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On Clifford's theorem for singular curves

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Abstract

Let C be a 2-connected projective curve either reduced with planar singularities or contained in a smooth algebraic surface and let S be a subcanonical cluster (that is, a zero-dimensional scheme such that the space $H^0(C, \mathscr{I}_S K_C)$ contains a generically invertible section). Under some general assumptions on S or C, we show that $h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \frac{1}{2}\deg(S)$ and if equality holds then either S is trivial or C is honestly hyperelliptic or 3-disconnected.

As a corollary, we give a generalization of Clifford's theorem for reduced curves with planar singularities.

1. *Introduction*

Since the early days of algebraic geometry, the rule of residual series has been fundamental in studying the geometry of a projective variety. The first results of the German school (Riemann, Roch, Brill, Noether, Klein, etc.) on special divisors were indeed based on the deep analysis of a linear series $|D|$ and its residual $|K - D|$.

The purpose of this paper is to extend this basic approach to the analysis of special linear series defined on an algebraic curve (possibly singular, non-reduced or reducible), giving applications to the case of semistable curves.

In this paper, in particular, we generalize the Theorem of Clifford, which states that

$$\dim |D| \leqslant \frac{\deg D}{2}$$

for every special effective divisor D on a smooth curve C (see [**7**]).

One can find in the literature many approaches that generalize Clifford's theorem and other classical results to certain kinds of singular curves, especially nodal ones. Important results were given by Eisenbud and Harris (see [**8**] and the appendix in [**9**]) and more recently by Esteves and Medeiros (see [**10**]), applying essentially degeneration techniques, in the case of reduced curves with two components. See also the case of graph curves by Bayer and Eisenbud [**2**]. Caporaso [**3**] gave a generalization of Clifford's theorem for certain line bundles on stable curves, in particular, line bundles of degree at most four and line bundles whose degree is bounded by $2p_a(\Gamma_i)$ for every component Γ_i.

Our approach is more general since we deal with rank 1 torsion-free sheaves on possibly reducible and non-reduced curves, without any bound on the number of components, but with very natural assumptions on the multidegree of the sheaves we consider.

Our analysis focuses on 2-connected curves, keeping in mind the classical characterization of special divisors on algebraic curves as effective divisors contained in the canonical system. To achieve this purpose, we introduce the notion of *subcanonical cluster*, that is, a zero-dimensional subscheme $S \subset C$ such that the space $H^0(C, \mathscr{I}_S K_C)$ contains a generically invertible section (see Section 2.3 for definition and main properties).

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We recall that a curve C is *m-connected* if $\deg_B K_C \geqslant m + (2p_a(B) - 2)$ for every subcurve $B \subset C$, or equivalently $B \cdot (C - B) \geqslant m$ if C is contained in a smooth surface.

From our point of view, it is fundamental to work only with subcanonical clusters since our aim is to consider only clusters truly contained in a canonical divisor. Moreover, we need to avoid clusters contained in a hyperplane canonical section but with uncontrolled behavior. For instance, by automatic adjunction (see [**5**, Lemma 2.4]) a section vanishing on a component A such that $C = A + B$ yields a section in $H^0(B, K_B)$, but considering the embedding $H^0(B, K_B) \hookrightarrow H^0(C, K_C)$, we can build clusters with unbounded degree on A such that every section in $H^0(C, K_C)$ vanishing on them vanishes on the entire subcurve A.

Our main result is the following theorem.

THEOREM A. *Let C be a projective curve either reduced with planar singularities or contained in a smooth algebraic surface. Assume C to be 2-connected and let $S \subset C$ be a subcanonical cluster. Assume that one of the following holds:*

(a) *S is a Cartier divisor;*
(b) *there exists $H \in H^0(C, \mathscr{I}_S K_C)$ such that* $\operatorname{div}(H) \cap \operatorname{Sing}(C_{\mathrm{red}}) = \emptyset$;
(c) *C_{red} is 4-connected.*

Then

$$h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \tfrac{1}{2}\deg(S).$$

Moreover, if equality holds, then the pair (S, C) satisfies one of the following assumptions:

(i) *$S = 0$, K_C;*
(ii) *C is honestly hyperelliptic and S is a multiple of the honest g^1_2;*
(iii) *C is 3-disconnected (that is, there is a decomposition $C = A + B$ with $A \cdot B = 2$).*

Let $\operatorname{Cliff}(\mathscr{I}_S K_C) := 2p_a(C) - \deg(S) - 2 \cdot h^0(\mathscr{I}_S K_C)$ be the Clifford index of the sheaf $\mathscr{I}_S K_C$. Note that if S is a Cartier divisor, then $\operatorname{Cliff}(\mathscr{I}_S K_C)$ is precisely the classical Clifford index for invertible sheaves. Theorem A is equivalent to the statement that the Clifford index is non-negative.

If C is a smooth curve, the theorem is equivalent to the classical *Clifford's theorem*, while if C is 1-connected but 2-disconnected, then $|K_C|$ has base points and, therefore, the cluster consisting of such base points does not satisfy the theorem. Moreover, without our assumptions the theorem is false even for subcanonical clusters contained in curves with very ample canonical sheaf. See for instance Example 5.2. However, we obtain a more general inequality by adding a correction term bounded by half of the number of irreducible components of C. See Theorem 3.8 for the full result.

The proof is based on the analysis of a cluster S of minimal Clifford index and maximal degree and of its residual S^* (see Subsection 2.3 for definitions and main properties). When considering the restriction to C_{red}, it may happen that every section in $H^0(C, \mathscr{I}_S K_C)$ decomposes as a sum of sections with small support. This behavior is completely new with respect to the smooth case and can even lead to the existence of clusters with negative Clifford index. This is the reason why in Subsection 2.3, we introduce the notion of splitting index of a cluster and we run our analysis by a stratification of the set of subcanonical clusters by their splitting index.

For clusters in each strata with minimal Clifford index, the following dichotomy holds: either $S^* \subset S$ or S and S^* are Cartier and disjoint. In the first case, we estimate the rank of the restriction of $H^0(C, \mathscr{I}_S K_C)$ to the curve supporting S, while in the second case we give a generalization of the classical techniques developed by Saint Donat [**16**].

As a corollary of Theorem A, we are able to analyze more deeply the case of reduced curves since the intersection products are always non-negative. The following results apply in particular to the case of 4-connected semistable curves.

Theorem B. *Let C be a projective 4-connected reduced curve with planar singularities. Let L be an invertible sheaf and S a cluster on C. Assume that*

$$0 \leqslant \deg[(\mathscr{I}_S L)_{|B}] \leqslant \deg K_{C|B}$$

for every subcurve $B \subset C$. Then

$$h^0(C, \mathscr{I}_S L) \leqslant \frac{\deg \mathscr{I}_S L}{2} + 1.$$

Moreover, if equality holds, then $\mathscr{I}_S L \cong \mathscr{I}_T \omega_C$ where T is a subcanonical cluster. The pair (T, C) satisfies one of the following assumptions:

(i) *$T = 0$, K_C;*
(ii) *C is honestly hyperelliptic and T is a multiple of the honest g^1_2.*

In the case of smooth curves, an effective divisor D either satisfies the assumptions of Clifford's theorem or it is non-special and $h^0(C, D)$ is computed easily by means of Riemann–Roch Theorem. If the curve C has many components, we may have a mixed behavior, which we deal with in the following theorem.

Theorem C. *Let C be a projective 4-connected reduced curve with planar singularities. Let L be an invertible sheaf and S a cluster on C such that*

$$0 \leqslant \deg[(\mathscr{I}_S L)_{|B}] \quad \text{for every subcurve } B \subset C.$$

Assume that there exists a subcurve $\Gamma \subset C$ such that $\deg(K_{C|\Gamma}) < \deg(\mathscr{I}_S L_{|\Gamma})$. and let C_0 be the maximal subcurve such that

$$\deg[(\mathscr{I}_S L)_{|B}] > \deg K_{C|B} \quad \text{for every subcurve } B \subset C_0.$$

Then

$$h^0(C, \mathscr{I}_S L) \leqslant \frac{\deg \mathscr{I}_S L}{2} + \frac{\deg(\mathscr{I}_S L - K_C)_{|C_0}}{2}.$$

We believe that the above results may be useful for the study of vector bundles on the compactification of the Moduli Space of genus g curves and in particular to the analysis of limit series. Moreover, they may be considered as a first step in order to develop a Brill–Noether-type analysis for semistable curves. Further applications will be given in a forthcoming article (see [**12**]) in which we analyze the normal generation of invertible sheaves on numerically connected curve. In particular, we are going to give a generalization of Noether's theorem.

Finally, as shown in [**5**], the study of invertible sheaves on curves lying on a smooth algebraic surface is rich in implications when Bertini's theorem does not hold or simply if one needs to consider every curve contained in a given linear system.

The paper is organized as follows. In Section 2, we set the notation and prove some preliminary results. In Section 3, we prove Theorem A, in Section 4 we study the case of reduced curves and prove Theorems B and C. Finally, in Section 5, we show some examples in which we illustrate that the Clifford index may be negative if our assumptions are not satisfied.

2. *Notation and preliminary results*

2.1. *Notation and conventions*

We work over an algebraically closed field $\mathbb{K}$ of characteristic $\geqslant 0$.

Throughout this paper a curve C will always be a Cohen–Macaulay scheme of pure dimension 1. Moreover, if not otherwise stated, a curve C will be *projective, either reduced with planar singularities* (that is, such that for every point $P \in C$, it is $\dim_{\mathbb{K}} \mathscr{M}/\mathscr{M}^2 \leqslant 2$, where $\mathscr{M}$ is the maximal ideal of $\mathscr{O}_{C,P}$) *or contained in a smooth algebraic surface* X, in which case we allow C to be reducible and non-reduced.

In both cases we will use the standard notation for curves lying on smooth algebraic surface, writing $C = \sum_{i=1}^{s} n_i \Gamma_i$, where Γ_i are the irreducible components of C and n_i are their multiplicities.

A subcurve $B \subseteq C$ *is a Cohen–Macaulay subscheme of pure dimension 1*; it will be written as $\sum m_i \Gamma_i$, with $0 \leqslant m_i \leqslant n_i$ for every i.

Given a sheaf $\mathscr{F}$ on C, we write $H^0(B, \mathscr{F})$ for $H^0(B, \mathscr{F}_{|B})$ and $H^0(C, \mathscr{F})_{|B}$ for the image of the restriction map $H^0(C, \mathscr{F}) \to H^0(B, \mathscr{F}_{|B})$.

ω_C denotes the dualizing sheaf of C (see [**13**, Chapter III, Section 7]) and $p_a(C)$ the arithmetic genus of C, $p_a(C) = 1 - \chi(\mathscr{O}_C)$. K_C denotes the canonical divisor.

By abuse of notation if $B \subset C$ is a subcurve of C, $C - B$ denotes the curve A such that $C = A + B$.

Note that under our assumptions every subcurve $B \subseteq C$ is *Gorenstein*, which is equivalent to say that ω_B is an invertible sheaf.

Throughout the paper, we will use the following exact sequences:

$$0 \longrightarrow \omega_A \longrightarrow \omega_C \longrightarrow \omega_{C|B} \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow \mathscr{O}_A(-B) \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_B \longrightarrow 0, \tag{2}$$

where $\mathscr{O}_A(-B) \cong \mathscr{O}_A \otimes \mathscr{O}_X(-B)$ if C is contained in a smooth surface X and corresponds to $\mathscr{I}_{A \cap B} \cdot \mathscr{O}_A$ if C is reduced; see [**1**, Proposition II.6.4, **15**, Chapter 3].

Definition 2.1. If A and B are subcurves of $C_0 \subseteq C$ such that $A + B = C_0 \subseteq C$, then their *intersection product* is

$$A \cdot B = \deg_B(K_{C_0}) - (2p_a(B) - 2) = \deg_A(K_{C_0}) - (2p_a(A) - 2).$$

If C is contained in a smooth algebraic surface X, this corresponds to the intersection product of curves as divisors on X.

If $A + B = C_0 \subseteq C$, we have the key formula (cf. [**13**, Exercise V.1.3])

$$p_a(C_0) = p_a(A) + p_a(B) + A \cdot B - 1. \tag{3}$$

Following the original definition of Franchetta, a curve C is (*numerically*) *m-connected* if $C_1 \cdot C_2 \geqslant m$ for every decomposition $C = C_1 + C_2$ in effective, both non-zero curves. See [**5**] for a more general definition in the case of Gorenstein curve. To avoid ambiguity between the various notions of connectedness for a curve, we will say that a curve is *numerically connected* if it is 1-connected, and *topologically connected* if it is connected as a topological space (with the Zariski topology).

Let $\mathscr{F}$ be a rank 1 torsion-free sheaf on C. We write $\deg \mathscr{F}_{|C}$ for the degree of $\mathscr{F}$ on C, $\deg \mathscr{F}_{|C} = \chi(\mathscr{F}) - \chi(\mathscr{O}_C)$. By Serre duality we mean the Grothendieck–Serre–Riemann–Roch duality theorem:

$$H^1(C, \mathscr{F}) \overset{\mathrm{d}}{=} \mathrm{Hom}(\mathscr{F}, \omega_C)$$

(where $\overset{\mathrm{d}}{=}$ denotes duality of vector spaces).

If $C = \sum n_i \Gamma_i$, then for each i the natural inclusion map $\epsilon_i : \Gamma_i \to C$ induces a map $\epsilon_i^* : \mathscr{F} \to \mathscr{F}_{|\Gamma_i}$. We denote by $d_i = \deg(\mathscr{F}_{|\Gamma_i}) = \deg_{\Gamma_i} \mathscr{F}$ the degree of $\mathscr{F}$ on each irreducible component and by $\mathbf{d} := (d_1, \ldots, d_s)$ the *multidegree of* $\mathscr{F}$ *on* C. If B is a subcurve of C, by $\mathbf{d}_B$ we mean the multidegree of $\mathscr{F}_{|B}$. We remark that there exists a natural partial ordering given by the multidegree.

$\mathscr{F}$ is numerically eventually free (NEF) if $d_i \geqslant 0$ for every i.

We say that two rank 1 torsion-free sheaves $\mathscr{F}$ and $\mathscr{G}$ are numerically equivalent if their degrees coincide on every subcurve and we will use the notation $\mathscr{F} \overset{\mathrm{num}}{\sim} \mathscr{G}$.

If S and S_1 are linearly equivalent Cartier divisor, we will use the notation $S \overset{\mathrm{lin}}{\sim} S_1$.

Definition 2.2. A curve C is *honestly hyperelliptic* if there exists a finite morphism $\psi\colon C \to \mathbb{P}^1$ of degree 2.

In this case C is either irreducible or of the form $C = \Gamma_1 + \Gamma_2$ with $p_a(\Gamma_i) = 0$ and $\Gamma_1 \cdot \Gamma_2 = p_a(C) + 1$ (see [**5**, Section 3] for a detailed treatment). For a given point $P \in \mathbb{P}^1$, $\psi^*(P)$ is a cluster of degree 2, which we will denote by a *honest* g_2^1.

Definition 2.3. A *cluster* Z of *degree* $\deg Z = r$ is a zero-dimensional subscheme with length $\mathscr{O}_Z = \dim_k \mathscr{O}_Z = r$. The multidegree of Z is defined as the opposite of the multidegree of $\mathscr{I}_Z$. We consider the empty set as the degree 0 cluster.

Definition 2.4. The Clifford index of a rank 1 torsion-free sheaf $\mathscr{F}$ on C is

$$\mathrm{Cliff}(\mathscr{F}) := \deg(\mathscr{F}) - 2h^0(C, \mathscr{F}) + 2.$$

If S is a cluster and $\mathscr{F} \cong \mathscr{I}_S K_C$, then the Clifford index of S may be defined as the Clifford index of $\mathscr{I}_S K_C$ and reads as follows:

$$\mathrm{Cliff}(\mathscr{I}_S K_C) := 2p_a(C) - \deg(S) - 2 \cdot h^0(\mathscr{I}_S K_C).$$

If $\mathscr{F}$ is an invertible sheaf (in particular, if S is a Cartier divisor), then $\mathrm{Cliff}(\mathscr{F})$, resp. $\mathrm{Cliff}(\mathscr{I}_S K_C)$ is precisely the classical Clifford index of the line bundle $\mathscr{F}$, resp. $\mathscr{I}_S K_C$.

2.2. *Preliminary results on projective curves*

In this section, we recall some useful results on invertible sheaves on projective curves.

In the following theorem, we summarize the main applications of the results proved in [**5**] on Cohen–Macaulay one-dimensional projective schemes. For a general treatment: see [**5**, Sections 2 and 3].

Theorem 2.5. *Let C be a Gorenstein curve, K_C the canonical divisor of C. Then the following properties hold.*

(i) *If C is 1-connected, then $H^1(C, K_C) \cong \mathbb{K}$.*
(ii) *If C is 2-connected and $C \ncong \mathbb{P}^1$, then $|K_C|$ is base point free.*
(iii) *If C is 3-connected and C is not honestly hyperelliptic (that is, there does not exist a finite morphism $\psi\colon C \to \mathbb{P}^1$ of degree 2), then K_C is very ample.*

(cf. [**5**, Theorems 1.1, 3.3, 3.6]).

The main instrument in the analysis of sheaves on projective curves with several components is the following proposition, which holds in a more general setup.

Proposition 2.6 [**5**, Lemma 2.4]. *Let C be a projective scheme of pure dimension 1 and let $\mathscr{F}$ be a coherent sheaf on C, and $\varphi\colon \mathscr{F} \to \omega_C$ a non-vanishing map of $\mathscr{O}_C$-modules. Set $\mathscr{J} = \operatorname{Ann}\varphi \subset \mathscr{O}_C$, and write $B \subset C$ for the subscheme defined by $\mathscr{J}$. Then B is Cohen–Macaulay and φ has a canonical factorization of the form*

$$\mathscr{F} \twoheadrightarrow \mathscr{F}_{|B} \hookrightarrow \omega_B = \mathscr{H}om_{\mathscr{O}_C}(\mathscr{O}_B, \omega_C) \subset \omega_C,$$

where $\mathscr{F}_{|B} \hookrightarrow \omega_B$ is generically onto.

A useful corollary of the above result is the following:

Corollary 2.7. *Let C be a pure one-dimensional projective scheme, let $\mathscr{F}$ be a rank 1 torsion-free sheaf on C. Assume that*

$$\deg(\mathscr{F})_{|B} \geqslant 2p_a(B) - 1$$

for every subcurve $B \subseteq C$.

Then $H^1(C, \mathscr{F}) = 0$.

Proof. The proof is a slight generalization of the techniques used in [**4**, Lemma 2.1].

Let us assume by contradiction that $H^1(C, \mathscr{F}) \neq 0$. Pick a non-vanishing section $\varphi \in \operatorname{Hom}(\mathscr{F}, \omega_C) \cong H^1(C, \mathscr{F})^*$. By Proposition 2.6, there exists a curve B such that φ induces an injective map $\mathscr{F}_{|B} \to \omega_B$. Thus,

$$\deg(\mathscr{F})_{|B} \leqslant \deg K_B = 2p_a(B) - 2,$$

which is impossible. □

During our analysis of the curve C we will need to estimate the dimension of $H^0(A, \mathscr{O}_A)$ for some subcurve $A \subset C$. To this purpose, we give a slight generalization of a result of Konno and Mendes Lopes (see [**14**, Lemma 1.4]).

Lemma 2.8. *Let C be a projective curve, either reduced with planar singularities or contained in a smooth algebraic surface and let $C = A + B$ a decomposition of C. Assume $A = \sum_{i=1}^h A_i$, where the A_i are the topologically connected components of A.*

(i) *If C is 1-connected, then $h^0(A, \mathscr{O}_A) \leqslant A \cdot B$.*
(ii) *If C is 2-connected, then $h^0(A, \mathscr{O}_A) \leqslant (A \cdot B)/2$.*
(iii) *If C is m-connected with $m \geqslant 3$, then $h^0(A, \mathscr{O}_A) \leqslant (A \cdot B)/2 - h \cdot (m-2)/2$, where $h = \#\{A_i\}$. Moreover, equality holds if and only if $h^0(A_i, \mathscr{O}_{A_i}) = 1$ and $A_i \cdot B = m$ for every component A_i.*

Proof. The 1-connected case is treated in [**14**, Lemma 1.4]. We will apply the same arguments for the m-connected case with $m \geqslant 2$.

If $h^0(A, \mathscr{O}_A) = 1$, the inequality holds trivially. If $h^0(A, \mathscr{O}_A) \geqslant 2$, then by [**14**, Lemma 1.2] there exist a decomposition $A = A_1 + A_2$ with $\mathscr{O}_{A_1}(-A_2)$ NEF and such that the restriction map $H^0(\mathscr{O}_{A_1}(-A_2)) \to H^0(\Gamma, \mathscr{O}_\Gamma(-A_2))$ is injective for every irreducible $\Gamma \subset A_1$. Since $\mathscr{O}_{A_1}(-A_2)$ is NEF, we can conclude that

$$h^0(\mathscr{O}_{A_1}(-A_2)) \leqslant h^0(\Gamma, \mathscr{O}_\Gamma(-A_2)) \leqslant 1 - A_2 \cdot \Gamma \leqslant 1 - A_1 \cdot A_2.$$

Therefore by induction on the number of irreducible components of A we get

$$\begin{aligned} h^0(A, \mathscr{O}_A) &\leqslant h^0(A_2, \mathscr{O}_{A_2}) + h^0(\mathscr{O}_{A_1}(-A_2)) \\ &\leqslant \frac{A_2 \cdot (C - A_2)}{2} - \frac{m-2}{2} + 1 - A_1 \cdot A_2 \\ &= \frac{A \cdot (C - A)}{2} - \frac{m-2}{2} + 1 - \frac{A_1 \cdot (C - A_1)}{2} \\ &\leqslant \frac{A \cdot (C - A)}{2} - \frac{m-2}{2} + 1 - \frac{m}{2}. \end{aligned}$$

This is enough to prove (ii). Applying the above dimension count to every topologically connected component of A, we get the inequality stated in (iii). Moreover, if $m \geqslant 3$ and $h^0(A_i, \mathscr{O}_{A_i}) \geqslant 2$ for a topologically connected component $A_i \subset A$, then by the above computation we have $h^0(A_i, \mathscr{O}_{A_i}) < (A_i \cdot B)/2 - (m-2)/2$. Therefore, equality holds if and only if for every A_i we have $h^0(A_i, \mathscr{O}_{A_i}) = 1$ and $A_i \cdot (C - A_i) = A_i \cdot B = m$. □

2.3. *Subcanonical clusters and Clifford index*

In this section, we introduce the notion of subcanonical cluster and we analyze its main properties. Note that our results work under the assumption C Gorenstein.

DEFINITION 2.9. Let C be a Gorenstein curve. A cluster $S \subset C$ is *subcanonical* if the space $H^0(C, \mathscr{I}_S K_C)$ contains a generically invertible section, that is, a section s_0 that does not vanish on any subcurve of C.

Note that if S is a general effective Cartier divisor such that the inequality $\deg_B(S) \leqslant \frac{1}{2}\deg_B(\omega_C)$ holds for every subcurve $B \subseteq C$ (or by duality such that its multidegree satisfies $\frac{1}{2}\deg_B(\omega_C) \leqslant \deg_B(S) \leqslant \deg_B(\omega_C)$ for every subcurve $B \subseteq C$), then by Franciosi [**11**, Section 3] S is a subcanonical cluster.

DEFINITION 2.10. Let C be a Gorenstein curve, $S \subset C$ be a subcanonical cluster and let $s_0 \in H^0(C, \mathscr{I}_S K_C)$ be a generically invertible section. The residual cluster S^* of S with respect to s_0 is defined by the following exact sequence

$$0 \longrightarrow \mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C) \xrightarrow{\alpha} \mathscr{H}om(\mathscr{O}_C, \omega_C) \longrightarrow \mathscr{O}_{S^*} \longrightarrow 0,$$

where the map α is defined by $\alpha(\varphi) : 1 \mapsto \varphi(s_0)$.

By duality it is $\mathscr{I}_{S^*}\omega_C \cong \mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C)$. Moreover, denoting by $\Lambda := \operatorname{div}(s_0)$ the effective divisor corresponding to s_0, we have the following exact sequence:

$$0 \longrightarrow \mathscr{I}_\Lambda\omega_C \longrightarrow \mathscr{I}_S\omega_C \longrightarrow \mathscr{O}_{S^*} \longrightarrow 0.$$

Therefore, S^* is subcanonical since $s_0 \in H^0(C, \mathscr{I}_{S^*} K_C)$ and it is straightforward to see that $(S^*)^* = S$.

Note that if C is contained in a smooth surface and s_0 is transverse to C at a point $P \in \operatorname{supp}(S)$ such that P is smooth for C_{red} and C has multiplicity n at P, writing $\mathscr{I}_\Lambda = (x)$ and $\mathscr{I}_S = (x, y^k) \subset \mathbb{K}[x, y]/(x, y^n)$, then $\mathscr{I}_{S^*} \cong (x, y^{n-k})$.

Remark 2.11. If S is a subcanonical cluster and S^* is its residual with respect to the section s_0, then the sheaf $\mathscr{I}_{S^*}\omega_C$ is the subsheaf of ω_C given as follows:

$$\mathscr{I}_{S^*}\omega_C = \{\varphi(s_0) \text{ s.t. } \varphi \in \mathscr{H}\mathrm{om}(\mathscr{I}_S\omega_C, \omega_C)\}. \tag{4}$$

This is clear from the analysis of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{H}\mathrm{om}(\mathscr{I}_S\omega_C, \omega_C) & \xrightarrow{\alpha} & \mathscr{H}\mathrm{om}(\mathscr{O}_C, \omega_C) & \longrightarrow & \mathscr{O}_{S^*} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \| & & \\ 0 & \longrightarrow & \mathscr{I}_{S^*}\omega_C & \longrightarrow & \omega_C & \longrightarrow & \mathscr{O}_{S^*} & \longrightarrow & 0 \end{array}$$

where the map α is defined by $\alpha(\varphi) : 1 \mapsto \varphi(s_0)$ and the maps β_1 and β_2 are isomorphisms.

Remark 2.12. The product map $H^0(C, \mathscr{I}_S K_C) \otimes H^0(C, \mathscr{I}_{S^*} K_C) \to H^0(C, 2K_C)$ satisfies the following commutative diagram:

$$\begin{array}{ccc} H^0(C, \mathscr{I}_S K_C) \otimes \mathrm{Hom}(\mathscr{I}_S K_C, K_C) & \xrightarrow{\mathrm{ev}} & H^0(C, K_C) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \cdot s_0} \\ H^0(C, \mathscr{I}_S K_C) \otimes H^0(C, \mathscr{I}_{S^*} K_C) & \longrightarrow & H^0(C, 2K_C) \end{array}$$

where the first row is the evaluation map $i \otimes \varphi \mapsto \varphi(i)$, the map β is the isomorphism defined by $\beta(i \otimes \varphi) = i \otimes \varphi(s_0)$, and the second column is the multiplication by the section s_0 defining the residual S^*.

The diagram is commutative: on the stalks the elements $s_0 \cdot \varphi(i)$ and $i \cdot \varphi(s_0)$ must coincide. In particular, consider $i \in H^0(C, \mathscr{I}_S K_C)$ and $j \in H^0(C, \mathscr{I}_{S^*} K_C)$: we can write $j = \varphi(s_0)$ for some $\varphi \in \mathrm{Hom}(\mathscr{I}_S K_C, K_C)$, from which we have

$$i \cdot j = i \cdot \varphi(s_0) = s_0 \cdot \varphi(i) \quad \text{in } H^0(C, 2K_C). \tag{5}$$

Remark 2.13. Note that, by Serre duality, it is $H^1(C, \mathscr{I}_S K_C) \stackrel{\mathrm{d}}{=} H^0(C, \mathscr{I}_{S^*} K_C)$, and $\mathrm{Cliff}(\mathscr{I}_S K_C) = \mathrm{Cliff}(\mathscr{I}_{S^*} K_C)$.

The following technical lemmas will be useful in the proof of Theorem A.

Lemma 2.14. *Let C be a Gorenstein curve. Let S, S^*, T, T^* subcanonical clusters such that*

(i) *S^* is the residual to S with respect to $H_0 \in H^0(C, \mathscr{I}_S K_C)$;*
(ii) *T^* is the residual to S with respect to $H_1 \in H^0(C, \mathscr{I}_S K_C)$;*
(iii) *T is the residual to S^* with respect to $H_2 \in H^0(C, \mathscr{I}_{S^*} K_C)$.*

Then the cluster U defined as the union of T and T^* (that is, $\mathscr{I}_U = \mathscr{I}_T \cap \mathscr{I}_{T^*}$) and the cluster defined by the intersection $R = T \cap T^*$ (that is, $\mathscr{I}_R = \mathscr{I}_T + \mathscr{I}_{T^*}$) are subcanonical.

Proof. R is obviously subcanonical, since it is contained in the cluster T.

Since $H_1 \in H^0(C, \mathscr{I}_S K_C)$, there exists an element $\varphi_1 \in \mathrm{Hom}(\mathscr{I}_{S^*} K_C, K_C)$ such that $H_1 = \varphi_1(H_0)$ by equation (4). Similarly, there exists $\psi_2 \in \mathrm{Hom}(\mathscr{I}_S K_C, K_C)$ such that $H_2 = \psi_2(H_0)$.

By equation (4), $\psi_2(H_1) \in H^0(C, \mathscr{I}_{T^*} K_C)$ and $\varphi_1(H_2) \in H^0(C, \mathscr{I}_T K_C)$.

By equation (5), we have

$$H_0 \cdot \psi_2(H_1) = H_1 \cdot \psi_2(H_0) = H_1 \cdot H_2 = \varphi_1(H_0) \cdot H_2 = H_0 \cdot \varphi_1(H_2) \tag{6}$$

and since H_0, H_1 and H_2 are generically invertible, we conclude that $\psi_2(H_1) = \varphi_1(H_2)$ in $H^0(C, K_C)$ and it is generically invertible. In particular,

$$\psi_2(H_1) = \varphi_1(H_2) \in H^0(C, \mathscr{I}_T K_C) \cap H^0(C, \mathscr{I}_{T^*} K_C) \subset H^0(C, \mathscr{I}_U K_C)$$

and we may conclude. □

REMARK 2.15. It is not difficult to prove that the clusters T and T^* defined in the previous lemma are reciprocally residual with respect to the section $H_3 = \psi_2(H_1) = \varphi_1(H_2) \in H^0(C, K_C)$. This induces an equivalence relation on the set of clusters with properties similar to the classical linear equivalence relation between divisors.

DEFINITION 2.16. A non-trivial subcanonical cluster S is called *splitting* for the linear system $|K_C|$ if for every $H \in H^0(C, \mathscr{I}_S K_C)$ there exists a decomposition $H = H_1 + H_2$ with H_1, $H_2 \in H^0(C, \mathscr{I}_S K_C)$ and a decomposition $C_{\mathrm{red}} = C_1 + C_2$ such that $\mathrm{supp}(H_{1|C_{\mathrm{red}}}) \subset C_1$ and $\mathrm{supp}(H_{2|C_{\mathrm{red}}}) \subset C_2$.

The *splitting index* of S is the minimal number k such that for every element $H \in H^0(C, \mathscr{I}_S K_C)$ there exists a decomposition $H = \sum_{i=0}^{k} H_i$ with $H_i \in H^0(C, \mathscr{I}_S K_C)$ and a decomposition $C_{\mathrm{red}} = \sum_{i=0}^{k} C_i$ such that $\mathrm{supp}(H_{i|C_{\mathrm{red}}}) \subset C_i$. We define the splitting index of the zero cluster to be zero.

PROPOSITION 2.17. *Let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be a Gorenstein curve and let S be a subcanonical cluster. Then the following properties hold.*

- (i) *If the splitting index of S is k, then there is a decomposition $C_{\mathrm{red}} = \sum_{i=0}^{k} C_i$ such that every $\bar{H} \in H^0(C, \mathscr{I}_S K_C)$ can be decomposed as $\bar{H} = \sum_{i=0}^{k} \bar{H}_i$ with $\mathrm{supp}(\bar{H}_{i|C_{\mathrm{red}}}) \subset C_i$. Moreover, if $\bar{H}$ is generic, then the sections $\bar{H}_i$ cannot be further decomposed.*
- (ii) *Given the above minimal decomposition $C_{\mathrm{red}} = \sum_{i=0}^{k} C_i$, we have that $C_i \cap C_j$ is in the base locus of $|\mathscr{I}_S K_C|$ for every i and j.*
- (iii) *If there exists a section $H \in H^0(C, \mathscr{I}_S K_C)$ such that $\mathrm{div}(H) \cap (\Gamma_i \cap \Gamma_j) = \emptyset$ for every $\Gamma_i \neq \Gamma_j$ irreducible components in C, then the splitting index of S is zero.*

Proof. To prove the first statement, since the possible decompositions of C_{red} are finite, there exists a decomposition $C_{\mathrm{red}} = \sum_{i=0}^{k} C_i$ such that the generic element $\bar{H} \in H^0(C, \mathscr{I}_S K_C)$ decomposes as $\bar{H} = \sum_{i=0}^{k} \bar{H}_i$, $\mathrm{supp}(\bar{H}_{i|C_{\mathrm{red}}}) \subset C_i$. Call Y the set of sections with this property, we are going to show that $Y = H^0(C, \mathscr{I}_S K_C)$. Y is obviously a linear subspace of $H^0(C, \mathscr{I}_S K_C)$ and, since it is dense, it must coincide with the entire space.

Similarly, we can prove that the subset X of $H^0(C, \mathscr{I}_S K_C)$ whose elements can be decomposed in at least $k+2$ summands is the union of a finite number of proper subspaces of $H^0(C, \mathscr{I}_S K_C)$, and hence, its complement is open.

To prove the second statement, assume that there exists a decomposition $H = H_1 + H_2$ with $H_1, H_2 \in H^0(C, \mathscr{I}_S K_C)$ and a decomposition $C_{\mathrm{red}} = C_1 + C_2$ such that $\mathrm{supp}(H_{1|C_{\mathrm{red}}}) \subset C_1$ and $\mathrm{supp}(H_{2|C_{\mathrm{red}}}) \subset C_2$. Then H_1 and H_2 vanish on $C_1 \cap C_2$, and hence, H vanishes there too.

In particular, if $\operatorname{div}(H) \cap (\Gamma_i \cap \Gamma_j) = \emptyset$ for every $\Gamma_i \neq \Gamma_j$, such a decomposition cannot exist. The third statement follows easily from the second. □

REMARK 2.18. If S is a subcanonical cluster and S^* is its residual with respect to a section H, then their splitting indexes are the same. Indeed, $H^0(C, \mathscr{I}_{S^*} K_C) = \{\varphi(H)\ s.t.\ \varphi \in \operatorname{Hom}(\mathscr{I}_S K_C, K_C)\}$ and if H can be decomposed as in Lemma 2.17, then the same is true for $\varphi(H)$. By the symmetry of the situation, we may conclude.

LEMMA 2.19. *Let C be a 1-connected Gorenstein curve and let S be a non-trivial subcanonical cluster with minimal Clifford index among the clusters with splitting index smaller than or equal to $k \in \mathbb{N}$. Then $H^0(C, \mathscr{I}_S K_C)$ is base point free, that is, for every $P \in C$, the evaluation map*

$$H^0(C, \mathscr{I}_S K_C) \otimes \mathscr{O}_{C,P} \longrightarrow \mathscr{I}_{S|P} \subset \mathscr{O}_{C,P}$$

generates the ideal $\mathscr{I}_{S|P}$ as $\mathscr{O}_{C,P}$-module.

Proof. The statement is equivalent to saying that for every subscheme T containing S with $\operatorname{length}(T) = \operatorname{length}(S) + 1$, it is $h^0(C, \mathscr{I}_T K_C) < h^0(C, \mathscr{I}_S K_C)$.

If T is not subcanonical, then by definition of subcanonical cluster there exists a decomposition $C = A + B$ and a suitable cluster T_A with support on A such that

$$H^0(A, \mathscr{I}_{T_A} \omega_A) \cong H^0(C, \mathscr{I}_T K_C)$$

and then we conclude since

$$H^0(A, \mathscr{I}_{T_A} \omega_A) \hookrightarrow H^0(A, \mathscr{I}_{S_A} \omega_A)$$

and $h^0(A, \mathscr{I}_{S_A} \omega_A) < h^0(C, \mathscr{I}_S K_C)$ because S is subcanonical and C is 1-connected.

If T is subcanonical and its splitting index is greater than k, then necessarily the vector spaces $H^0(C, \mathscr{I}_T K_C)$ and $H^0(C, \mathscr{I}_S K_C)$ cannot be equal.

If T is subcanonical and its splitting index is smaller than or equal to k, then

$$\operatorname{Cliff}(\mathscr{I}_T K_C) = 2p_a(C) - \deg(S) - 1 - 2h^0(C, \mathscr{I}_T K_C) \geqslant \operatorname{Cliff}(\mathscr{I}_S K_C)$$

if and only if $h^0(C, \mathscr{I}_T K_C) < h^0(C, \mathscr{I}_S K_C)$. □

3. *Clifford's theorem*

In this section, we will prove Theorem A. The proof of the theorem is given arguing by contradiction by assuming the existence of a very special cluster for which its Clifford index is non-positive.

The first two lemmas work under the assumption C Gorenstein. The rest of the section needs an assumption on the singularities of C, namely C with planar singularities, or C contained in a smooth algebraic surface if non-reduced.

In the following lemma, we will show that there exists a special relation between a maximal cluster with non-positive Clifford index and its residual with respect to a generic section.

LEMMA 3.1. *Let C be a 2-connected Gorenstein curve. Fix $k \in \mathbb{N}$ and let S be a non-trivial subcanonical cluster with minimal non-positive Clifford index and maximal total degree among* *the clusters with splitting index smaller than or equal to k. Let S^*, T, T^* be subcanonical clusters such that*

(i) *S^* is the residual to S with respect to a generic section $H_0 \in H^0(C, \mathscr{I}_S K_C)$;*
(ii) *T^* is the residual to S with respect to a generic section $H_1 \in H^0(C, \mathscr{I}_S K_C)$;*
(iii) *T is the residual to S^* with respect to a generic section $H_2 \in H^0(C, \mathscr{I}_{S^*} K_C)$.*

Then either $T^* \cap T = \emptyset$ and $\mathrm{Cliff}(\mathscr{I}_S K_C) = 0$ or $T^* \subset T$.

Proof. Let Σ_k be the set of clusters with splitting index smaller than or equal to k.

Note at first that $\deg T = \deg S$, $h^0(C, \mathscr{I}_T K_C) = h^0(C, \mathscr{I}_S K_C)$ and similarly for S^* and T^* by Remark 2.13.

$\mathrm{Cliff}(\mathscr{I}_S K_C)$ is minimal non-positive if and only if $h^0(C, \mathscr{I}_S K_C) = p_a(C) - \deg S/2 + M$ with $M \geqslant 0$ maximal.

Call R the intersection of the two clusters T and T^*, that is, the subscheme defined by the ideal $\mathscr{I}_T + \mathscr{I}_{T^*}$, and U the minimal cluster containing both, that is, $\mathscr{I}_U = \mathscr{I}_T \cap \mathscr{I}_{T^*}$. Then R and U are subcanonical clusters by Lemma 2.14 and they belong to Σ_k. Indeed by Proposition 2.17 and Remark 2.18, the splitting indexes of T and T^* are equal to the one of S. Regarding U, by equation (6), we know that there is a section $H_3 \in H^0(C, K_C)$ vanishing on U such that $H_0 \cdot H_3 = H_1 \cdot H_2$. Notice that, since H_0 and H_1 are generic, H_3 can be seen as a deformation of $H_1 \in H^0(C, \mathscr{I}_{T^*} K_C)$, hence it is generic too in $H^0(C, \mathscr{I}_{T^*} K_C)$. Thus, the splitting index of U is smaller than or equal to the splitting index of T^*. With regard to R, with a similar argument we can prove that $R^* \in \Sigma_k$ and hence $R \in \Sigma_k$ too.

Moreover, we have the following exact sequence:

$$0 \longrightarrow \mathscr{I}_U \omega_C \longrightarrow \mathscr{I}_T \omega_C \oplus \mathscr{I}_{T^*} \omega_C \longrightarrow \mathscr{I}_R \omega_C \longrightarrow 0.$$

Thus, we know that

$$h^0(C, \mathscr{I}_T K_C) + h^0(C, \mathscr{I}_{T^*} K_C) \leqslant h^0(C, \mathscr{I}_R K_C) + h^0(C, \mathscr{I}_U K_C).$$

By Riemann–Roch and Serre duality the LHS is equal to $p_a(C) + 1 + 2M$, whilst the RHS is $\leqslant p_a(C) - \deg U/2 + M + p_a(C) - \deg R/2 + M = p_a(C) + 1 + 2M$.

By the maximality of the degree of T then one of the following must hold:

(i) $U = K_C$, $R = 0$, whence $T \cap T^* = \emptyset$ and $M = 0$, that is, $\mathrm{Cliff}(\mathscr{I}_T K_C) = 0$; moreover, it is $h^0(C, \mathscr{I}_S K_C) + h^0(C, \mathscr{I}_{S^*} K_C) = h^0(C, K_C) + 1$;
(ii) $U = T$, $R = T^*$ and in particular $T^* \subseteq T$. □

LEMMA 3.2. *Let C be a 2-connected Gorenstein curve and S be a subcanonical cluster. Assume that there is an irreducible component $\Gamma \subset C$ such that*

$$\dim[H^0(C, \mathscr{I}_S K_C)_{|\Gamma}] \geqslant 2.$$

Then for a generic $P \in \Gamma$, the cluster $S + P$ is still subcanonical.

Proof. We argue by contradiction.

If S is subcanonical but $P+S$ is not, that is, $H^0(C,\mathscr{I}_{S+P}K_C)_{|B}=0$ for some subcurve $B\subset C$ (clearly, $\Gamma\nsubseteq B$ since $H^0(C,\mathscr{I}_{S+P}K_C)_{|\Gamma}\neq 0$ by our assumption), we consider the following commutative diagram:

$$\begin{array}{ccccc}
H^0(C-B,\mathscr{I}_P\mathscr{I}_1K_{C-B}) & \longrightarrow & H^0(C,\mathscr{I}_{S+P}K_C) & \longrightarrow & H^0(B,\mathscr{I}_{S+P}K_C)_{|B}=0\\
\downarrow & & \downarrow & & \\
H^0(C-B,\mathscr{I}_1K_{C-B}) & \longrightarrow & H^0(C,\mathscr{I}_SK_C) & \longrightarrow & H^0(B,\mathscr{I}_SK_C)_{|B}=\mathbb{K}\\
\downarrow & & \downarrow & & \\
H^0(P,\mathscr{O}_P) & \xrightarrow{=} & H^0(P,\mathscr{O}_P) & &
\end{array}$$

where $\mathscr{I}_1$ is the ideal sheaf on $C-B$ given as the kernel of the map $\mathscr{I}_S\to(\mathscr{I}_S)_{|B}$.

By a simple diagram chase the restriction map $H^0(C-B,\mathscr{I}_1K_{C-B})\to H^0(P,\mathscr{O}_P)$ must be zero, and hence, by genericity of the point P, the global restriction map from $H^0(C-B,\mathscr{I}_1K_{C-B})$ to Γ must be zero. This is impossible, since this would imply that the restriction of the global space $H^0(C,\mathscr{I}_SK_C)$ to Γ would be at most one-dimensional, contradicting our assumption. □

The following lemma generalizes the classical techniques showed by Saint Donat [**16**].

LEMMA 3.3. *Let C be a 2-connected projective curve, either reduced with planar singularities or contained in a smooth algebraic surface.*

Fix $k\in\mathbb{N}$ and let S be a non-trivial subcanonical cluster with minimal non-positive Clifford index and maximal total degree among the clusters with splitting index smaller than or equal to k. Let S^* be the residual to S with respect to a generic hyperplane section H.

Suppose that there is an irreducible component $\Gamma\subset C$ such that

$$\dim[H^0(C,\mathscr{I}_SK_C)_{|\Gamma}]\geqslant 2,$$
$$\dim[H^0(C,\mathscr{I}_{S^*}K_C)_{|\Gamma}]\geqslant 2.$$

Then S^* is a length 2 cluster such that $h^0(C,\mathscr{I}_{S^*}K_C)=p_a(C)-1$. In particular, C is either honestly hyperelliptic or 3-disconnected.

Proof. We divide the proof in four steps.

Let Σ_k be the set of clusters with splitting index smaller than or equal to k. By Remark 2.18, we know that $S^*\in\Sigma_k$.

Note that since C is 2-connected, then $2\leqslant\deg(S)\leqslant\deg(K_C)-2$.

Step 1 : S and S^* are Cartier divisor and non-splitting. Consider a generic point $P\in\Gamma$. In particular, $P\notin S$. By Lemma 3.2, $P+S$ is subcanonical and by Lemma 2.19 $h^0(C,\mathscr{I}_P\mathscr{I}_SK_C)=h^0(C,\mathscr{I}_SK_C)-1$.

Consider a generically invertible section H in $H^0(C,\mathscr{I}_SK_C)$ vanishing at P and the residual S^* with respect to H. We have $P\in S^*$ and we can apply Lemma 3.1 because P is general, and hence, the corresponding invertible section is general as well. Since $S^*\not\subset S$ we have $S^*\cap S=\emptyset$ and both are Cartier divisors.

S and S^* Cartier with minimal Clifford indexes among the clusters in Σ_k implies that both the linear systems $|K_C(-S)|$ and $|K_C(-S^*)|$ are base point free by Lemma 2.19. Hence, we can find a divisor $S^*\in|K_C(-S)|$ not passing through the singular locus of C_{red}. This implies that the splitting index of S^* is zero by Proposition 2.17 and Remark 2.18 shows that the splitting index of S is zero as well.

Step 2 : $h^0(C, \mathscr{I}_S K_C)_{|D} \leqslant h^0(C, \mathscr{I}_{S^*} K_C)_{|D}$ *for any* $D \subset C$. Consider again a generic point $P \in \Gamma$, $P \notin S$ and $P \notin S^*$. With the same argument adopted in step 1, we take a cluster S_1^* residual to S such that $P \in S_1^*$ and a second cluster S_2 residual to S^* such that $P \in S_2$. By Lemma 3.1, $S_1^* \subset S_2$ since their intersection contains P. This gives us the following inequality for every subcurve $D \subset C$:

$$\begin{aligned}\dim[H^0(C, \mathscr{I}_{S^*} K_C)_{|D}] &= \dim[H^0(C, \mathscr{I}_{S_1^*} K_C)_{|D}] \geqslant \dim[H^0(C, \mathscr{I}_{S_2} K_C)_{|D}] \\ &= \dim[H^0(C, \mathscr{I}_S K_C)_{|D}]. \end{aligned} \tag{7}$$

Step 3 : $h^0(C, \mathscr{I}_S K_C) = 2$. We argue by contradiction, assuming that $h^0(C, \mathscr{I}_S K_C) \geqslant 3$.

Case (a):

$$\exists \text{ irreducible } \Gamma \subset C \text{ s.t. } \dim[H^0(C, \mathscr{I}_S K_C)_{|\Gamma}] \geqslant 3.$$

We may apply Lemma 3.2 twice to conclude that, given two generic points P and Q in Γ, the cluster $P + Q + S$ is subcanonical and the points impose independent conditions to $H^0(C, \mathscr{I}_S K_C)$. Hence, there exists a generically invertible $H \in H^0(C, \mathscr{I}_S K_C)$ passing through $P + Q$. Consider T^*, the residual to S with respect to H: $P + Q \subset T^*$.

Step 2 allows us to apply Lemma 3.2 to the cluster S^* as well, and hence, $P + S^*$ is subcanonical and P and Q impose independent conditions to $H^0(C, \mathscr{I}_{S^*} K_C)$. Hence, there exists a generically invertible section $H_1 \in H^0(C, \mathscr{I}_P \mathscr{I}_{S^*} K_C)$ but $H_1 \notin H^0(C, \mathscr{I}_Q \mathscr{I}_P \mathscr{I}_{S^*} K_C)$. Let T_1 be the residual to S^* with respect to this section. We have that $P \in T_1$ but $Q \notin T_1$.

This is impossible: $P \in T_1 \cap T^*$ but $Q \in T^*$, $Q \notin T_1$. Thus, $\emptyset \neq T_1 \cap T^* \subsetneq T^*$, contradicting Lemma 3.1.

Hence, this case cannot happen, that is, for every irreducible component Γ, the restriction of $H^0(C, \mathscr{I}_S K_C)$ to Γ is at most two-dimensional.

Case (b):

$$\begin{cases} \dim[H^0(C, \mathscr{I}_S K_C)_{|C_{\text{red}}}] \geqslant 3, \\ \dim[H^0(C, \mathscr{I}_S K_C)_{|\Gamma_0}] \leqslant 2 \qquad \text{for every irreducible } \Gamma_0 \subset C. \end{cases}$$

We want to argue as in case (a) finding two points P and Q which lead to the same contradiction.

Since case (a) cannot happen, we know that $\dim[H^0(C, \mathscr{I}_S K_C)_{|\Gamma}] = 2$, and hence, there must exist a topologically connected reduced subcurve $D \supset \Gamma$, minimal up to inclusion, such that

$$\dim[H^0(C, \mathscr{I}_S K_C)_{|D}] \geqslant 3.$$

By minimality of D, there exists an irreducible component $\Gamma_1 \subset D$, with $\Gamma_1 \neq \Gamma$, and a section $H_0 \in H^0(C, \mathscr{I}_S K_C)$ such that $H_{0|\Gamma_1} \neq 0$ while $H_{0|D-\Gamma_1} = 0$. In particular, H_0 vanishes on $\Gamma_1 \cap (D - \Gamma_1)$.

We consider a generic point $P \in \Gamma$. Thanks to Lemma 3.2 and Step 1, there exists a generically invertible section $H \in H^0(C, \mathscr{I}_{S+P} K_C)$ not vanishing on any singular point of C_{red}.

Hence, we know that the sections H and H_0 span a two-dimensional subspace of $H^0(C, \mathscr{I}_{S+P} K_C)_{|\Gamma_1}$. We apply Lemma 3.2 to Γ_1 taking a point Q generic in Γ_1 such that $S + P + Q$ is subcanonical and P and Q impose independent conditions on $H^0(C, \mathscr{I}_S K_C)$.

We may conclude as in case (a) that this case cannot happen.

Case (c):

$$\begin{cases} \dim[H^0(C, \mathscr{I}_S K_C)_{|C_{\text{red}}}] = 2, \\ \dim[H^0(C, \mathscr{I}_S K_C)] \geqslant 3. \end{cases}$$

Consider a generic point $P \in \Gamma$. By Lemma 3.2, $S+P$ is subcanonical and by genericity of P

$$H^0(C, \mathscr{I}_{S+P}K_C)_{|C_{\mathrm{red}}} = \langle H \rangle,$$

where H is generically invertible and does not vanish on any singular point of C_{red}. In particular, $P+S$ is non-splitting.

We want to show that $(P+S)_{|C_{\mathrm{red}}} = K_{C|C_{\mathrm{red}}}$. If not, there would exists a point Q in C_{red} not imposing any condition on $H^0(C, \mathscr{I}_S K_C)$, that is, the unique non-zero section $H \in H^0(C, \mathscr{I}_{S+P}K_C)_{|C_{\mathrm{red}}}$ would vanish at Q. In particular, $S+P+Q$ would be subcanonical, since the section H must be generically invertible. But our assumptions are that S has maximal degree among the non-splitting non-trivial cluster of minimal Clifford index. Therefore, since $P+Q+S \neq K_C$ (otherwise $\dim[H^0(C, \mathscr{I}_S K_C)] \leqslant 2$), we should have

$$\mathrm{Cliff}(\mathscr{I}_{S+P+Q}K_C) > \mathrm{Cliff}(\mathscr{I}_S K_C),$$

which is equivalent to

$$h^0(C, \mathscr{I}_{S+P+Q}K_C) < h^0(C, \mathscr{I}_S K_C) - 1$$

contradicting our hypotheses.

Thus, $(P+S)_{|C_{\mathrm{red}}} = K_{C|C_{\mathrm{red}}}$ and we can argue as in Step 1 taking a cluster S_1^* residual to S with respect to a generic section and passing through P. Hence $S^*_{1|C_{\mathrm{red}}} = P$ and the multiplicity of Γ in C is at least 2 since $\deg S_1^* > 1$.

In this case, we consider a generic length 2 cluster σ_0 supported at P. Since S and S^* are Cartier and supported on smooth points of C_{red}, it is easy to check by semicontinuity that σ_0 imposes independent conditions on $H^0(C, \mathscr{I}_S K_C)$ and $H^0(C, \mathscr{I}_{S^*} K_C)$, and we can treat σ_0 as we did with the length 2 cluster $P+Q$ in the previous case, that is, we take T_1 and T^* such that $P \in T_1 \cap T^*$ but $\sigma_0 \not\subset T_1 \cap T^*$. By Lemma 3.1 this is a contradiction.

Hence, we are allowed to conclude that

$$\dim[H^0(C, \mathscr{I}_S K_C)] = 2.$$

Step 4 : $\deg S^* = 2$ *and* $h^0(C, \mathscr{I}_{S^*} K_C) = p_a(C) - 1$. By our assumptions and Step 3

$$0 \geqslant \mathrm{Cliff}(\mathscr{I}_S K_C) = \deg(\mathscr{I}_S K_C) - 2h^0(C, \mathscr{I}_S K_C) + 2 = \deg(\mathscr{I}_S K_C) - 2,$$

which implies that

$$\deg S^* = \deg(\mathscr{I}_S K_C) \leqslant 2.$$

But if $\deg S^* = 1$, then the point S^* would be a base point for K_C, which is absurd by Theorem 2.5 since C is 2-connected and has genus at least 2 since $p_a(C) = h^0(C, K_C) \geqslant \dim[H^0(C, \mathscr{I}_S K_C)_{|\Gamma}] \geqslant 2$.

Finally, Riemann–Roch Theorem and Serre duality implies that

$$h^0(C, \mathscr{I}_{S^*} K_C) = p_a(C) - 1,$$

hence S^* is a length 2 cluster not imposing independent condition on K_C. This happens if and only if C is honestly hyperelliptic or C is 3-disconnected. □

The following three technical lemmas will be used in the proof of Theorem 3.7 in order to give estimates for the rank of the restriction map $r : H^0(C, \mathscr{I}_S K_C) \to H^0(B, \mathscr{I}_S K_C)$ for some particular subcurves $B \subset C$.

LEMMA 3.4. *Let C be a 2-connected curve contained in a smooth algebraic surface and S a non-trivial subcanonical cluster with minimal Clifford index among the clusters with splitting index smaller than or equal to $k \in \mathbb{N}$.*

If there is an irreducible component Γ *and a point* $P \in \Gamma$ *such that* $S_{|P}$ *is not contained in* C_{red}*, then the restriction map* $H^0(C, \mathscr{I}_S K_C) \to H^0(m\Gamma, \mathscr{I}_S K_C)$ *has rank* 1*, where* m *is the minimal integer such that* $S_{|P} \subset m\Gamma$.

Proof. Let S be a non-trivial subcanonical cluster with minimal Clifford index and let $P \in C$ be a point such that $S_{|P}$ is not contained in C_{red}.

Let $\mathscr{O}_{C,P}$ be the local ring of C at P, $\mathscr{N}$ be the maximal ideal of $\mathscr{O}_{C,P}$ and $\mathscr{M}$ be the maximal ideal of $\mathscr{O}_{C\text{red},P}$.

Thanks to Lemma 2.19, locally at P the ideal $\mathscr{I}_{S|P} \subset \mathscr{O}_{C,P}$ can be written as

$$\mathscr{I}_{S|P} = (H, H_1, \ldots, H_k, p_1, \ldots, p_l),$$

where $H, H_1, \ldots, H_k, p_1, \ldots, p_l$ are linearly independent sections in $H^0(C, \mathscr{I}_S K_C)$.

Moreover, we ask $H, H_1, \ldots, H_k$, to be of minimal degree when restricted to S_{red}, whereas $p_1, \ldots, p_l$ must have degree strictly bigger. Algebraically, if $\mathscr{I}_{S_{\text{red}}|P} \subset \mathscr{M}^n$ but $\mathscr{I}_{S_{\text{red}}|P} \not\subset \mathscr{M}^{n+1}$, then we ask $H, H_1, \ldots, H_k$ to be a basis of the $\mathbb{K}$-vector space $\mathscr{I}_{S_{\text{red}}}/(\mathscr{I}_{S_{\text{red}}} \cap \mathscr{M}^{n+1})$ and $p_1, \ldots, p_l$ to satisfy $p_{i|C_{\text{red}}} \in \mathscr{M}^{n+1}$.

Let us consider a subcluster $\hat{S} \subset S$ of colength $=1$, such that $\hat{S} \neq S$ precisely at P. In particular, we ask the ideal $\mathscr{I}_{\hat{S}}$ to coincide with $(\mathscr{I}_S, H_\infty)$, where $H_\infty \in \mathscr{I}_{(m-1)\Gamma_{|P}}$.

Define now a one-dimensional family $\{S_\lambda\}$ of clusters, each of them given locally at P by the ideal

$$\mathscr{I}_{S_\lambda} = (H + \lambda H_\infty, H_1, \ldots, H_k, p_1, \ldots, p_l)$$

and coinciding with $\hat{S}$ elsewhere. By construction every S_λ contains $\hat{S}$ and we have $H \notin H^0(C, \mathscr{I}_{S_\lambda} K_C)$, which implies $H^0(C, \mathscr{I}_{S_\lambda} K_C) \subsetneq H^0(C, \mathscr{I}_{\hat{S}} K_C)$ for every $\lambda \neq 0$. Indeed, if locally $H \in \mathscr{I}_{S_\lambda|P}$, there would exist elements $\alpha, \alpha_i, \beta_i \in \mathscr{O}_{C,P}$ such that

$$H = \alpha(H + \lambda H_\infty) + \sum \alpha_i H_i + \sum \beta_i p_i.$$

Since $\{H, H_1, \ldots, H_k\}$ represents a basis for the $\mathbb{K}$-vector space $\mathscr{I}_{S_{\text{red}}}/(\mathscr{I}_{S_{\text{red}}} \cap \mathscr{M}^{n+1})$, we should have $\alpha \cong 1 \operatorname{mod} \mathscr{N}$, the maximal ideal of $\mathscr{O}_{C,P}$. In particular, α should be invertible in $\mathscr{O}_{C,P}$ and, since $\lambda \in \mathbb{C}^*$, the above equation should imply

$$H_\infty \in (H, H_1, \ldots, H_k, p_1, \ldots p_l) = \mathscr{I}_{S|P},$$

that is, $\mathscr{I}_{\hat{S}|P} \cong \mathscr{I}_{S|P}$, which is impossible by construction of H_∞.

On the contrary, since Cliff $\mathscr{I}_S K_C$ is minimal, it is $H^0(C, \mathscr{I}_S K_C) = H^0(C, \mathscr{I}_{\hat{S}} K_C)$ by our numerical assumptions. Indeed, let us consider the residual to S, respectively $\hat{S}$, with respect to a section in $H^0(C, \mathscr{I}_S K_C)$. We have $S^* \subset \hat{S}^*$ and we know that S^* satisfies the assumptions of Lemma 2.19 since S does. Hence, $h^0(C, \mathscr{I}_{\hat{S}^*} K_C) < h^0(C, \mathscr{I}_{S^*} K_C)$ and, in particular, $h^0(C, \mathscr{I}_S K_C) = h^0(C, \mathscr{I}_{\hat{S}} K_C)$ by Riemann–Roch theorem and Serre duality for residual clusters.

To conclude the proof, we are going to show that this vector space is spanned by H and a codimension 1 subspace given by sections vanishing on $m\Gamma$.

Our claim is that for every $\lambda \neq 0$ every section in $H^0(C, \mathscr{I}_{S_\lambda} K_C)$ vanishes on the curve $m\Gamma$.

Fix a cluster S_λ, let $\sigma \in H^0(C, \mathscr{I}_{S_\lambda} K_C)$ and consider a generic S_μ. Since both $H^0(C, \mathscr{I}_{S_\lambda} K_C)$ and $H^0(C, \mathscr{I}_{S_\mu} K_C)$ are codimension 1 subspaces of the same vector space, then there exists a linear combination $\sigma + b_\mu H \in H^0(C, \mathscr{I}_{S_\mu} K_C)$.

Localizing at P, we can write $\sigma = \sum \alpha_i p_i + \alpha(H + \lambda H_\infty) + \sum \gamma_i H_i$. Since $\sigma + b_\mu H$ belongs to $\mathscr{I}_{S_\mu}$, there exists elements β_i, δ_i and $\beta \in \mathscr{O}_{C,P}$ such that

$$\alpha(H + \lambda H_\infty) + b_\mu H = \sum \beta_i p_i + \beta(H + \mu H_\infty) + \sum \delta_i H_i.$$

Both the polynomials are in $\mathscr{I}_{\hat{S}}$. By the description above, we must have

$$\alpha + b_\mu = \beta \mod \mathscr{N},$$
$$\alpha\lambda = \beta\mu \mod \mathscr{N},$$

where $\mathscr{N}$ as above is the maximal ideal of $\mathscr{O}_{C,P}$. This forces

$$b_\mu = \alpha(\mathrm{mod}\ \mathscr{N})\left(\frac{\lambda}{\mu} - 1\right).$$

Suppose now that $\alpha \notin \mathscr{N}$. Then, apart from H, any element in $\langle \sigma, H\rangle$ should be written as $a(\sigma + b_\mu H)$ for some μ. In particular, for $c \neq 0$ every ideal of the form

$$(c\sigma + dH, H_1, \ldots, H_k, p_1, \ldots, p_l)$$

is contained in some $\mathscr{I}_{S_\mu}$.

This implies that length $\mathscr{O}_{C,P}/(c\sigma + dH, H_1, \ldots, H_k, p_1, \ldots, p_l)$ is at least length $S+1$ since the ideal vanishes on S and S_μ (since $\sigma \in H^0(C, \mathscr{I}_{S_\lambda}K_C) \subset H^0(C, \mathscr{I}_S K_C)$).

But its degeneration $\mathscr{O}_{C,P}/(H, H_1, \ldots, H_k, p_1, \ldots, p_l) = \mathscr{O}_{C,P}/\mathscr{I}_S = \mathscr{O}_S$ has strictly smaller length. This is impossible since the length is upper semicontinuous.

We must conclude that $\alpha \in \mathscr{N}$ and that $b_\mu = 0$. This means that the original $\sigma \in H^0(C, \mathscr{I}_{S_\lambda}K_C)$ belongs to $H^0(C, \mathscr{I}_{S_\mu}K_C)$, that is, $H^0(C, \mathscr{I}_{S_\lambda}K_C) = H^0(C, \mathscr{I}_{S_\mu}K_C)$ for every $\lambda, \mu \in \mathbb{C}^*$.

In particular, every section in $H^0(C, \mathscr{I}_{S_\lambda}K_C)$ must vanish on every S_μ, and in particular, it vanishes on the scheme theoretic union $\bigcup_{\mu \in \mathbb{K}} S_\mu$ which has infinite length. This may happen only if $H^0(C, \mathscr{I}_{S_\lambda}K_C))_{|m\Gamma} = \{0\}$. □

LEMMA 3.5. *Let C be a 2-connected projective curve either reduced with planar singularities or contained in a smooth algebraic surface. Let $B \subset C$ be a subcurve such that the restriction map*

$$H^0(C, \mathscr{I}_{K_{C|B}}K_C) \longrightarrow H^0(m\Gamma, \mathscr{O}_{m\Gamma})$$

has rank 1 for every subcurve $m\Gamma \subset B$.

If $B = \sum_{j=1}^{l} B_j$ is the decomposition of B in topologically connected component, then the restriction map

$$H^0(C, \mathscr{I}_{K_{C|B}}K_C) \longrightarrow H^0(B, \mathscr{O}_B)$$

has rank $\leqslant l$ (where l is the number of components).

Proof. The lemma follows from Lemma 3.4 since the restriction map has rank 1 on every topologically connected component. □

LEMMA 3.6. *Let C be a 2-connected projective curve either reduced with planar singularities or contained in a smooth algebraic surface. Suppose that C_{red} is μ-connected. Let S be a subcanonical cluster, and assume that there exists a subcurve B such that $C_{\mathrm{red}} \subset B$ and the restriction map*

$$H^0(C, \mathscr{I}_S K_C) \longrightarrow H^0(m\Gamma, \mathscr{I}_S K_C)$$

has rank 1 for every subcurve $m\Gamma \subset B$. Then the following hold.

(i) *The restriction map $H^0(C, \mathscr{I}_S K_C) \to H^0(B, \mathscr{I}_S K_C)$ has rank $k+1$ (where k is the splitting index of S).*

(ii) *If $k > 0$, we have $\deg K_{C|B} - \deg S_{|B} \geqslant \max\{k; (\mu/2)(k+1)\}$.*

Proof. Since the restriction map to every $m\Gamma$ has rank 1, it is generated by the restriction of a generically invertible section $H \in H^0(C, \mathscr{I}_S K_C)$. By genericity, we may assume that H verifies the minimum for the splitting index, that is, $H = \sum_{i=0}^{k} H_i$ with $H_i \in H^0(C, \mathscr{I}_S K_C)$ and there is a maximal decomposition $C_{\mathrm{red}} = \sum_{i=0}^{k} C_i$ with $\mathrm{supp}(H_{i|C_{\mathrm{red}}}) = C_i$ and H cannot be further decomposed.

(i) To prove the first part of the statement, notice that the restriction map

$$H^0(C, \mathscr{I}_S K_C)_{|B} \longrightarrow H^0(C, \mathscr{I}_S K_C)_{|C_{\mathrm{red}}}$$

is an isomorphism. Indeed the above restriction map is obviously onto. It is injective as well, since otherwise there would be a section $\hat{H}$ in $H^0(C, \mathscr{I}_S K_C)$ vanishing on C_{red} but not on B, that is, there would be a subcurve $m\Gamma \subset B$ such that $\hat{H}$ vanishes on Γ but not on $m\Gamma$. But the rank of the restriction $H^0(C, \mathscr{I}_S K_C) \to H^0(m\Gamma, \mathscr{I}_S K_C)$ is 1 by our assumptions, as well as the rank of $H^0(C, \mathscr{I}_S K_C) \to H^0(\Gamma, \mathscr{I}_S K_C)$, and hence, the section $\hat{H}$ cannot exist.

Thus, without loss of generality, we can assume $B = C_{\mathrm{red}}$ and we take the decomposition $C_{\mathrm{red}} = \sum_{i=0}^{k} C_i$.

The first statement follows if we prove that for every C_i it is $H^0(C, \mathscr{I}_S K_C)_{|C_i} = \langle H_i \rangle$. For simplicity, we are going to prove it for C_1.

Write $C_1 = \sum_{j=1}^{J_0} \Gamma_j$, where the curves Γ_j are the irreducible components. Note that C_1 is connected, hence 1-connected, since the decomposition of C is maximal. We are going to prove by induction that there exists a decomposition sequence

$$\Gamma_1 = B_1 \subset B_2 \subset \cdots \subset B_{J_0} = C_1$$

such that $H^0(C, \mathscr{I}_S K_C)_{|B_J} = \langle H_1 \rangle$ for every $J \leqslant J_0$.

The first case, $J = 1$, follows from our assumptions. Assume now that it holds for B_{J-1}. Since C_1 is 1-connected, then $B_{J-1} \cap (C_1 - B_{J-1}) \neq \emptyset$. Take H_1 and evaluate it on $B_{J-1} \cap (C_1 - B_{J-1})$. If it is zero, then H_1 can be decomposed as the sum of two sections of $H^0(C, \mathscr{I}_S K_C)_{|C_1}$, one supported on B_{J-1} the other on $C_1 - B_{J-1}$. But then we may apply Proposition 2.17, part 1, to conclude that this would force H_1, and H as well, to be decomposed as the sum of more sections than allowed.

Hence, there exists at least one component, say Γ_J, such that H_1 does not vanish on $B_{J-1} \cap \Gamma_J$. Define $B_J := B_{J-1} + \Gamma_J$. Our claim is that $H^0(C, \mathscr{I}_S K_C)_{|B_J} = \langle H_1 \rangle$. If not there would exist $\bar{H} \in H^0(C, \mathscr{I}_S K_C)_{|D_J}$ linearly independent of H_1 such that $\bar{H}_{|D_{J-1}} = 0$ (possibly after a linear combination of sections). Moreover, we would have $\bar{H}_{|\Gamma_J} = H_{1|\Gamma_J}$ up to rescaling by our assumptions, and hence, H_1 must vanish on $D_{J-1} \cap \Gamma_J$, which is absurd.

(ii) Suppose now that the splitting index k is at least 1. We are going to study $\deg K_C - \deg S$ on B.

Assume at first that $B = C_{\mathrm{red}}$. Consider a decomposition sequence $C_0 = D_0 \subset D_1 \subset \ldots \subset D_k = B = C_{\mathrm{red}}$, where $D_i - D_{i-1} = C_i$. Up to reindexing the subcurve C_i we can suppose that the curves D_i are topologically connected, hence 1-connected since they are reduced.

We prove by induction that $\deg(\mathscr{I}_S K_C)_{|D_i} \geqslant i$. For $i = 0$ it is obvious. For $i > 0$ consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{N} & \longrightarrow & (\mathscr{I}_S K_C)_{|D_i} & \xrightarrow{\pi_i} & (\mathscr{I}_S K_C)_{|D_{i-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{C|C_i}(-D_{i-1}) & \longrightarrow & K_{C|D_i} & \longrightarrow & K_{C|D_{i-1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & S_{|D_i} & \longrightarrow & S_{|D_{i-1}} & \longrightarrow & 0
\end{array}$$

where $\mathscr{N}$ is the kernel of π_i and Z a subsheaf of $S_{|D_i}$, both considered as sheaves with support on C_i. Note that by our assumptions the section H_i restricts to a non-zero generically invertible section of $\mathscr{N}$, and thus, $\deg_{C_i} \mathscr{N} \geqslant 0$. Computing degrees, we obtain

$$\begin{aligned}\deg_{C_i} Z &= \deg K_{C|C_i}(-D_{i-1}) - \deg \mathscr{N} = \deg K_{C|C_i} - C_i \cdot D_{i-1} - \deg \mathscr{N} \\ &\leqslant \deg K_{C|C_i} - C_i \cdot D_{i-1} \leqslant \deg K_{C|C_i} - 1. \end{aligned} \tag{8}$$

But $\deg S_{|D_i} = \deg S_{|D_{i-1}} + \deg_{C_i} Z$, and by induction hypothesis, we may assume $\deg(\mathscr{I}_S K_C)_{|D_{i-1}} \geqslant (i-1)$. Hence,

$$\begin{aligned}\deg(\mathscr{I}_S K_C)_{|D_i} &= \deg K_{C|D_i} - \deg S_{|D_i} \\ &= (\deg K_{C|D_{i-1}} - \deg S_{|D_{i-1}}) + (\deg K_{C|C_i} - \deg_{C_i} Z) \\ &\geqslant (i-1) + 1 = i.\end{aligned}$$

In particular, we have the first inequality we wanted to prove, that is,

$$\deg K_{C|B} - \deg S_{|B} \geqslant k.$$

Moreover, equation (8) yields $\deg K_{C|C_i} - \deg S_{|C_i} \geqslant C_i \cdot D_{i-1}$. Taking sum over all the irreducible components we obtain

$$\deg K_{C|C_{\mathrm{red}}} - \deg S_{|C_{\mathrm{red}}} \geqslant \frac{1}{2}\sum_{i=0}^{k} C_i \cdot (C_{\mathrm{red}} - C_i).$$

Thus, if the reduced curve C_{red} is μ-connected, we have

$$\deg K_{C|B} - \deg S_{|B} \geqslant \frac{\mu}{2}(k+1).$$

We deal now with the case $C_{\mathrm{red}} \subsetneq B$. We just proved that

$$\deg(\mathscr{I}_S K_C)_{|C_{\mathrm{red}}} = \deg K_{C|C_{\mathrm{red}}} - \deg S_{|C_{\mathrm{red}}} \geqslant \max\left\{k, \frac{\mu}{2}(k+1)\right\}.$$

Consider the following diagram, which exists and commute since S is subcanonical:

$$\begin{array}{ccccc} \mathscr{O}_{B-C_{\mathrm{red}}}(-(C_{\mathrm{red}})) & \hookrightarrow & \mathscr{O}_B & \twoheadrightarrow & \mathscr{O}_{C_{\mathrm{red}}} \\ \downarrow & & \downarrow & & \downarrow \\ \ker(\rho) & \hookrightarrow & (\mathscr{I}_S K_C)_{|B} & \overset{\rho}{\twoheadrightarrow} & (\mathscr{I}_S K_C)_{|C_{\mathrm{red}}} \end{array}$$

Computing degrees, we may conclude by the following equation

$$\begin{aligned}\deg(\mathscr{I}_S K_C)_{|B} &= \chi((\mathscr{I}_S K_C)_{|B}) - \chi(\mathscr{O}_B) \\ &= \chi((\mathscr{I}_S K_C)_{|C_{\mathrm{red}}}) + \chi(\ker(\rho)) - \chi(\mathscr{O}_{C_{\mathrm{red}}}) - \chi(\mathscr{O}_{B-C_{\mathrm{red}}}(-(C_{\mathrm{red}}))) \\ &= \deg(\mathscr{I}_S K_C)_{|C_{\mathrm{red}}} + \chi(\ker(\rho))) - \chi(\mathscr{O}_{B-C_{\mathrm{red}}}(-(C_{\mathrm{red}}))) \\ &\geqslant \max\left\{k, \frac{\mu}{2}(k+1)\right\}.\end{aligned}$$

□

Our main result follows from the following theorem.

THEOREM 3.7. *Let C be a projective curve either reduced with planar singularities or contained in a smooth algebraic surface. Assume C to be 2-connected and C_{red} μ-connected.*

Let $S \subset C$ be a subcanonical cluster of splitting index k. Then

$$h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \frac{1}{2}\deg(S) + \frac{k}{2}. \tag{9}$$

The following holds:

(i) *if* $C_{\rm red}$ *is* 2*-connected, then* $h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \frac{1}{2}\deg(S) + \max\{0, \frac{k}{2} - \frac{1}{2}\}$;
(ii) *if* $C_{\rm red}$ *is* 3*-connected, then* $h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \frac{1}{2}\deg(S) + \max\{0, \frac{k}{4} - \frac{3}{4}\}$;
(iii) *if* $C_{\rm red}$ *is* 4*-connected, then*

$$h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \tfrac{1}{2}\deg(S). \tag{10}$$

Moreover, if equality holds in equation (9) *or in equation* (10)*, then the pair* (S, C) *satisfies one of the following assumptions*:

(i) $S = 0$, K_C *and* $k = 0$;
(ii) C *is honestly hyperelliptic,* S *is a multiple of the honest* g^1_2 *and* $k = 0$;
(iii) C *is* 3*-disconnected* (*that is, there is a decomposition* $C = A + B$ *with* $A \cdot B = 2$).

Proof. Fix $k \in \mathbb{N}$ and let Σ_k be the set of clusters with splitting index smaller than or equal to k.

Then equation (9) is equivalent to

$$\mathrm{Cliff}(\mathscr{I}_S K_C) := 2p_a(C) - \deg(S) - 2 \cdot h^0(\mathscr{I}_S K_C) \geqslant -k$$

for every cluster $S \in \Sigma_k$.

If the Clifford index of non-trivial clusters is always positive, the claim is trivially true. Suppose then the existence of a non-trivial subcanonical cluster with non-positive Clifford index in Σ_k.

Step 1 : *Clusters of minimal Clifford index and maximal degree.* We are going to prove at first that *the claim is true for a cluster* S *of minimal Clifford index and maximal degree*, more precisely that the required inequalities hold for such a cluster and if equality holds in equation (9) or in equation (10), then the pair (S, C) satisfies one of the conditions listed in the statement.

Let S be a non-trivial subcanonical cluster in Σ_k with minimal Clifford index and maximal total degree. Let S^* be its residual with respect to a generic hyperplane section H. Without loss of generality, we can suppose that the splitting index of S is precisely k. We have

$$h^0(C, \mathscr{I}_S K_C) = p_a(C) - \frac{\deg S}{2} + M \tag{11}$$

with $M \geqslant 0$ maximal in Σ_k.

By Lemma 3.1, we know that either S^* is contained in S or S is disjoint from S^* and $\mathrm{Cliff}(\mathscr{I}_S K_C) = 0$; in the second case, S and S^* are Cartier divisors since they are locally isomorphic to K_C.

Case 1 : *There exists an irreducible component* $\Gamma \subset C$ *such that*

$$\dim[H^0(C, \mathscr{I}_S K_C)_{|\Gamma}] \geqslant 2 \quad \text{and} \quad \dim[H^0(C, \mathscr{I}_{S^*} K_C)_{|\Gamma}] \geqslant 2.$$

By Lemma 3.3, we know that $\deg(S^*) = 2$, that is, C is 3-disconnected or honestly hyperelliptic and that $h^0(C, \mathscr{I}_S K_C) = p_a(C) - \deg S/2$.

Case 2 : $S^* \subset S$ *and the restriction map* $H^0(C, \mathscr{I}_S K_C) \to H^0(\Gamma, \mathscr{I}_S K_C)$ *has rank 1 for every irreducible* $\Gamma \subset C$.

Let $B = \sum m_i \Gamma_i$ be the minimal subcurve of C containing S and every irreducible component Γ_i such that $\deg_{\Gamma_i} K_C = 0$.

First of all note that $S \cap \Gamma \neq \emptyset$ for every irreducible component $\Gamma \subset C$ such that $K_{C|\Gamma} \neq 0$ because $S^* \subseteq S$. Thus, $C_{\rm red} \subset B$.

By Lemma 3.4, the restriction map $H^0(C, \mathscr{I}_S K_C) \to H^0(m_i\Gamma_i, \mathscr{I}_S K_C)$ has rank 1 for every irreducible $\Gamma_i \subset B$ with multiplicity $m_i > 1$ in B. We apply Lemma 3.6 and we may conclude that the restriction map $H^0(C, \mathscr{I}_S K_C) \to H^0(B, \mathscr{I}_S K_C)$ has rank $k + 1$.

Suppose at first that $B \neq C$ (in particular, C is not reduced). Consider the following exact sequence

$$0 \longrightarrow \omega_{C-B} \longrightarrow \mathscr{I}_S\omega_C \longrightarrow \mathscr{I}_S\omega_{C|B} \longrightarrow 0.$$

In particular,

$$h^0(C, \mathscr{I}_S K_C) = h^0(C-B, K_{C-B}) + \dim \operatorname{Im}\{r_B : H^0(C, \mathscr{I}_S K_C) \longrightarrow H^0(B, \mathscr{I}_S K_C)\}.$$

Since the restriction map r_B has rank $k+1$, then

$$h^0(C, \mathscr{I}_S K_C) = h^0(C-B, K_{C-B}) + k + 1.$$

Equations (3) and (11) imply that

$$M = k - \left(\frac{\deg K_{C|B}}{2} - \frac{\deg S}{2}\right) - \left(\frac{B\cdot(C-B)}{2} - h^0(C-B, \mathscr{O}_{C-B})\right).$$

If $k = 0$, that is, the cluster S is not splitting, every summand in the above formula cannot be positive since by Lemma 2.8 $(B\cdot(C-B))/2 - h^0(C-B, \mathscr{O}_{C-B}) \geqslant 0$. Thus, we have $M = 0$, $S = K_{C|B}$ and $(B\cdot(C-B))/2 = h^0(C-B, \mathscr{O}_{C-B})$ and, still by Lemma 2.8 we know that the curve C is not 3-connected.

If $k > 0$, assume C_{red} to be μ-connected but not $(\mu+1)$-connected. By Lemma 3.6, we know that $\deg K_{C|B} - \deg S \geqslant \max\{k, (\mu/2)(k+1)\}$, thus by Lemma 2.8

$$\begin{aligned} 0 \leqslant M &\leqslant \min\left\{\frac{k}{2}, \left(1 - \frac{\mu}{4}\right)k - \frac{\mu}{4}\right\} - \left(\frac{B\cdot(C-B)}{2} - h^0(C-B, \mathscr{O}_{C-B})\right) \\ &\leqslant \min\left\{\frac{k}{2}, \left(1 - \frac{\mu}{4}\right)k - \frac{\mu}{4}\right\}. \end{aligned}$$

Since M is non-negative, we have that $\mu \leqslant 3$.

If $\mu \geqslant 2$, then $\min\{k/2, (1-\mu/4)k - \mu/4\} = (1-\mu/4)k - \mu/4$ and

$$\begin{aligned} h^0(C, \mathscr{I}_S K_C) &\leqslant p_a(C) - \frac{\deg S}{2} + \left(1 - \frac{\mu}{4}\right)k - \frac{\mu}{4} \\ &\quad - \left(\frac{B\cdot(C-B)}{2} - h^0(C-B, \mathscr{O}_{C-B})\right) \\ &\leqslant p_a(C) - \frac{\deg S}{2} + \left(1 - \frac{\mu}{4}\right)k - \frac{\mu}{4} \end{aligned}$$

and if equality holds, then C is 3-disconnected thanks to Lemma 2.8. If C_{red} is 2-disconnected, that is, $\mu = 1$, we know that $\min\{k/2, (1-\mu/4)k - \mu/4\} = k/2$ and

$$\begin{aligned} h^0(C, \mathscr{I}_S K_C) &\leqslant p_a(C) - \frac{\deg S}{2} + \frac{k}{2} - \left(\frac{B\cdot(C-B)}{2} - h^0(C-B, \mathscr{O}_{C-B})\right) \\ &\leqslant p_a(C) - \frac{\deg S}{2} + \frac{k}{2} \end{aligned}$$

and if equality holds, then C is 3-disconnected.

We have still to study the case in which $B = C$. With the same argument, we have

$$M = k - \left(\frac{\deg K_C}{2} - \frac{\deg S}{2}\right).$$

We can argue as before: if $k = 0$ and $M \geqslant 0$, we have $S = K_C$, which is impossible since we asked S to be non-trivial. If $k > 0$ by Lemma 3.6 we conclude that $M \leqslant \min\{k/2, (1-\mu/4)k - \mu/4\}$ and C_{red} is not 4-connected. Moreover, if $M = k/2$, then $\deg K_C - \deg$

$S = k$. But $\deg S^* = \deg K_C - \deg S = k$ by its definition. This forces $h^0(C, \mathscr{I}_{S^*}K_C) = p_a(C) - \deg S^*/2 + k/2 = p_a(C)$, which is impossible since $k > 0$ and K_C ample. Thus, $M \leqslant \min\{k/2 - \frac{1}{2}, (1 - \mu/4)k - \mu/4\}$ and

$$h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \frac{\deg S}{2} + \left(1 - \frac{\mu}{4}\right) k - \frac{\mu}{4}$$

if $\mu = 2, 3$ while

$$h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \frac{\deg S}{2} + \frac{k}{2} - \frac{1}{2}$$

if $\mu = 1$.

Case 3 : S is a Cartier divisor, $\mathrm{Cliff}(\mathscr{I}_S K_C) = 0$ *and there exists a decomposition $C = A + B$ such that A and B have no common components, $S = K_{C|B}$ and $S^* = K_{C|A}$.*

If Case 1 and 2 do not hold, we may conclude by Lemma 3.1 that S and S^* are disjoint Cartier divisor, that their Clifford index is zero and that for every irreducible $\Gamma \subset C$, one of the restriction maps to $H^0(\Gamma, \mathscr{I}_S K_C)$ and $H^0(\Gamma, \mathscr{I}_{S^*}K_C)$ has rank 1.

If for an irreducible $\Gamma \subset C$ the restriction map $H^0(C, \mathscr{I}_S K_C) \to H^0(\Gamma, \mathscr{I}_S K_C)$ has rank 1, since $\mathscr{I}_S K_C$ is base point free by Proposition 2.19, then $S_{|\Gamma} = K_{C|\Gamma}$ and moreover $S_{|n\Gamma} = K_{C|n\Gamma}$ for Γ of multiplicity n since S is Cartier. Thus, $S^*_{|n\Gamma} = \emptyset$.

The same holds for S^*. Therefore, there exists a decomposition $C = A + B$ such that A and B have no common components, $S = K_{C|B}$ and $S^* = K_{C|A}$.

Note that in this case, since S and S^* are Cartier divisor with minimal Clifford index, by Proposition 2.19 we know that $|\mathscr{I}_S K_C|$ and $|\mathscr{I}_{S^*}K_C|$ are base point free, and in particular, the generic section does not pass through the singularities of C_{red}. Thus the splitting index k is 0.

In this situation, we consider the following exact sequences

$$0 \longrightarrow \omega_A \longrightarrow \mathscr{I}_S \omega_C \xrightarrow{r_B} \mathscr{O}_B \longrightarrow 0,$$
$$0 \longrightarrow \omega_B \longrightarrow \mathscr{I}_{S^*} \omega_C \xrightarrow{r_A} \mathscr{O}_A \longrightarrow 0.$$

Since $h^0(C, \mathscr{I}_S K_C) = h^0(A, K_A) + \mathrm{rank}(r_B)$ (and similarly for S^*), the conditions $\mathrm{Cliff}(\mathscr{I}_S K_C) = \mathrm{Cliff}(\mathscr{I}_{S^*}K_C) = 0$ imply that

$$h^0(A, \mathscr{O}_A) + \mathrm{rank}(r_B) = \frac{A \cdot B}{2} + 1,$$
$$h^0(B, \mathscr{O}_B) + \mathrm{rank}(r_A) = \frac{A \cdot B}{2} + 1.$$

Hence,

$$h^0(A, \mathscr{O}_A) + h^0(B, \mathscr{O}_B) + \mathrm{rank}(r_A) + \mathrm{rank}(r_B) = A \cdot B + 2. \tag{12}$$

Write $A = \sum_{i=1}^{h} A_i$ and $B = \sum_{j=1}^{l} B_j$ where the A_i and B_j are the topologically connected components of A and B, respectively.

By Lemma 3.4 and 3.5, we know that $\mathrm{rank}(r_A) \leqslant h$ and $\mathrm{rank}(r_B) \leqslant l$.

If $h^0(A, \mathscr{O}_A) \leqslant (A \cdot B)/2 - h$ and $h^0(B, \mathscr{O}_B) \leqslant (A \cdot B)/2 - l$, equation (12) implies that

$$A \cdot B + 2 \leqslant \frac{A \cdot B}{2} - h + \frac{A \cdot B}{2} - l + h + l,$$

which is impossible.

Thus, we have either $h^0(A, \mathscr{O}_A) > (A \cdot B)/2 - h$ or $h^0(B, \mathscr{O}_B) > (A \cdot B)/2 - l$. Let us suppose that the first inequality is true.

By Lemma 2.8, we have $h^0(A, \mathscr{O}_A) \leqslant (A \cdot B)/2 - (m-2)/2 \cdot h$ assuming C m-connected (with $m \geqslant 3$). Therefore, we know that C is 4-disconnected and moreover there must be a topologically connected component of A, say A_1, such that $h^0(A_1, \mathscr{O}_{A_1}) > (A_1 \cdot B)/2 - 1$.

If C is 3-connected, Lemma 2.8 says that $h^0(A_1, \mathscr{O}_{A_1}) \leqslant (A_1 \cdot B)/2 - \frac{1}{2}$ and we must conclude that $h^0(A_1, \mathscr{O}_{A_1}) = 1$ and $A_1 \cdot B = A_1 \cdot (C - A_1) = 3$. This forces $C - A_1$ to be 2-connected by Ciliberto, Francia and Mendes Lopes [**6**, Lemma A.4] and allows us to consider the subcanonical cluster $\tilde{S}^* := S^* \cap (C - A_1) = K_{(C-A_1)|(A-A_1)}$. It is

$$h^0(C, \mathscr{I}_{S^*} K_C) = h^0(C - A_1, \mathscr{I}_{\tilde{S}^*} K_{C-A_1}) + 1.$$

By an induction argument, we apply Clifford's theorem to the curve $C - A_1$ and the cluster $\tilde{S}^*$ which can be easily seen to be subcanonical for the system $|K_{(C-A_1)}|$ since $\mathscr{O}_{C-A_1} \subset \mathscr{I}_{\tilde{S}^*} K_{C-A_1}$. Moreover, the splitting index of $\tilde{S}^*$ is zero since it is clear that $H^0(C - A_1, \mathscr{I}_{\tilde{S}^*} K_{C-A_1})$ does not have any base point in $\mathrm{Sing}((C - A_1)_{\mathrm{red}})$. Thus, we have

$$h^0(C, \mathscr{I}_{S^*} K_C) = h^0(C - A_1, \mathscr{I}_{\tilde{S}^*} K_{C-A_1}) + 1 \leqslant p_a(C - A_1) - \frac{\deg(\tilde{S}^*)}{2} + 1.$$

Since $p_a(C - A_1) = p_a(C) - p_a(A_1) - 2$ and $\deg(\tilde{S}^*) = \deg(S^*) - (2p_a(A_1) + 1)$, we conclude that

$$\begin{aligned} h^0(C, \mathscr{I}_{S^*} K_C) &\leqslant (p_a(C) - p_a(A_1) - 2) - \frac{\deg(S^*)}{2} + \left(p_a(A_1) + \frac{1}{2}\right) + 1 \\ &= p_a(C) - \frac{\deg(S^*)}{2} - \frac{1}{2}. \end{aligned}$$

Therefore, $M = -\frac{1}{2}$, but we were asking $M \geqslant 0$, and hence C is 3-disconnected.

Step 2 : *Clusters of minimal Clifford index of any degree.* We deal now with the case of a *cluster* S *of minimal Clifford index, without any assumption on its degree.*

If there exists a non-trivial cluster with minimal non-positive Clifford index $S \in \Sigma_k$, there exists as well a non-trivial cluster $S_{\max}$ of maximal degree with the same Clifford index. In particular, a straightforward computation shows that the inequalities of the statement hold for $\mathscr{I}_S K_C$ if and only if they hold for $\mathscr{I}_{S_{\max}} K_C$, and similarly for the equalities.

We just showed that $\mathscr{I}_{S_{\max}} K_C$, and thus $\mathscr{I}_S K_C$ as well, satisfies the inequalities of the statement, hence proving the first part of the statement.

Moreover, if equality holds in equation (9) or in equation (10) for $\mathscr{I}_S K_C$ (and, equivalently, for $\mathscr{I}_{S_{\max}} K_C$), then the pair $(S_{\max}, C)$ satisfies one of the conditions listed in the statement. If C is 3-disconnected, there is nothing more to prove.

If, instead, C is 3-connected, then case (ii) must hold, and hence, C is honestly hyperelliptic. We can repeat verbatim the classical idea of Clifford's theorem for a smooth hyperelliptic curve of Saint Donat (see [**16**] or [**13**, Lemma IV.5.5]) and conclude that S is a multiple of a honest g^1_2. □

As a corollary we obtain the following result in which the computation of the splitting index, usually tricky, is avoided by the count of the number of irreducible components.

THEOREM 3.8. *Let* $C = \sum_{i=0}^{s} n_i \Gamma_i$ *be a projective curve either reduced with planar singularities or contained in a smooth algebraic surface with* $(s + 1)$ *irreducible components. Assume* C *to be* 2*-connected and let* $S \subset C$ *be a subcanonical cluster. Then*

$$h^0(\mathscr{I}_S K_C) \leqslant p_a(C) - \frac{1}{2}\deg(S) + \frac{s}{2}.$$

Proof. If follows immediately from Theorem 3.7 since the splitting index of every cluster is at most the number of irreducible components of C minus 1. □

If S is a Cartier divisor, we have the following theorem.

THEOREM 3.9. *Let C be a projective curve either reduced with planar singularities or contained in a smooth algebraic surface. Assume C to be 2-connected and let $S \subset C$ be a subcanonical Cartier cluster. Then*

$$h^0(C, \mathscr{I}_S K_C) \leqslant p_a(C) - \tfrac{1}{2}\deg(S).$$

Moreover, if equality holds, then the pair (S, C) satisfies one of the following assumptions:

(i) $S = 0$, K_C;
(ii) *C is honestly hyperelliptic and S is a multiple of the honest g^1_2;*
(iii) *C is 3-disconnected (that is, there is a decomposition $C = A + B$ with $A \cdot B = 2$).*

Proof. If S is not splitting, the results follows from Theorem 3.7. Thus, we can suppose that S has splitting index $k > 0$. By Proposition 2.17, we know that there is a decomposition $C_{\mathrm{red}} = \sum_{i=0}^k C_i$ such that every $H \in H^0(C, \mathscr{I}_S K_C)$ can be written as $H = \sum_{i=0}^k H_i$ with $H_i \in H^0(C, \mathscr{I}_S K_C)$ and $\operatorname{supp} H_i \subset C_i$.

In particular, every section $H \in H^0(C, \mathscr{I}_S K_C)$ vanishes on $C_i \cap C_j$ and we can decompose $H^0(C, \mathscr{I}_S K_C)_{|C_{\mathrm{red}}}$ as the direct sum of proper subspaces.

$$H^0(C, \mathscr{I}_S K_C)_{|C_{\mathrm{red}}} = \bigoplus_{i=0}^k H^0(C, \mathscr{I}_S K_C)_{|C_i}$$

such that the following diagram holds:

$$\begin{array}{ccccc}
\bigoplus_{i=0}^k H^0(C, \mathscr{I}_S K_C)_{|C_i} & \xrightarrow{\cong} & H^0(C, \mathscr{I}_S K_C)_{|C_{\mathrm{red}}} & & \\
\downarrow & & \downarrow & & \\
\bigoplus_{i=0}^k H^0(C_i, \mathscr{I}_S \cap \mathscr{I}_{C_i \cap (C_{\mathrm{red}} - C_i)} K_{C|C_i}) & \longrightarrow & H^0(C_{\mathrm{red}}, \mathscr{I}_S K_{C|C_{\mathrm{red}}}) & \longrightarrow & H^0(Z, \mathscr{O}_Z)
\end{array}$$

Since the map $\bigoplus_{i=0}^k \mathscr{I}_S \cap \mathscr{I}_{C_i \cap (C_{\mathrm{red}} - C_i)} \omega_{C|C_i} \to \mathscr{I}_S \omega_{C|C_{\mathrm{red}}}$ is generically an isomorphism, its cokernel is a skyscraper sheaf $\mathscr{O}_Z$. Since S is Cartier, it is not difficult to verify that $\mathscr{O}_Z$ is isomorphic, as sheaf on C_{red}, to the structure sheaf of the scheme $\bigcup_{i,j} C_i \cap C_j$, thus it has length $\frac{1}{2}\sum_{i=0}^k C_i \cdot (C_{\mathrm{red}} - C_i)$.

Let $\bar{S}$ be the base locus of $H^0(C, \mathscr{I}_S K_C)$. We have the following exact sequence

$$0 \longrightarrow \mathscr{I}_{\bar{S}} \longrightarrow \mathscr{I}_S \longrightarrow \mathscr{F} \longrightarrow 0$$

and $\mathscr{F} \cong \mathscr{O}_\xi$, where ξ is a cluster. It is clear from the above diagram that there is a natural surjective morphism $\mathscr{O}_\xi \twoheadrightarrow \mathscr{O}_Z$. In particular, the colength of $S \subset \bar{S}$ is at least $\frac{1}{2}\sum_{i=0}^k C_i \cdot (C_{\mathrm{red}} - C_i) \geqslant k$.

Since $H^0(C, \mathscr{I}_S K_C) = H^0(C, \mathscr{I}_{\bar{S}} K_C)$, the splitting index of $\hat{S}$ is still k and we can apply Theorem 3.7:

$$\begin{aligned}
h^0(C, \mathscr{I}_S K_C) = h^0(C, \mathscr{I}_{\bar{S}} K_C) &\leqslant p_a(C) - \frac{1}{2}\deg \bar{S} + \frac{k}{2} \\
&= p_a(C) - \frac{1}{2}\deg S - \frac{1}{2}\operatorname{colength}(\bar{S} \supset S) + \frac{k}{2} \\
&\leqslant p_a(C) - \frac{1}{2}\deg S - \frac{k}{2} + \frac{k}{2} = p_a(C) - \frac{1}{2}\deg S.
\end{aligned}$$

Note that if equality holds, $h^0(C, \mathscr{I}_{\bar{S}} K_C) = p_a(C) - \frac{1}{2}\deg \bar{S} + \frac{k}{2}$, and thus by Theorem 3.7 we know that one of the three cases listed ($\bar{S}$ trivial, or C honestly hyperelliptic, or C 3-disconnected) must hold. Since we are assuming that the splitting index k is strictly positive, we are forced to conclude that case (iii) of Theorem 3.7 holds, that is, C is 3-disconnected. □

Proof of Theorem A. It is a straightforward corollary of Theorem 3.7 if C_{red} is 4-connected; of Theorem 3.9 if S is Cartier; of Proposition 2.17 and Theorem 3.7 if there is a section $H \in H^0(C, \mathscr{I}_S K_C)$ avoiding the singularities of C_{red} since in this case S is not splitting. □

4. *Clifford's theorem for reduced curves*

In this section, we will prove Clifford's theorem for reduced 4-connected curves with planar singularities. Theorem B works under the assumptions that the sheaves $\mathscr{I}_S L$ and its dual $\mathscr{H}\mathrm{om}(\mathscr{I}_S L, \omega_C)$ are NEF.

In Theorem C, we deal with the case in which the second sheaf is not NEF. We split the curve in $C_0 + C_1$, where C_1 is the NEF part. It is still possible to find a Clifford type bound for $h^0(C, \mathscr{I}_S L)$ with a correction term which corresponds to a Riemann–Roch estimate over C_0. In the extremal case in which $C = C_0$, we recover Riemann–Roch theorem since $h^1(C, \mathscr{I}_S L) = 0$.

The inequality of Theorem C can be written also as

$$h^0(C, \mathscr{I}_S L) \leqslant \frac{\deg(I_S L)_{|C_1}}{2} + \deg(I_S L)_{|C_0} - \frac{\deg(K_C)_{|C_0}}{2}.$$

The following trivial remark will be useful in the proof of Theorems B and C.

Remark 4.1. Let C be a reduced projective curve with planar singularities. Let $C = A + B$ be an effective decomposition of C in non-trivial subcurves. Consider two rank 1 torsion-free sheaves $\mathscr{I}_{S_A} L_A$ and $\mathscr{I}_{S_B} L_B$ supported, respectively, on A and B with the property that $A \cap B \subset S_A$ and $A \cap B \subset S_B$. Then the sheaf on C defined as $\mathscr{I}_{S_A} L_A \oplus \mathscr{I}_{S_B} L_B$ is a rank 1 torsion-free sheaf as well, since the sheaves living on the two curves can be glued together as they both vanish on the intersection.

Proof of Theorem B. If $H^0(C, \mathscr{I}_S L) = 0$ or $H^1(C, \mathscr{I}_S L) = 0$, the result follows from Riemann–Roch theorem and the positivity of $\deg \mathscr{I}_S L$. We will assume from now on that both spaces are non-trivial.

We are going to show that if the sheaf $\mathscr{I}_S L$ attains the minimal Clifford index among the sheaves satisfying the assumption of Theorem B, then $\mathscr{I}_S L$ is a subcanonical sheaf.

With this aim we prove firstly that there exists an inclusion $\mathscr{O}_C \hookrightarrow \mathscr{I}_S L$ and secondly that there exists an inclusion $\mathscr{I}_S L \hookrightarrow \omega_C$.

If $\mathscr{O}_C \not\hookrightarrow \mathscr{I}_S L$, let $B \subset C$ be the maximal subcurve which annihilates every section in $H^0(C, \mathscr{I}_S L)$ and let $A = C - B$. Then there is a cluster S_A on A such that

$$0 \longrightarrow \mathscr{I}_{S_A} L_{|A}(-B) \longrightarrow \mathscr{I}_S L \longrightarrow (\mathscr{I}_S L)_{|B} \longrightarrow 0$$

and moreover there is an isomorphism between vector spaces:

$$H^0(A, \mathscr{I}_{S_A} L_{|A}(-B)) \cong H^0(C, \mathscr{I}_S L).$$

If $A \neq C$, consider the sheaf $\mathscr{F} = \mathscr{I}_{S_A} L_{|A}(-B) \oplus \mathscr{O}_B(A)(-A)$. By Remark 4.1, $\mathscr{F}$ is a rank 1 torsion-free sheaf and it is immediately seen that

$$0 \leqslant \deg \mathscr{F}_{|C_0} \leqslant K_{C|C_0} \quad \text{for every subcurve } C_0 \subset C.$$

Since $\mathrm{Cliff}(\mathscr{I}_S L)$ is minimum by our assumption then

$$\mathrm{Cliff}(\mathscr{I}_S L) \leqslant \mathrm{Cliff}(\mathscr{F}). \tag{13}$$

But, by our construction $h^0(C,\mathscr{F}) = h^0(A, \mathscr{I}_{S_A} L_{|A}(-B)) + h^0(B, \mathscr{O}_B)$ and by definition of degree we have

$$\begin{aligned}\deg(\mathscr{I}_S L) &= \chi(\mathscr{I}_S L) - \chi(\mathscr{O}_C) = \chi(\mathscr{I}_{S_A} L_{|A}(-B)) + \chi((\mathscr{I}_S L)_{|B}) - \chi(\mathscr{O}_C)\\ &= \chi(\mathscr{I}_{S_A} L_{|A}(-B)) + \deg((\mathscr{I}_S L)_{|B}) + \chi(\mathscr{O}_B) - \chi(\mathscr{O}_C)\\ &\geqslant \chi(\mathscr{I}_{S_A} L_{|A}(-B)) + \chi(\mathscr{O}_B) - \chi(\mathscr{O}_C) = \deg(\mathscr{F}).\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Cliff}(\mathscr{F}) &= \deg(\mathscr{F}) - 2h^0(C,\mathscr{F}) + 2\\ &\leqslant \deg(\mathscr{I}_S L) - 2h^0(A, \mathscr{I}_{S_A} L_{|A}(-B)) - 2h^0(B,\mathscr{O}_B) + 2\\ &\leqslant \deg(\mathscr{I}_S L) - 2h^0(A, \mathscr{I}_{S_A} L_{|A}(-B)) = \mathrm{Cliff}(\mathscr{I}_S L) - 2.\end{aligned}$$

This contradicts equation (13), hence $A = C$, that is, there exist sections not vanishing on any subcurve, or, equivalently, $\mathscr{O}_C \hookrightarrow \mathscr{I}_S L$.

Now we show that $\mathscr{I}_S L \hookrightarrow \omega_C$. The dual sheaf $\mathscr{H}om(\mathscr{I}_S L, \omega_C)$ satisfies the assumption of Theorem B and by Serre duality it has the same Clifford index of $\mathscr{I}_S L$, and hence, thanks to the previous step $\mathscr{O}_C \hookrightarrow \mathscr{H}om(\mathscr{I}_S L, \omega_C)$. In particular, $H^0(C,\mathscr{O}_C) \hookrightarrow H^0(C, \mathscr{H}om(\mathscr{I}_S L, \omega_C)) = \mathrm{Hom}(\mathscr{I}_S L, \omega_C)$. Hence, there is a map from $\mathscr{I}_S L$ to ω_C not vanishing on any component, and by automatic adjunction (Proposition 2.6) we may conclude that $\mathscr{I}_S L \hookrightarrow \omega_C$.

We proved that any sheaf $\mathscr{I}_S L$ with minimal Clifford index satisfies $\mathscr{O}_C \hookrightarrow \mathscr{I}_S L \hookrightarrow \omega_C$, and hence $\mathscr{I}_S L \cong \mathscr{I}_T \omega_C$, where T is a subcanonical cluster. But Theorem A holds for $\mathscr{I}_T \omega_C$, which concludes the proof. □

Remark 4.2. If $\mathscr{I}_S L$ is not isomorphic to a sheaf of the form $\mathscr{I}_T K_C$ for a subcanonical T, then the proof of Theorem B shows that we have the stricter inequality $h^0(C, \mathscr{I}_S L) \leqslant \deg(\mathscr{I}_S L)/2$.

Proof of Theorem C. If $H^0(C, \mathscr{I}_S L) = 0$ or $H^1(C, \mathscr{I}_S L) = 0$, the result follows from Riemann–Roch theorem and the positivity of $\deg \mathscr{I}_S L$. We will assume from now on that both spaces are non-trivial.

Let C_0 be the maximal subcurve such that

$$\deg[(\mathscr{I}_S L)]_{|B} > \deg K_{C|B}$$

for every subcurve $B \subset C_0$.

Consider a cluster T on C_0 such that $(\mathscr{I}_T \mathscr{I}_S L)_{|C_0} \overset{\text{num}}{\sim} K_{C|C_0}$. Such cluster must exist by our degree assumptions. The sheaf $\mathscr{I}_T \mathscr{I}_S L$ satisfies the assumptions of Theorem B, and thus,

$$h^0(C, \mathscr{I}_T \mathscr{I}_S L) \leqslant \frac{\deg \mathscr{I}_S L}{2} - \frac{\deg T}{2} + 1.$$

Moreover, if $\mathscr{I}_T \mathscr{I}_S L \cong \mathscr{I}_Z \omega_C$ with Z subcanonical cluster, we have

$$h^1(C, \mathscr{I}_T \mathscr{I}_S L) > h^1(C, \mathscr{I}_S L).$$

This follows from the analysis of the following commutative diagram:

$$\begin{array}{ccc} H^1(C,\mathscr{I}_S L)^* \cong \mathrm{Hom}_C(\mathscr{I}_S L, \omega_C) & \hookrightarrow & \mathrm{Hom}_C(\mathscr{I}_T \mathscr{I}_S L, \omega_C) \cong H^1(C, \mathscr{I}_T \mathscr{I}_S L)^* \\ \downarrow & & \downarrow r_0 \\ 0 = \mathrm{Hom}_{C_0}(\mathscr{I}_S L, \omega_C) & \longrightarrow & \mathrm{Hom}_{C_0}(\mathscr{I}_T \mathscr{I}_S L, \omega_C) \end{array}$$

We have that $\mathrm{Hom}_{C_0}(\mathscr{I}_S L, \omega_C) = H^1(C_0, \mathscr{I}_S L)^* = 0$ by Corollary 2.7. The map r_0 corresponds to the restriction map $H^0(C, \mathscr{I}_{Z^*} K_C) \to H^0(C_0, \mathscr{I}_{Z^*} K_C)$, which is non-zero since Z^* is subcanonical.

In this case, we may conclude since

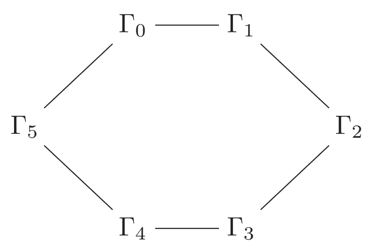
$$h^0(C, \mathscr{I}_S L) \leqslant h^0(C, \mathscr{I}_T \mathscr{I}_S L) + \deg T - 1 \leqslant \frac{\deg \mathscr{I}_S L}{2} + \frac{\deg(\mathscr{I}_S L - K_C)_{|C_0}}{2}.$$

If $\mathscr{I}_T \mathscr{I}_S L$ is not subcanonical, by Remark 4.2 it is $h^0(C, \mathscr{I}_T \mathscr{I}_S L) \leqslant \deg \mathscr{I}_S L/2 - \deg T/2$ and we have the same inequality. □

5. *Examples*

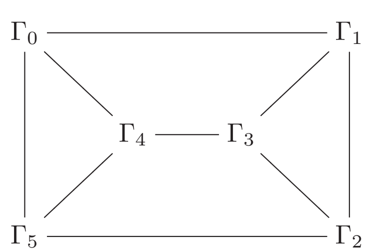
In this section, we will illustrate some examples in which the estimates of Theorem 3.7 and Theorems B and C are sharp. The first two examples concern Theorem 3.7 and show that the Clifford index can be negative when C_{red} is 4-disconnected. Examples 5.3 and 5.4 regard Theorem A and in particular they show how to build 3-disconnected curves and non-trivial and non-splitting subcanonical clusters with vanishing Clifford index. The final example (first given by Caporaso [**3**, Example 4.3.4]) shows a case in which $H^0(C, \mathscr{I}_S L) \neq 0$, $H^1(C, \mathscr{I}_S L) \neq 0$ and equality holds in Theorem C.

Example 5.1. Let $C = \sum_{i=0}^k \Gamma_i$ such that $\Gamma_i \cdot \Gamma_{i+1} = 1$, $\Gamma_0 \cdot \Gamma_k = 1$ and all the other intersection products are 0. Suppose that $p_a(\Gamma_i) \geqslant 2$. In the case $k = 5$, its dual graph is the following:



Take $S^* = \bigcup_{i,j}(\Gamma_i \cap \Gamma_j)$, which is a degree $(k+1)$ cluster. Since $\Gamma_i \cdot (C - \Gamma_i) = 2$ every section in $H^0(C, K_C)$ vanishing on a point of $\Gamma_i \cap (C - \Gamma_i)$ must vanish on the other, hence if a section $H^0(C, K_C)$ vanishes on any singular point of C, it must vanish on every singular point. In particular, $h^0(C, \mathscr{I}_{S^*} K_C) = p_a(C) - 1 = p_a(C) - \deg S^*/2 + k/2 - \frac{1}{2}$. It is clear that the splitting index of S^* is precisely k.

Example 5.2. Let $C = \sum_{i=0}^5 \Gamma_i$ and suppose that $p_a(\Gamma_i) \geqslant 2$. Suppose moreover that the intersection products are defined by the following dual graph, where the existence of the simple line means that the intersection product between the two curves is 1.

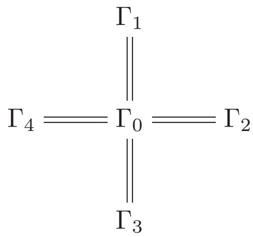


Take $S^* = \bigcup_{i,j}(\Gamma_i \cap \Gamma_j)$, which is a degree 9 cluster. It is easy to check that one can decompose $H^0(C, \mathscr{I}_{S^*}K_C) \cong \bigoplus_{i=0}^5 H^0(\Gamma_i, K_{\Gamma_i})$ and that the splitting index of S^* is $k = 5$. Thus, we have

$$h^0(C, \mathscr{I}_{S^*}K_C) = \sum_{i=0}^{s} p_a(\Gamma_i) = p_a(C) - \frac{\deg S^*}{2} + \frac{1}{2}$$

and note that $\frac{1}{2}$ is precisely $k/4 - \frac{3}{4}$, which means that equality can hold when C_{red} is 3-connected but 4-disconnected.

Example 5.3. Let $C = \Gamma_0 + \sum_{i=1}^n \Gamma_i$ with $\Gamma_0 \cdot \Gamma_i = 2$ for every $i \geqslant 1$ and $\Gamma_i \cdot \Gamma_j = 0$ for $i > j \geqslant 1$ (possibly $\Gamma_i = \Gamma_j$ for some i, j).



C is 2-connected but 3-disconnected. Taking $S = K_{C|C-\Gamma_0}$, we have

$$h^0(C, \mathscr{I}_S K_C) = h^0(\Gamma_0, K_{\Gamma_0}) + h^0(C - \Gamma_0, \mathscr{O}_{C-\Gamma_0}) = p_a(C) - \frac{\deg S}{2}$$

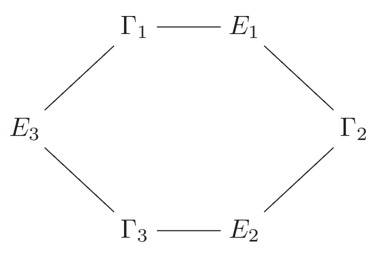
since $h^0(C - \Gamma_0, \mathscr{O}_{C-\Gamma_0}) = n = \Gamma_0 \cdot (C - \Gamma_0)/2$.

Example 5.4. Let $C = \Gamma_0 + \Gamma_1$ with Γ_1 irreducible and Γ_0 irreducible and hyperelliptic. Suppose that $\mathscr{O}_{\Gamma_0}(\Gamma_1)$ is a g^1_2 divisor on Γ_0.

Let S be another divisor in the linear series g^1_2 on Γ_0. Then $h^0(\Gamma_0, \mathscr{I}_S K_C) = p_a(\Gamma_0)$, and thus, $h^0(C, \mathscr{I}_S K_C) = p_a(\Gamma_1) + p_a(\Gamma_0) = p_a(C) - 1 = p_a(C) - \deg S/2$.

We believe that if C is 2-connected but 3-disconnected and $\text{Cliff}(\mathscr{I}_S K_C) = 0$ for a subcanonical non-splitting cluster S, then S must be the sum of clusters shaped as the two above, that is, a linear combination of a sum of g^1_2 plus a term of the form $K_{C|B}$ with $h^0(B, \mathscr{O}_B) = (B \cdot (C - B))/2$ or $h^0(C - B, \mathscr{O}_{C-B}) = (B \cdot (C - B))/2$.

Example 5.5. Let $C = \sum_{i=1}^k \Gamma_i + \sum_{j=1}^k E_j$, where $p_a(\Gamma_i) = 0$, $p_a(E_j) = 1$. Moreover, $\Gamma_i \cdot E_i = \Gamma_i \cdot E_{i-1} = \Gamma_1 \cdot E_k = 1$ and every other intersection number is 0.



Take a smooth point P_i over each Γ_i and consider the sheaf $L = \mathscr{O}_C(\sum_i P_i)$. Under the notation of Theorem C, we have that $C_0 = \sum \Gamma_i$ and $C_1 = \sum E_j$. A straightforward computation shows that $h^0(C, L) = k = \deg L/2 + \deg(L - K_C)_{|C_0}/2$.

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