)$.

By $\mathrm{Pic}^{\mathbf{d}}(C)$ we denote the Picard scheme which parametrizes the classes of invertible sheaves of multidegree $\mathbf{d} = (d_1, \ldots, d_s)$ (see [5]).

Two invertible sheaves $\mathcal{F}$, $\mathcal{F}'$ are said to be numerically equivalent on C (notation: $\mathcal{F} \overset{\text{num}}{\sim} \mathcal{F}'$) if $\deg \mathcal{F}_{|\Gamma_i} = \deg \mathcal{F}'_{|\Gamma_i}$ for all Γ_i.

By a general local transverse cut Δ_i we mean a 0–dimensional subscheme of C with support a general smooth point Q of C_{red} such that $\mathcal{O}_C(\Delta_i)$ is invertible. As a scheme, $\Delta_i \cong \mathbb{K}[x]/(x^\nu)$, where $\nu = \mathrm{mult}_Q(C)$.

C is said to be *numerically m-connected* if $C_1 \cdot C_2 \geq m$ for every effective decomposition $C = C_1 + C_2$, where $C_1 \cdot C_2$ denotes their intersection number as divisor on S. A curve C is said to be *numerically connected* if it is 1-connected.

We recall that a curve C is *honestly hyperelliptic* if there exists a finite morphism $\psi \colon C \to \mathbb{P}^1$ of degree 2. In this case C is either irreducible, or of the form $C = \Gamma_1 + \Gamma_2$ with $p_a(\Gamma_i) = 0$ and $\Gamma_1 \cdot \Gamma_2 = p_a(C) + 1$. (See [2] for details)

2.2 General divisors of low degree

We recall two vanishing theorem proved in [5] on "general " invertible sheaves of low degree.

Theorem 3. *Assume $C = \sum_{i=1}^{s} n_i \Gamma_i$ to be a curve contained in smooth algebraic surface. Let $\mathbf{d} = (d_1, ..., d_s) \in \mathbb{N}^s$ be such that for each invertible sheaf $\mathcal{G}'$ of multidegree $\mathbf{d}$ we have*

$$\deg \mathcal{G}'_{|B} \geq \ p_a(B) \quad \forall \textit{ subcurve } \ B \subseteq C$$

Then for $[\mathcal{G}]$ general in $\mathrm{Pic}^{\mathbf{d}}(C)$, $H^1(C, \mathcal{G}) = 0$.

Theorem 4. *Assume $C = \sum_{i=1}^{s} n_i\Gamma_i$ to be a curve contained in smooth algebraic surface. Let $\mathbf{d} = (d_1, ..., d_s) \in \mathbb{N}^s$ be such that for each invertible sheaf $\mathcal{F}'$ of multidegree $\mathbf{d}$ we have*

$$\deg \mathcal{F}'_{|B} \geq \ p_a(B) + 1 \quad \forall \ \ \textit{subcurve} \ \ B \subseteq C$$

Then for $[\mathcal{F}]$ general in $\mathrm{Pic}^{\mathbf{d}}(C)$, $|\mathcal{F}|$ is a base-point free system.

For a proof of the above theorems see [5, Thm. 3.1, 3.2].

2.3 Mumford's argument

To prove the surjection of a certain multiplication map Mumford's argument consists in finding a useful subsheaf and then analyzing the natural decomposition which come out.

Proposition 5 (Mumford's argument)**.** *Let $\mathcal{L}$, $\mathcal{H}$ be invertible sheaves on a curve C with $|\mathcal{H}|$ and $|\mathcal{L}|$ base point free systems.*

Let $\mathcal{F} \cong \mathcal{H}(-\Delta)$ (Δ a 0-dimensional scheme) be an invertible subsheaf of $\mathcal{H}$ so that $|\mathcal{F}|$ is a base point free system on C. Assume furthermore that the sequence

$$0 \to \mathcal{F} \to \ \mathcal{H} \to \mathcal{O}_\Delta \to 0$$

is exact on global sections and that the multiplication map $p_1 : H^0(\mathcal{F}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{F} \otimes \mathcal{L})$ is onto. Then we have a surjection

$$r_1 : H^0(\mathcal{H}) \otimes H^0(\mathcal{L}) \twoheadrightarrow H^0(\mathcal{H} \otimes \mathcal{L})$$

For a proof see [9, Thm. 6]. Applying Prop. 8 and Thm. 3–4 as a corollary of we get the following (see. [5, Thm. A])

Theorem 6. *Let $\mathcal{H}$ be an invertible sheaf on C such that $\mathcal{H} \overset{\mathrm{num}}{\sim} \mathcal{F} \otimes \mathcal{G}$, where $\mathcal{F}, \mathcal{G}$ are invertible sheaves such that*

$$\deg \mathcal{G}_{|B} \geq \ p_a(B) \quad \forall \ \ \textit{subcurve} \ \ B \subseteq C \tag{1}$$

$$\deg \mathcal{F}_{|B} \geq \ p_a(B) + 1 \ \ \forall \ \ \textit{subcurve} \ \ B \subseteq C \tag{2}$$

Then the natural multiplication map $(H^0(\mathcal{H}))^{\otimes 2} \to H^0(\mathcal{H}^{\otimes 2})$ is onto. Moreover, if C is numerically connected then $\mathcal{H}$ is normally generated on C.

3 Koszul cohomology groups of algebraic curves

3.1 Definition and basic results

In this section we recall the notion of Koszul cohomology groups as introduced and developed by Green in [6], and we focus on some applications of Koszul cohomology to the analysis of invertible sheaves on a numerically connected curve C. We recall the duality between certain Koszul cohomology groups proved in [5] and we prove a slightly generalization of Green's H^0-Lemma.

Let $\mathcal{H}$, $\mathcal{F}$ be invertible sheaves on C and let $W \subseteq H^0(C, \mathcal{F})$ be a subspace which yields a base point free system of projective dimension r.

Let $M_{\mathcal{F}}$ be the Kernel of the evaluation map $W \otimes \mathcal{O}_C \stackrel{\text{ev}}{\to} \mathcal{F}$. Twisting with $\mathcal{H} \otimes \mathcal{F}^q$ and taking exterior powers we get the following exact sequence

$$0 \to M_{p,q} \to \bigwedge^p W \otimes \mathcal{H} \otimes \mathcal{F}^q \to M_{p-1,q+1} \to 0$$

where $M_{p,q}$ denotes the sheaf $\bigwedge^p M_{\mathcal{F}} \otimes \mathcal{H} \otimes \mathcal{F}^q$. Taking cohomology we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & H^0(M_{p+1,q-1}) \arrow[d] & & \\
 & \bigwedge^{p+1} W \otimes H^0(\mathcal{H} \otimes \mathcal{F}^{q-1}) \arrow[d, "\varphi_{p,q}"'] \arrow[dr, "d_{p+1,q-1}"] & & 0 \arrow[d] \\
0 \arrow[r] & H^0(M_{p,q}) \arrow[r] \arrow[d] & \bigwedge^p W \otimes H^0(\mathcal{H} \otimes \mathcal{F}^q) \arrow[r] \arrow[dr, "d_{p,q}"'] & H^0(M_{p-1,q+1}) \arrow[d] \\
 & H^1(M_{p+1,q-1}) \arrow[d] & & \bigwedge^{p-1} W \otimes H^0(\mathcal{H} \otimes \mathcal{F}^{q+1}) \\
 & \bigwedge^{p+1} W \otimes H^1(\mathcal{H} \otimes \mathcal{F}^{q-1}) & &
\end{tikzcd}$$

where $d_{p,q}$ are the Koszul differentials, defined as follows:

$$\begin{array}{l} d_{p,q} : \bigwedge^p W \otimes H^0(C, \mathcal{H} \otimes \mathcal{F}^q) \longrightarrow \bigwedge^{p-1} W \otimes H^0(C, \mathcal{H} \otimes \mathcal{F}^{q+1}) \\ \sum s_{i_1} \wedge s_{i_2} \wedge \ldots \wedge s_{i_p} \otimes \alpha_{i_1 i_2 \ldots i_p} \mapsto \sum s_{i_1} \wedge \ldots \hat{s}_{i_j} \ldots \wedge s_{i_{r-1}} \otimes \alpha_{i_1 \ldots \hat{i}_j \ldots i_p} \cdot s_{i_j} \end{array}$$

(here $\{s_0, \ldots, s_r\}$ is a basis for W.)

The Koszul groups $\mathcal{K}_{p,q}(C, W, \mathcal{H}, \mathcal{F})$ are defined by $\ker d_{p,q} / \operatorname{im} d_{p+1,q-1}$.

If $W = H^0(C, \mathcal{F})$ they are usually denoted by $\mathcal{K}_{p,q}(C, \mathcal{H}, \mathcal{F})$, while if $\mathcal{H} \cong \mathcal{O}_C$ the usual notation is $\mathcal{K}_{p,q}(C, \mathcal{F})$. Notice that the multiplication map

$$W \otimes H^0(C, \mathcal{H}) \to H^0(C, \mathcal{F} \otimes \mathcal{H})$$

is surjective if and only if $\mathcal{K}_{0,1}(C, W, \mathcal{H}, \mathcal{F}) = 0$ and that $\mathcal{F}$ is normally generated if and only if $\mathcal{K}_{0,q}(C, \mathcal{F}) = 0 \ \forall \ q \geq 1$.

For our analysis the main applications of Koszul cohomology are the following propositions.

Proposition 7 (Duality). *Let $W \subseteq |\mathcal{F}|$ be a base point free system of dimension r. Then*

$$\mathcal{K}_{p,q}(C,W,\mathcal{H},\mathcal{F}) \overset{\mathrm{d}}{=} \mathcal{K}_{r-p-1,2-q}(C,W,\omega_C \otimes \mathcal{H}^{-1},\mathcal{F})$$

(where $\overset{\mathrm{d}}{=}$ means duality of vector space).

For a proof see [5, Prop. 1.4].

Proposition 8 (H^0-Lemma). *Let $\mathcal{F}$, $\mathcal{H}$ be invertible sheaves on C and assume $W \subseteq H^0(C,\mathcal{F})$ to be a subspace of* $\dim = r+1$ *which yields a base point free system. If either*

(i) $H^1(C,\mathcal{H}\otimes\mathcal{F}^{-1}) = 0$,
or

(ii) C is numerically connected, $\omega_C \cong \mathcal{H}\otimes\mathcal{F}^{-1}$ and $r \geq 2$,
or

(iii) C is numerically connected, $h^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F}) \leq r-1$ and there exists a reduced subcurve $B \subseteq C$ such that:

- $W \hookrightarrow W_{|B}$,
- $H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F}) \hookrightarrow H^0(B,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$,
- *every non–zero section of $H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$ does not vanish identically on any component of B;*

then $\mathcal{K}_{0,1}(C,W,\mathcal{H},\mathcal{F}) = 0$, that is, the multiplication map

$$W\otimes H^0(C,\mathcal{H}) \to H^0(C,\mathcal{F}\otimes\mathcal{H})$$

is surjective.

Proof. By duality we need to prove that $\mathcal{K}_{r-1,1}(C,W,\omega_C\otimes\mathcal{H}^{-1},\mathcal{F}) = 0$. To this aim let $\{s_0,\ldots,s_r\}$ be a basis for W and let $\alpha = \sum s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_{r-1}}\otimes\alpha_{i_1 i_2\ldots i_{r-1}} \in \bigwedge^{r-1} W\otimes H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$ be an element in the Kernel of the Koszul map $d_{r-1,1}$.

In cases (i) and (ii) obviously $\alpha = 0$ (see [5] for details).

In the latter case by our assumptions we can restrict to the curve B. Since B is reduced we can choose $r+1$ "sufficiently general points" on B so that $s_j(P_i) = \delta^i_j$. But then $\alpha \in ker(d_{r-1,1})$ implies for every multiindex $\mathbf{I} = \{i_1,\ldots i_{r-2}\}$ the following equations (up to sign)

$$\alpha_{j_1 i_1\ldots i_{r-2}}\cdot s_{j_1} + \alpha_{j_2 i_1\ldots i_{r-2}}\cdot s_{j_2} + \alpha_{j_3 i_1\ldots i_{r-2}} s_{j_3} = 0.$$

(where $\{i_1,\ldots i_{r-2}\}\cup\{j_1,j_2,j_3\} = \{0,\ldots,r+1\}$).

Evaluating at $P_j' s$ and reindexing we get

$$\alpha_{i_1 \dots i_{r-1}}(P_{i_k}) = 0 \ \ \forall k = 1, \dots, r-1.$$

But now, since the $P_j' s$ are in general positions and every section of $H^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F})$ does not vanish identically on any component of B, we may assume that any $(r-1)$-tuple of points $P_{i_1}, \dots, P_{i_{r-1}}$ imposes independent conditions on $H^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F})$.

The proposition then follows by a dimension count since by assumptions $h^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F}) \leq h^0(C, \mathcal{F}) - 2 = r - 1$. □

Sometimes it turns out to be very useful the classical base point free pencil trick. For curves with several components we need some accuracy. It can be stated as follows:

Proposition 9 (Base point free pencil trick)**.** *Let C be a numerically connected curve and let $\mathcal{F}$, $\mathcal{H}$ be invertible sheaves on C. Assume that s_1 and s_2 are two sections in $H^0(\mathcal{F})$ having no common zeros and not vanishing identically on any subcurve of C. Assume furthermore that for every nonzero section $t \in H^0(\mathcal{H})$, t does not vanish identically on any subcurve of C.*

Then the kernel of the map

$$\begin{array}{ccc} H^0(\mathcal{H})s_1 \oplus H^0(\mathcal{H})s_2 & \longrightarrow & H^0(\mathcal{F} \otimes \mathcal{H}) \\ (\ t_1 \ \ , \ \ t_2\ ) & \mapsto & t_1 \otimes s_1 + t_2 \otimes s_2 \end{array}$$

is isomorphic to $H^0(\mathcal{H} \otimes \mathcal{F}^{-1})$.

Proof. With this given hypotheses the proof works essentially in the same way as the classical proof. Indeed, by our assumptions the kernel is isomorphic to the space of pairs t_1, t_2 such that $t_1 \otimes s_1 = -t_2 \otimes s_2$. Since these sections do not vanish identically on any subcurve of C, and s_1, s_2 have no common zeros, this is equivalent to say that t_1 is zero at the zeros Λ_2 of s_2, i.e., we can write $t_1 = \lambda \cdot s_2$ and then $t_2 = \lambda \cdot s_1$ with $\lambda \in H^0(C, \mathcal{H}(-\Lambda_2)) \cong H^0(\mathcal{H} \otimes \mathcal{F}^{-1})$. □

4 Even curves and even divisors

Even divisors and even curves can be defined in a very natural way as follows.

Definition 10. Let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be a curve contained in a smooth algebraic surface and let $\mathcal{H}$ be a numerically effective invertible sheaf on C. Then $\mathcal{H}$ is said to be *even* if

$$\deg \mathcal{H}_{|\Gamma_i} \text{ is even } \ \forall \ \text{ irreducible } \ \Gamma_i \subset C$$

Definition 11. Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a numerically connected curve contained in a smooth algebraic surface. Then C is said to be *even* if

$$\deg \omega_{C|\Gamma_i} \text{ is even } \ \forall \ \text{ irreducible } \ \Gamma_i \subset C$$

(this is equivalent to say $\Gamma_i \cdot (C - \Gamma_i)$ even $\forall i$.)

Even curves and even divisors appear for instance if you consider the canonical system $|K_S|$ for a surface S of general type and may have useful applications to the analysis of pluricanonical maps of algebraic surfaces. Indeed, by adjunction, for every curve $C \in |K_S|$ we have $|(2K_S)_{|C}| = |K_C|$, that is every curve in the canonical system is even.

4.1 Combinatorial properties of even curves

Now we analyze some useful combinatorial properties of numerically connected even curves.

First of all we have

Remark 12. *Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a numerically connected even curve. Then there exists an invertible sheaf $\mathcal{G}$ of multidegree $\delta = (\delta_1, \ldots, \delta_s)$ such that $\mathcal{G}^{\otimes 2} \overset{\text{num}}{\sim} \omega_C$.*

The following technical lemmas will turn out to be useful in the analysis of the particular cases of Theorem 1.

Lemma 13. *Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a numerically connected curve contained in a smooth algebraic surface.*

Suppose that there exists an invertible sheaf $\mathcal{G}$ of multidegree $\delta = (\delta_1, \ldots, \delta_s)$ such that $\mathcal{G}^{\otimes 2} \overset{\text{num}}{\sim} \omega_C$.

Then for a general effective divisor $\mathcal{O}_C(\Delta)$ in Pic^{δ} every non–zero section in $H^0(C, \mathcal{O}_C(\Delta))$ does not vanish identically on any proper subcurve of C.

Proof. The lemma follows since for every decomposition $C = A + B$ we have

$$0 \to H^0(B, \mathcal{O}_B(\Delta) \otimes \mathcal{O}_B(-A)) \to H^0(C, \mathcal{O}_C(\Delta)) \to H^0(A, \mathcal{O}_A(\Delta)) \to 0$$

and by a degree argument it is $H^0(B, \mathcal{O}_B(\Delta) \otimes \mathcal{O}_B(-A)) = 0$. Indeed by duality and adjunction it is equivalent to prove that $H^1(B, \omega_C \otimes \mathcal{O}_B(\Delta)^{-1}) = 0$. This follows by Thm. 3 since by numerically connectedness $\deg(\omega_C \otimes \mathcal{O}_C(\Delta)^{-1})_{|B'} = \frac{1}{2} \deg \omega_{C|B'} \geq p_a(B')$ for every $B' \subseteq B$. $\square$

Finally we consider the particular situation where $\Gamma \cdot (C - \Gamma) = 2$ for every irreducible component Γ

Lemma 14. *Let C be 2- connected, $C \neq n\Gamma$ and assume that for all irreducible $\Gamma \subset C$, $\Gamma \cdot (C - \Gamma) = 2$. Then*

(i) for every Γ, $\mathrm{mult}_C(\Gamma) \leq 2$;

(ii) if C is nonreduced we can write $C = C_{red} + B$ where $C_{red} = \sum_{i=1}^{s} \Gamma_i$ and $B = \sum_{i=1}^{k} \Gamma_i$ $(k \leq s)$ are chains of reduced and irreducible components such that:

- *$C_{red} \cdot B = 2$;*
- *both C_{red} and B are numerically connected (and their dual graphs are simply connected);*
- *$C_{red} \cdot \Gamma_1 = C_{red} \cdot \Gamma_k = 1$, while $C_{red} \cdot \Gamma_i = 0$ $\forall$ $i = 2, \ldots, k-1$.*

Proof. (i) Let $n = \mathrm{mult}_C(\Gamma)$ and let $D = C - n\Gamma$. Assume that $n \geq 3$. By hypotheses

$$\Gamma \cdot (D + (n-1)\Gamma) = 2$$

and $\forall$ $m,$ $1 \leq m \leq n-2$ we have $(n-m)\Gamma \cdot (D + m\Gamma) \geq 2$ i.e.,

$$\Gamma^2 \geq \frac{1 - \Gamma \cdot D}{m} \ .$$

Writing $\Gamma^2 = \frac{2 - \Gamma \cdot D}{(n-1)}$ then we have

$$\begin{array}{rcll} \frac{2-\Gamma \cdot D}{(n-1)} & \geq & \frac{1 - \Gamma \cdot D}{m} & \Leftrightarrow \\ \frac{2m-(n-1)}{(n-1)m} & \geq & \left(\ \Gamma \cdot D \ \right) \left(\ \frac{m-(n-1)}{(n-1)m} \ \right) & \Leftrightarrow \\ 1 > 1 - \frac{m}{(n-1)-m} & \geq & \Gamma \cdot D & \end{array}$$

which is absurd since $\Gamma \cdot D \geq 1$ by numerically connectedness.

(ii) Let $\Gamma \subset C$ be an irreducible subcurve of multiplicity 2 and let Γ' be another irreducible component which intersects Γ. If $\mathrm{mult}_C(\Gamma') = 1$ then $\Gamma' \cdot (2\Gamma) \geq 2$ implies $\Gamma' \cdot (2\Gamma) = \Gamma' \cdot (C - \Gamma') = 2$ and then Γ' is extremal in the graph.

Now let us consider the nonreduced part of C, say $C' = \sum_{i=1}^{k} 2\Gamma_i$ and write $C = C' + C''$, where C'' consists of disconnected extremal components of multiplicity 1, so that $B = C'_{red} = \sum_{i=1}^{k} \Gamma_i$ and $C_{red} = C'_{red} + C''$.

Notice that, by the above remark, for every $\Gamma_i \subset C'$ there exists at least a $\Gamma_j \subset C'$ which intersects Γ_i. Moreover by 2-connectedness

$$B \cdot C_{red} = (\sum_{i=1}^{k} \Gamma_i)(\sum_{j=1}^{k} \Gamma_j + C'') \geq 2.$$

Since for every i we have $\Gamma_i(C-\Gamma_i)=2$ this inequality can be read as

$$\sum_{i=1}^{k}\Gamma_i\cdot\left[(C-\Gamma_i)-\sum_{\substack{j=1\\ j\neq i}}^{k}\Gamma_j\right]=\sum_{i=1}^{k}\left[2-\sum_{\substack{j=1\\ j\neq i}}^{k}\Gamma_i\cdot\Gamma_j\right]\geq 2$$

which is equivalent to

$$\sum_{\substack{i,j=1\\ j\neq i}}^{k}2\Gamma_i\cdot\Gamma_j\leq 2k-2$$

i.e., $\forall\ i=1,\dots,k\quad \exists$ unique $j>i$ s.t. $\Gamma_i\cdot\Gamma_j=1$ and equalities hold throughout.

This exactly means that B and then C_{red} are chains of irreducible and reduced components, such that $C_{red}\cdot B=2$. In particular B and C_{red} are numerically connected since C is 2-connected (cf. [4]).

The last assertion follows since for every i we have $\Gamma_i(C-\Gamma_i)=2$. □

4.2 Numerically connected even irreducible curves

Next we focalize on irreducible but nonreduced curves.

Lemma 15. *Let $C=3\Gamma$ be an irreducible but reduced curve of multiplicity 3 contained in a smooth algebraic surface. Suppose C 3-connected.*

Then $h^1(C,\omega_C(-\Delta))=1$ for a general effective divisor Δ of (multi)degree $\delta\leq\frac{1}{2}(\deg_\Gamma\omega_C)$.

Proof. First of all notice that ω_C is very ample and yields an embedding $\varphi_{\omega_C}(C)\subset\mathbb{P}^N$ since C is 3-connected and not honestly hyperelliptic (see [2, §3]), and that for a general Δ of degree $\delta\leq\frac{1}{2}(\deg_\Gamma\omega_C)$, by [5, Thm. 3.1] it is

$$H^0(C,\omega_C)\twoheadrightarrow H^0(2\Gamma,\omega_C)\twoheadrightarrow\mathcal{O}_{\Delta_{|2\Gamma}}$$

The proof will be made by induction on δ.

For $\delta=1$ take a point Q. Δ will be chosen among all the Cartier divisors of degree 1 with support Q.

We recall that the set of all Cartier divisor of multidegree $=1$ concentrated at Q is isomorphic to

$$\mathbf{B}_Q=\{[g]\mid g\in\mathcal{O}_{Q,C}\quad;\quad v_Q(\sigma(g))=1\}$$

where $\sigma:C\to\Gamma$ is the reduction morphism and $v_Q(\cdot)$ is the valuation at Q in $\mathcal{O}_{Q,\Gamma}$ (see [10, §4]). Furthermore if $C=n\Gamma$ is contained in a smooth algebraic surface then $\mathbf{B}_Q\cong\mathbb{A}^{n-1}$ (see e.g. [10, 4.3.1]).

Now it is $H^0(C,\omega_C)\twoheadrightarrow\mathbb{K}_Q$ and $H^0(C,\omega_C)\twoheadrightarrow\Delta_{|2\Gamma}$ for every local transverse cut Δ supported at Q since ω_C is very ample and length$(\Delta_{|2\Gamma})=2$. Finally, two sections in $H^0(C,\omega_C)$ vanishing at Q, vanish contemporarily on Δ if and only if they define the same element in $\mathbf{B}_Q$.

Thus, if $H^0(C,\omega_C) \twoheadrightarrow \mathbf{B}_Q$ we can take Δ to be any local transverse cut. Otherwise we take Δ not in the image of $H^0(C,\omega_C) \to \mathbf{B}_Q$, which exactly means that there exists a section of $H^0(C,\omega_C)$ which vanish on $\Delta_{|2\Gamma}$ but not on Δ, i.e. $H^0(C,\omega_C) \twoheadrightarrow \mathcal{O}_\Delta$.

Now assume the theorem for $\delta' < \delta$ and consider a general divisor Δ' of degree $\delta - 1$. By induction we may assume $H^1(C,\omega_C(-\Delta')) = 1$ and the restriction map

$$H^0(C,\omega_C(-\Delta')) \to H^0(2\Gamma,\omega_C(-\Delta'))$$

to be of maximal rank.

Thus, we simply take a general point Q not in the base locus of $|\omega_C(-\Delta')|$ and a general local transverse cut ζ supported at Q.

We have $H^0(C,\omega_C(-\Delta')) \twoheadrightarrow \mathbb{K}_Q$ and, by degree consideration, there exist at least two independente sections $s_1, s_2 \in H^0(C,\omega_C(-\Delta'))$ vanishing at Q but not vanishing on any subcurve of C. Whence, arguing as above, for ζ general we have $H^0(C,\omega_C(-\Delta')) \twoheadrightarrow \zeta_{|2\Gamma}$, and then we can repeat the argument adopted for the case $\delta = 1$.

Taking $\Delta = \Delta' + \zeta$ we obtain the required surjection.

□

5 Normally generated adjoint divisors

This section is devoted to the proof of Theorem 1 stated in the introduction.

We remark that the first half page of the proof is exactly the same of [5, Thm. B]. For completeness and for the reader's benefit we have decided to rewrite here also this part.

Proof of Theorem 1. For all $k \in \mathbb{N}$ we have to show the surjectivity of the maps

$$\rho_k : (H^0(C,\mathcal{H}))^{\otimes k} \longrightarrow H^0(C,\mathcal{H}^{\otimes k})$$

For $k = 0, 1$ it is obvious since C is numerically connected. For $k \geq 3$ we use induction applying Proposition 8 to the sheaves $\mathcal{H}^{\otimes(k-1)}$ and $\mathcal{H}$.

Now we treat the case $k = 2$.

Let $C = \sum_{i=1}^s n_i\Gamma_i$ be our curve, and for all $i = 1,\ldots,s$ set $\gamma_i = \deg \omega_{C|\Gamma_i}$, $d_i = \deg \mathcal{H}_{|\Gamma_i}$. By our assumptions for all $i = 1,\ldots,s$ there exists an integer δ_i such that $\gamma_i \leq 2\delta_i \leq d_i$. Taking δ_i general local transverse cuts on each irreducible component Γ_i, then we find an invertible sheaf $\mathcal{G}$ of multidegree $(\delta_1,\ldots,\delta_s)$ such that

$$\deg \omega_{C|B} \leq \deg \mathcal{G}_{|B}^{\otimes 2} \leq \deg \mathcal{H}_{|B}$$

i.e., by 1-connectedness $\deg \mathcal{G}_{|B} \geq p_a(B)$ for all $B \subset C$, except possibly for $B = C$.

Set $\mathcal{F} := \mathcal{H}(-\Delta) \cong \mathcal{H}\otimes\mathcal{G}^{-1}$. Then we have $\deg \mathcal{F}^{\otimes 2}_{|\Gamma_i} \geq \deg \mathcal{H}_{|\Gamma_i}\ \ \forall i$, whence $\deg \mathcal{F}_{|B} \geq p_a(B) + 1$ for all $B \subseteq C$.

Case 1. There exists an index h s.t. $\gamma_h < 2\delta_h \leq d_h$.

In this case $\mathcal{F}$ and $\mathcal{G}$ satisfy the hypotheses of Theorem 6 and then we can conclude.

Case 2. For all $i = 1, \dots, s$ $\begin{cases} \deg \omega_{C|\Gamma_i} \ \text{ is even} \\ \deg \mathcal{A}_{|\Gamma_i} = 1 \end{cases}$

Here we will treat firstly the general situation, showing that we may find a new decomposition $\mathcal{H} \overset{\text{num}}{\sim} \mathcal{F}' \otimes \mathcal{G}'$, with $\mathcal{F}'$ and $\mathcal{G}'$ as in Theorem 6, and then we will treat the exceptional cases.

To this aim let us consider the following list

(a) $C = n\Gamma,\ n \geq 3,\ \Gamma^2$ even;

(b) $C = \Gamma_1 + 2\Gamma_2, \Gamma_1 \cdot \Gamma_2 = 1$;

(c) For all irreducible $\Gamma \subset C$, $\Gamma \cdot (C - \Gamma) = 2$.

Claim 16. *Assume that for all $i = 1, \dots, s$ $\deg \omega_{C|\Gamma_i}$ is even and $\deg \mathcal{A}_{|\Gamma_i} = 1$. If C does not belong to the above list then there exists a decomposition $\mathcal{H} \overset{\text{num}}{\sim} \mathcal{F}' \otimes \mathcal{G}'$, with $\mathcal{F}'$ and $\mathcal{G}'$ as in Theorem 6.*

Proof. By hypotheses there exists an irreducible component Γ_h of multiplicity n_h such that $\Gamma_h \cdot (C - \Gamma_h) \geq 4$.

Taking $\mathcal{A}_h$ a general transverse cut on this component, we let $\mathcal{G}' := \mathcal{G} \otimes \mathcal{A}_h$, $\mathcal{F}' := \mathcal{F} \otimes \mathcal{A}_h^{-1}$, and we infer that $\mathcal{G}'$ and $\mathcal{F}'$ satisfy respectively condition (3) and condition (4) of Theorem 6.

For $\mathcal{G}'$ this is obvious. About $\mathcal{F}'$ we have

$$\deg \mathcal{F}'_{|B} = \frac{\deg \omega_{C|B}}{2} + \deg \mathcal{A}_{|B} - \deg \mathcal{A}_{h|B}$$

Thus, if $B = C$ the required inequality holds since $\deg \mathcal{A}_{|B} - \deg \mathcal{A}_{h|B} \geq 2$ because case (a) and (b) do not occur, while if $B \subset C$ and $B \neq m_h\Gamma_h$ it holds because C is 2-connected and $\deg \mathcal{A}_{|B} - \deg \mathcal{A}_{h|B} \geq 1$. Finally, if $B = m_h\Gamma_h$ $(1 \leq m_h \leq n_h)$ it holds thanks to our choice of Γ_h. $\square$

The exceptional configurations.

In the configurations (a), (b), (c) listed above we can no longer find a suitable decomposition with $\mathcal{F} \hookrightarrow \mathcal{H}$, $|\mathcal{F}|$ b.p.f. and $H^1(\mathcal{F}) = H^1(\mathcal{H}\otimes\mathcal{F}^{-1}) = 0$.

In these particular configurations we have

$$\mathcal{G} \cong \mathcal{O}_C(\Delta) \text{ is an invertible sheaf such that } \omega_C \overset{\text{num}}{\sim} \mathcal{G}^{\otimes 2} \tag{3}$$

$$\mathcal{F}=\mathcal{H}(-\Delta)\cong\mathcal{H}\otimes\mathcal{G}^{-1} \text{ is an invertible subsheaf of } \mathcal{H} \text{ s. t. } \mathcal{F}\overset{\text{num}}{\sim}\mathcal{G}\otimes\mathcal{A} \tag{4}$$

In particular $\deg\mathcal{G}_{|C}=p_a(C)-1$ and $\deg\mathcal{F}_{|C}\geq p_a(C)+2$, while

$$\text{For every subcurve } B\subseteq C \begin{cases} \deg\mathcal{G}_{|B}\geq p_a(B) \\ \deg\mathcal{F}_{|B}\geq p_a(B)+\deg(\mathcal{A}_{|B}) \end{cases}$$

Hence by theorems 3, 4 and Lemma 13, for a general Δ we may assume $H^1(\mathcal{F})=0$, $|\mathcal{F}|$ b.p.f. and $H^1(B,\mathcal{G})=0$ for every proper subcurve $B\subset C$.

By Mumford's argument the theorem will follow if we show that $\mathcal{F}$ and $\mathcal{H}$ satisfy the hypotheses of the H^0-Lemma (Prop. 8).

Now we are able to consider the exceptional configurations.

Configuration (a). $C=n\Gamma,\ n\geq 3,\ \Gamma^2$ even.

We will consider firstly the case $n=3$ and then we will apply induction for $n\geq 4$.

Assume that $C=3\Gamma$, Γ^2 even. Let us consider $\mathcal{F}=\mathcal{H}(-\Delta),\mathcal{G}=\mathcal{O}_C(\Delta)$. For a general choice of Δ, $|\mathcal{F}|$ is a base point free system of degree $p_a(C)+2$ and projective dimension 2 such that $|\mathcal{F}|\hookrightarrow|\mathcal{F}_{|\Gamma}|$. Furthermore $(\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})\overset{\text{num}}{\sim}\mathcal{G}$ i.e., $h^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})=1$ by Lemma 15. Thus H^0-lemma holds and we can conclude .

Now let us consider the case $C=n\Gamma,\ n\geq 4,\ \Gamma^2$ even.

Here we can proceed by induction taking the splitting $C=C'+C''$, where $C'=3\Gamma$ and $C''=(n-3)\Gamma$. Let us consider the following commutative diagram:

$$\begin{array}{ccccc}
H^0(C',\omega_{C'}\otimes\mathcal{A})\otimes H^0(\mathcal{H}) & \hookrightarrow & H^0(\mathcal{H})\otimes H^0(\mathcal{H}) & \twoheadrightarrow & H^0(C'',\mathcal{H})\otimes H^0(\mathcal{H}) \\
t_1\downarrow & & \rho_2\downarrow & & t_3\downarrow \\
H^0(C',\omega_{C'}\otimes\mathcal{A}\otimes\mathcal{H}) & \hookrightarrow & H^0(C,\mathcal{H}^{\otimes 2}) & \twoheadrightarrow & H^0(C'',\mathcal{H}^{\otimes 2})
\end{array}$$

(where $H^0(\mathcal{H})=H^0(C,\mathcal{H})$).

On C'' we can apply the standard argument since we are in Case 1 discussed above, and then we get the surjectivity of t_3. On $C'=3\Gamma$, we have the multiplication of two adjoint divisors $\omega_{C'}\otimes\mathcal{A}$, with $\deg\mathcal{A}=3$ and $\mathcal{H}\cong\omega_{C'}\otimes\mathcal{A}'$ with $\deg\mathcal{A}'\geq 6$, and then we can proceed as in the previous cases applying the H^0-lemma.

Configuration (b). $C=2\Gamma_1+\Gamma_2,\Gamma_1\cdot\Gamma_2=1$.

Let $\mathcal{F}=\mathcal{H}(-\Delta),\mathcal{G}=\mathcal{O}(\Delta)$ and let us consider $C_{red}=\Gamma_1+\Gamma_2$. Notice that $\deg\mathcal{F}_{|\Gamma_i}=p_a(\Gamma_i)+1$ and $\deg\mathcal{G}_{|\Gamma_1}=p_a(\Gamma_i)$ for $i=1,2$.

We claim that $\mathcal{F}$ and $\mathcal{H}$ satisfy (iii) of the Proposition 8. Indeed for a general Δ we have $h^0(C,\mathcal{F})=3$ and by 2-connectedness we may assume $h^0(\Gamma_1,\mathcal{F}\otimes\mathcal{O}_{\Gamma_1}(-C_{red}))=0$, i.e., $H^0(C,\mathcal{F})\hookrightarrow H^0(C_{red},\mathcal{F})$. Moreover, $(\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})^{\otimes 2}\overset{\text{num}}{\sim}\mathcal{G}^{\otimes 2}\overset{\text{num}}{\sim}\omega_C$, thus by Lemma 13 each non-zero section of $H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$ does not vanish identically on any component of C_{red}.

To apply Proposition 8 we need an estimate for $h^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F})$. This follows since $\deg(\omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F})_{|\Gamma_2} = p_a(\Gamma_2)$, and furthermore $\omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F} \overset{\text{num}}{\sim} \mathcal{G}$: thus by Lemma 13, for a general Δ, $H^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F}) \hookrightarrow H^0(\Gamma_2, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F})$ and we may assume this vector space being of dimension 1.

Configuration (c). For all irreducible $\Gamma \subset C$, $\Gamma \cdot (C - \Gamma) = 2$.

Let $\mathcal{G} = \mathcal{O}_C(\Delta)$ and $\mathcal{F} = \mathcal{H}(-\Delta)$. Notice that for all irreducible $\Gamma \subset C$, we have $\deg \mathcal{F}_{|\Gamma} = p_a(\Gamma) + 1$ and $\deg \mathcal{G}_{|\Gamma} = p_a(\Gamma)$.

If C is reduced $|\mathcal{F}|$ is a base point free system of projective dimension $r \geq 2$ (since by numerically conditions $\deg \mathcal{A} \geq 3$) while $\omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F} \overset{\text{num}}{\sim} \mathcal{G}$ satisfies Lemma 13. To conclude we only need $h^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F}) \leq h^0(C, \mathcal{F}) - 2$. This can be easily seen since, as above, for a general Δ we have $H^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F}) \hookrightarrow H^0(\Gamma, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F}) \cong \mathbb{C}$.

Thus H^0-lemma holds and we can conclude .

Now let us assume C non reduced. By the above analyses we may assume that C_{red} has at least 3 component.

Let us consider the splitting of Lemma 14, $C = C_{red} + B$ where $B = \sum_{i=1}^{k} \Gamma_i$ is a chain of reduced curve such that

$$C_{red} \cdot \Gamma_1 = C_{red} \cdot \Gamma_k = 1 \ , \qquad C_{red} \cdot \Gamma_i = 0 \quad \forall \ \ i = 2, \ldots, k-1.$$

Now, by degree consideration (since we have $\deg \mathcal{F}_{|\Gamma_i} = p_a(\Gamma_i) + 1$,) for $\mathcal{F}$ sufficiently general, we may assume $h^1(B, \mathcal{O}_B(\mathcal{F}) \otimes \mathcal{O}_B(-C_{red})) = 0$ i.e., the exact sequence

$$0 \to \mathcal{O}_B(\mathcal{F}) \otimes \mathcal{O}_B(-C_{red}) \to \ \mathcal{F} \to \mathcal{O}_{C_{red}}(\mathcal{F}) \to 0$$

is exact on global sections.

This means that we can pick a subspace $W \subseteq H^0(C, \mathcal{F})$ such that W is isomorphic to $H^0(C_{red}, \mathcal{F})$ (hence of dimension $r + 1 \geq 3$) and W yields a base point free subsystem of $|\mathcal{F}|$ (since it is base point free on C_{red}).

Moreover, $(\omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F})^{\otimes 2} \overset{\text{num}}{\sim} \omega_C$, i.e. $\deg(\omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F})_{|\Gamma_i} = p_a(\Gamma_i)$ for every i. Thus by the particular configuration of C_{red} (it is a chain of curves), applying an induction argument on the number of components we get $h^0(C_{red}, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F}) = h^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F}) = 1$.

Finally, by Lemma 13 every section of $H^0(C, \omega_C \otimes \mathcal{H}^{-1} \otimes \mathcal{F})$ does not vanish on any component of C_{red}.

Thus H^0-Lemma applies and then the theorem follows.

Q.E.D. for Theorem 1

6 Projectively normal tricanonical embeddings of algebraic surfaces

In this section we prove that the image of the tricanonical embedding of a surface of general type is projectively normal.

The arguments we adopt to show the projective normality are very classical and based on the restriction of a curve in the canonical system. The only novelty is that now we do not make any requests on such a curve (i.e. we allow the curve to be singular and with many components) since we can apply Thm. 1.

Proof of Thm. 2. The proof follows the ideas and the arguments illustrated in [3, §2].

By Hypotheses we can take a section $s \in H^0(S, K_S)$. Let $C \in |K_S|$ be the (numerically connected) curve defined by $(s) = 0$.

To simplify the notations we write $\mathcal{H} = K_S^{\otimes 3}$.

Whence, by adjunction and our numerical assumptions we have $(\mathcal{H})_{|C} \cong \omega_C \otimes \mathcal{A}$, with $\mathcal{A}$ ample of degree $= (K_S)^2 \geq 3$. Thus C and $(\mathcal{H})_{|C}$ satisfy the hypotheses of Theorem 1 and then for every integer k the map

$$\overline{\rho}_k : H^0(C, \mathcal{H}) \otimes H^0(C, \mathcal{H}^{\otimes(k-1)}) \longrightarrow H^0(C, \mathcal{H}^{\otimes k})$$

is surjective. The theorem now follows by the following commutative diagram (where $\mathcal{H} = K_S^{\otimes 3}$)

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^0(S, K_S^{\otimes 2}) \otimes H^0(S, \mathcal{H}^{\otimes(k-1)}) \arrow[r, "A_k"] \arrow[d, "C_k"] & H^0(S, K_S^{\otimes(3k-1)}) \arrow[d, "c_k"] \\
H^0(S, \mathcal{H}) \otimes H^0(S, \mathcal{H}^{\otimes(k-1)}) \arrow[r, "\rho_k"] \arrow[d, "R_k"] & H^0(S, \mathcal{H}^{\otimes k}) \arrow[d, "r_k"] \\
H^0(C, \mathcal{H}) \otimes H^0(S, \mathcal{H}^{\otimes(k-1)}) \arrow[r, "\overline{\rho}_k{}'"] \arrow[d] & H^0(C, \mathcal{H}^{\otimes k}) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Indeed, for $k = 2$ the map A_k is surjective by [3, Lemma 3.3] since we have assumed $p_g \geq 2, q = 0$; for $k \geq 3$ the map A_k is surjective by the part (a) of the proof of Thm. 4.7 in [3]. Moreover, for every integer k, the map $\overline{\rho}_k{}'$ is onto since $\overline{\rho}_k$ and the retriction $H^0(S, \mathcal{H}^{\otimes(k-1)}) \to H^0(C, \mathcal{H}^{\otimes(k-1)})$ are both surjective by our numerical assumptions.

Whence for every integer k the multiplication map ρ_k is surjective, which is what we wanted to prove.

Q.E.D. for Theorem 2

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