Continuous LTI systems defined on L^p functions and $\mathscr{D}'_{L^p}$ distributions: analysis by impulse response and convolution

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***Abstract*—In this paper it is shown that every continuous LTI (linear time-invariant) system $\mathscr{L}$ defined either on L^p or on $\mathscr{D}'_{L^p}$ ($1 \leqslant p \leqslant \infty$) admits an impulse response $\Delta \in \mathscr{D}'_{L^{p'}}$ ($1 \leqslant p' \leqslant \infty,\ 1/p + 1/p' = 1$).**

Schwartz' extension to $\mathscr{D}'_{L^p}$ distributions of the usual notion of convolution product for L^p functions is used to prove that (apart some restrictions for $p = \infty$) for every f either in L^p or in $\mathscr{D}'_{L^p}$ we have $\mathscr{L}(f) = \Delta * f$.

Perspectives of applications to linear differential equations are shown by one example.

I. Introduction

IN SIGNAL processing theory, a linear, time-invariant (LTI), continuous-time system is a map

$$\mathscr{L} : \mathscr{I} \to \mathscr{O}$$

where: $\mathscr{I}$ (input space) and $\mathscr{O}$ (output space) are linear spaces of signals defined on $\mathbb{R}$, both closed under translation, and $\mathscr{L}$ is a linear map which commutes with translation. If moreover $\mathscr{I}$ and $\mathscr{O}$ are equipped with notions of convergence and limit for sequences (denoted $\mathscr{I}$-$\lim$ and $\mathscr{O}$-$\lim$ respectively) and for every $f \in \mathscr{I}$ and every *sequence* $f_k \in \mathscr{I}$ such that $\mathscr{I}\text{-}\lim_{k\to\infty} f_k = f$ it is $\mathscr{O}\text{-}\lim_{k\to\infty} \mathscr{L}(f_k) = \mathscr{L}(f)$, then $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is said to be *continuous*.

For continuous LTI systems a crucial role is plaid by the so called *impulse response*. In recent papers (see [1], [2] and [5]) Sandberg pointed out that, even if an impulse response Δ may be defined for $\mathscr{L}$, we cannot always expect that the knowledge of Δ shall determine the behavior of $\mathscr{L}$. Indeed he proved that there exist different continuous causal LTI systems $\mathscr{L}_1, \mathscr{L}_2 : \mathscr{C} \to \mathscr{C}$ with the same impulse response $\Delta = 0$, where $\mathscr{C}$ is the space of bounded uniformly continuous complex valued functions defined on $\mathbb{R}$.

Moreover in [1]–[6] Sandberg faced the problems of defining in a natural and correct way the impulse response in the setting of ordinary functions and how to represent a linear system via convolution.

In particular in [4] he showed that every continuous LTI system $\mathscr{L} : \mathscr{C}_0 \to L^\infty$ (where $\mathscr{C}_0$ denotes the space of continuous functions with limit 0 at ∞) admits a general representation as a uniform limit of a convolution, giving moreover necessary and sufficient conditions under which this limit reduces to an ordinary convolution. In [3], [4] Sandberg considered continuous LTI systems where the inputs are drawn from L^p (with $1 \leqslant p < \infty$) and the outputs are bounded functions, showing that an impulse response Δ always exists; moreover he pointed out that $\Delta \in L^{p'}$, where p' is the conjugate index of p, and that for every input $f \in L^p$ the corresponding output is given by $\Delta * f$, where "$*$" is the usual convolution of functions. Finally in [6] Sandberg considered also continuous LTI systems $\mathscr{L} : L^\infty \to L^\infty$ (where the input and output spaces are equipped with particular notions of convergence and limit for sequences); in the same reference he described $\mathscr{L}$ as an iterated limit of a convolution and he gave necessary and sufficient conditions under which this limit can be written as a convolution with an integrable impulse response function.

This paper and the previous one [7] must be considered as an attempt to face in the realm of distribution theory some of the problem posed by Sandberg.

Our analysis is based on the ideas and on the language of L. Schwartz, treated for instance in its classical book on distribution theory [8]. In particular, Thm. XXIII, Ch. 6 of [8] allowed us to start our study, proving that for every continuous LTI system $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ it can be defined an impulse response $\Delta \in \mathscr{D}'$, i.e., in the space of all distributions (for a brief survey on distributions see *Notation and conventions* at the end of this Section), as far as $\mathscr{I}$ and $\mathscr{O}$ verify two simple assumptions which merely exclude too strange input and output spaces (see [7], Section I for details). Our definition agrees with the notion of impulse response adopted by Sandberg in [1]–[6] and with $\mathscr{L}(\delta)$ whenever $\delta \in \mathscr{I}$ and $\mathscr{I}$ satisfies some natural conditions of convergence.

To be more precise Δ is the unique element in $\mathscr{D}'$ such that, for every sequence $\varphi_k \in \mathscr{D}$ with $\mathscr{E}'\text{-}\lim_{k\to\infty} \varphi_k = \delta$, it is $\Delta = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}(\varphi_k)$, where $\mathscr{E}'$ denotes the space of distributions with compact support (for a detailed description of Δ we refer to [7], Section II).

In [7] there are also shown the following results:

- $\mathscr{L}(f) = \Delta * f$ for every $f \in \mathscr{D}$, where $\mathscr{D}$ denotes the space of all C^∞ complex-valued functions defined on $\mathbb{R}$ with compact support and $*$ denotes the convolution product between a distribution and a test function (see [7], Theorem 2.1);
- if $\mathscr{L}_1 : \mathscr{I} \to \mathscr{O}$ is another continuous LTI system with

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the same impulse response Δ, then

$$\mathscr{L}_1(f) = \mathscr{L}(f) \quad \text{for every } f \in \Sigma(\mathscr{D}, \mathscr{I})$$

where $\Sigma(\mathscr{D}, \mathscr{I})$ is the set of all input signals related to $\mathscr{D}$ by limits of sequences (for the formal definition, see [7], Section III)

Since, apart few pathologies, we have $\Sigma(\mathscr{D}, \mathscr{I}) = \mathscr{I}$, our conclusion was then that, apart few pathologies, the impulse response Δ of a continuous LTI systems $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ uniquely determines $\mathscr{L}$. The spontaneous questions is: "When $f \in \mathscr{I}$ but $f \notin \mathscr{D}$, how can we obtain $\mathscr{L}(f)$ by means of Δ and f ?"

In this paper we focus on the usual Banach spaces L^p and on the distributional spaces $\mathscr{D}'_{L^p}$, where $\mathscr{D}'_{L^p}$ denotes the subspace of $\mathscr{D}'$ spanned by L^p itself and by the distributional derivatives (of any order) of its elements.

If $\mathscr{I} = L^p$ or $\mathscr{D}'_{L^p}$ in this paper we show that (apart pathologies in the case $p = \infty$) the behavior of $\mathscr{L}$ on all of $\mathscr{I}$ can be completely understood as a convolution product (in a suitable sense that will be clarified in Section IV) with the impulse response $\Delta \in \mathscr{D}'$. In this way we extend our description of $\mathscr{L}$ as convolution product with Δ from "smooth" signals $f \in \mathscr{D}$ to every signal $f \in \mathscr{I}$.

To be more precise in this paper we consider continuous LTI systems $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ where

- either $\mathscr{I} = L^p$ or $\mathscr{I} = \mathscr{D}'_{L^p}$, with $1 \leqslant p \leqslant \infty$
- $\mathscr{O} = \mathscr{D}'$

We point out that in L^p and $\mathscr{D}'$ we will consider the usual notions of convergence for sequences, while for $\mathscr{D}'_{L^p}$ we will always specify whether the weak convergence or the strong convergence has to be considered.

Notice that the choice $\mathscr{O} = \mathscr{D}'$ allows the widest possible range of behaviors for $\mathscr{L}$ (for instance it ensures that no continuous LTI system $L^p \to L^p$ or $L^p \to L^1_{\mathrm{loc}}$ is lost).

In Section II we get by an extremely technical proof that the impulse response Δ of $\mathscr{L}$ is in $\mathscr{D}'_{L^{p'}}$ where as usual $1/p + 1/p' = 1$.

In Section III we take into account the extension of the usual notion of convolution between L^p and L^q functions, to convolution between $\mathscr{D}'_{L^p}$ and $\mathscr{D}'_{L^q}$ distributions (see [8], Chapter VI, Section 1 and 8). In this way, given $\Delta \in \mathscr{D}'_{L^{p'}}$ the convolution $\Delta * f$ became meaningful for every $f \in \mathscr{D}'_{L^p}$. This allows us to introduce for every $\Delta \in \mathscr{D}'_{L^{p'}}$ a LTI system

$$\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'$$

defined by $\mathscr{L}_\Delta(f) = \Delta * f$. In Section IV we prove that $\mathscr{L}_\Delta$ is a continuous LTI system.

In Section V thanks to the comparison results obtained in [7], we compare $\mathscr{L}$ with $\mathscr{L}_\Delta$. As a corollary we give a complete analysis of $\mathscr{L}$ by means of its impulse response and the notion of convolution product.

Going in details, for $\mathscr{L} : L^p \to \mathscr{D}'$ we prove that

- $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$ for every $f \in L^p$ if $1 \leqslant p < \infty$
- $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$ for every $f \in \mathscr{C}_0$ (where $\mathscr{C}_0$ is the space of continuous functions null at infinity) if $p = \infty$

while for $\mathscr{L} : \mathscr{D}'_{L^p} \to \mathscr{D}'$ we prove that

- $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$ for every $f \in \mathscr{D}'_{L^p}$ if $1 \leqslant p < \infty$
- $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$ for every $f \in \mathscr{D}'_{L^\infty}$ if $p = \infty$ and $\mathscr{L}$ is continuous with respect to the weak convergence in $\mathscr{D}'_{L^\infty}$
- $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$ for every $f \in \dot{\mathscr{D}}'_{L^\infty}$ (where $\dot{\mathscr{D}}'_{L^\infty}$ is the space of distributions null at infinity) if $p = \infty$ and $\mathscr{L}$ is continuous with respect to the strong convergence in $\mathscr{D}'_{L^\infty}$

The most relevant consequence of these results is that, again except pathologies, the family of continuous LTI systems

$$\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}', \quad \Delta \in \mathscr{D}'_{L^{p'}}$$

coincides with the family of all continuous LTI systems defined on $\mathscr{D}'_{L^p}$, while its restriction to L^p coincides with the family of all continuous LTI systems defined on L^p.

Finally, in Section VI, perspectives of applications to linear differential equations are shown by an intentionally simple example.

Notation and conventions

To improve readability, we give here a brief survey of the spaces of distributions we use, and of the definitions we adopt.

$\mathscr{D}$ denotes the space of all C^∞ complex-valued functions defined on $\mathbb{R}$ with compact support. Given a sequence $\varphi_k \in \mathscr{D}$ and $\varphi \in \mathscr{D}$ we write $\mathscr{D}\text{-}\lim_{k\to\infty} \varphi_k = \varphi$ if there is a compact subset K of $\mathbb{R}$ such that $\operatorname{supp} \varphi_k \subset K$ for every k, and moreover for every $h \in \mathbb{N}$ the sequence $D^h\varphi_k$ converges to $D^h\varphi$ uniformly on $\mathbb{R}$.

A subset B of $\mathscr{D}$ is called *bounded* if there are a compact subset K of $\mathbb{R}$ and positive real numbers $M_0, M_1, \dots$ such that

$$\begin{aligned} \operatorname{supp} \varphi &\subset K && \text{for every } \varphi \in B \\ \sup\left\{ \left\|D^h\varphi\right\|_\infty : \varphi \in B \right\} &\leqslant M_h && \text{for every } h \in \mathbb{N} \end{aligned}$$

A linear functional $f : \mathscr{D} \to \mathbb{C}$ (as usual, $f(\varphi)$ is denoted by $\langle f, \varphi \rangle$) is called *continuous* if for every $\varphi \in \mathscr{D}$ and every sequence $\varphi_k \in \mathscr{D}$ such that $\mathscr{D}\text{-}\lim_{k\to\infty} \varphi_k = \varphi$ it is $\lim_{k\to\infty} \langle f, \varphi_k \rangle = \langle f, \varphi \rangle$. A continuous linear functional $f : \mathscr{D} \to \mathbb{C}$ is called a *distribution* on $\mathbb{R}$.

$\mathscr{D}'$ denotes the space of all distributions on $\mathbb{R}$. Every $f \in L^1_{\mathrm{loc}}$, i.e, every functions $f : \mathbb{R} \to \mathbb{C}$ which is integrable on every compact subset of $\mathbb{R}$, becomes a distribution on $\mathbb{R}$ by setting

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(t)\varphi(t)\mathrm{d}t$$

In $\mathscr{D}'$ two notions of convergence for sequences are considered: a weak convergence and a strong one. Given a sequence $f_k \in \mathscr{D}'$ and $f \in \mathscr{D}'$ we say that f_k *weakly* (resp. *strongly*) converges to f, and write

$$w\text{-}\mathscr{D}'\text{-}\lim_{k\to\infty} f_k = f \quad (\text{resp. } s\text{-}\mathscr{D}'\text{-}\lim_{k\to\infty} f_k = f)$$

if $\lim_{k\to\infty}\langle f_k,\varphi\rangle = \langle f,\varphi\rangle$ for every $\varphi\in\mathscr{D}$ (resp. for every $\varphi\in\widetilde{\mathscr{D}}$ and uniformly on every bounded subset B of $\mathscr{D}$). Obviously

$$s\text{-}\mathscr{D}'\text{-}\lim_{k\to\infty} f_k = f \Rightarrow w\text{-}\mathscr{D}'\text{-}\lim_{k\to\infty} f_k = f$$

It is worth to remark that a deep result (which holds only for sequences and no more for filters, see [8] Chapter III, Theorem XIII) proves that

$$w\text{-}\mathscr{D}'\text{-}\lim_{k\to\infty} f_k = f \iff s\text{-}\mathscr{D}'\text{-}\lim_{k\to\infty} f_k = f$$

As a consequence, for the convergence of sequences in $\mathscr{D}'$, the specifications "weak, strong" and the prefixes "w-, s-" will be omitted.

In order to handle linear changes of variables for distributions, we agree to denote an element $f\in\mathscr{D}'$ by a function-like symbol $f(t)$, so that the name "t" of the current variable is pointed out. In this way, for every pairs of real numbers $\lambda, a\in\mathbb{R}$ such that $\lambda\neq 0$, we denote by $f(\lambda t+a) = f(a+\lambda t)$ the distribution defined by

$$\langle f(\lambda t+a),\varphi(t)\rangle = |\lambda|^{-1}\langle f(t),\varphi\left(\lambda^{-1}(t-a)\right)\rangle$$

for every $\varphi\in\mathscr{D}$. In particular, for $\lambda = 1, a = -\tau$, we obtain $f(t-\tau)$ defined by

$$\langle f(t-\tau),\varphi(t)\rangle = \langle f(t),\varphi\left(t+\tau\right)\rangle$$

and, for $\lambda = -1, a = \tau$, we obtain $f(\tau-t)$ defined by

$$\langle f(\tau-t),\varphi(t)\rangle = \langle f(t),\varphi\left(\tau-t\right)\rangle$$

For every $f\in\mathscr{D}', \varphi\in\mathscr{D}$, the convolution $f*\varphi$ is the C^∞ function defined, for every $t\in\mathbb{R}$, by

$$(f*\varphi)(t) = \langle f(x),\varphi(t-x)\rangle = \langle f(t-x),\varphi(x)\rangle$$

Observe that, whenever f is a locally integrable function, this definition agrees with the usual definition

$$(f*\varphi)(t) = \int_{-\infty}^{+\infty} f(x)\varphi(t-x)\mathrm{d}x = \int_{-\infty}^{+\infty} f(t-x)\varphi(x)\mathrm{d}x$$

For every distribution $f(t)$, we denote by $\widetilde{f}(t)$ the distribution defined by $\widetilde{f}(t) = f(-t)$.

Now we illustrate the definition and some properties of the distributional spaces $\mathscr{D}'_{L^p}$.

For $1\leqslant p\leqslant\infty$, $\mathscr{D}'_{L^p}$ denotes the subspace of $\mathscr{D}'$ spanned by L^p itself and by the derivatives (of any order) of its elements. In particular every $f\in\mathscr{D}'_{L^p}$ may be written as a finite sum of the following form

$$f = \sum_h f_h^{(h)} \qquad \text{with } f_h\in L^p \text{ for every } h$$

where $f_h^{(h)}$ means distributional derivative of order h of the function f_h.

The meaning of $\mathscr{D}'_{L^p}$ rests on this definition: for instance, if L^p voltages across a capacitor are accepted, then also $\mathscr{D}'_{L^p}$ currents through the same capacitor must be accepted. For a deeper understanding and an easier handling, two other equivalent definitions of $\mathscr{D}'_{L^p}$ are needed.

Firstly, $\mathscr{D}'_{L^p}$ may be introduced as the space of distributions $f\in\mathscr{D}'$ such that, for every $\varphi\in\mathscr{D}$, it is $f*\varphi\in L^p$ (see [8], Chapter VI, Theorem XXV).

Secondly, just as $\mathscr{D}'$, also $\mathscr{D}'_{L^p}$ may be introduced via a duality pairing as a space of functionals as follows (see [8], Chapter VI, Sect. 8 and in particular Theorem XXV).

For $1\leqslant p\leqslant\infty$, let $\mathscr{D}_{L^p}$ be the space of all C^∞ complex-valued functions φ defined on $\mathbb{R}$, such that $D^h\varphi\in L^p$ for every $h\in\mathbb{N}$. Given a sequence φ_k of members of $\mathscr{D}_{L^p}$, and $\varphi\in\mathscr{D}_{L^p}$ we write $\mathscr{D}_{L^p}\text{-}\lim_{k\to\infty}\varphi_k = \varphi$ if for every $h\in\mathbb{N}$ the sequence $D^h\varphi_k$ converges to $D^h\varphi$ in L^p. For $p=\infty$, $\dot{\mathscr{D}}_{L^\infty}$ denotes the subspace of $\mathscr{D}_{L^\infty}$, whose elements are the φ such that $\lim_{|t|\to\infty} D^h\varphi(t) = 0$ for every $h\in\mathbb{N}$, equipped with a similar notion of convergence and $\dot{\mathscr{D}}_{L^\infty}\text{-}\lim$ for sequences. We point out that if $1<p<q<\infty$ there are the following inclusions

$$\mathscr{D}_{L^1}\subset\mathscr{D}_{L^p}\subset\mathscr{D}_{L^q}\subset\dot{\mathscr{D}}_{L^\infty}\subset\mathscr{D}_{L^\infty}$$

A subset B of $\mathscr{D}_{L^p}$ is called *bounded* if there are positive real numbers $M_0, M_1, \ldots$ such that for every $h\in\mathbb{N}$ it is

$$\sup\left\{\|D^h\varphi\|_p : \varphi\in B\right\}\leqslant M_h$$

For $p=\infty$, *bounded* subsets of $\dot{\mathscr{D}}_{L^\infty}$ have a similar definition.

For $1 < p\leqslant\infty$, $\mathscr{D}'_{L^p}$ is the space of linear and continuous functional f from $\mathscr{D}_{L^{p'}}$ into $\mathbb{C}$, where p' is defined by $1/p'+1/p = 1$. For $p=1$, $\mathscr{D}'_{L^1}$ is the space of linear and continuous functionals from $\dot{\mathscr{D}}_{L^\infty}$ into $\mathbb{C}$. As usual, for every $f\in\mathscr{D}'_{L^p}, \varphi\in\mathscr{D}_{L^{p'}}$ if $1<p\leq\infty$ and for every $f\in\mathscr{D}'_{L^1}$, $\varphi\in\dot{\mathscr{D}}_{L^\infty}$ if $p=1$ the complex number $f(\varphi)$ is denoted by $\langle f,\varphi\rangle$, and whenever $f\in L^p$ it is

$$\langle f,\varphi\rangle = \int_{-\infty}^{+\infty} f(t)\varphi(t)\mathrm{d}t$$

Notice that also in $\mathscr{D}'_{L^p}$ two notions of convergence for sequences need to be considered, a *weak* and a *strong*. Now we illustrate what do they mean.

Let $1 < p\leqslant\infty$. Given a sequence $f_k\in\mathscr{D}'_{L^p}$ and $f\in\mathscr{D}'_{L^p}$ we say that f_k *weakly* (resp. *strongly*) converges to f, and write

$$w\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty} f_k = f \quad (\text{resp. } s\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty} f_k = f)$$

if $\lim_{k\to\infty}\langle f_k,\varphi\rangle = \langle f,\varphi\rangle$ for every $\varphi\in\mathscr{D}_{L^{p'}}$ (resp. for every $\varphi\in\widetilde{\mathscr{D}}_{L^{p'}}$ and uniformly on every bounded subset B of $\mathscr{D}_{L^{p'}}$). Weak and strong convergence for sequences in $\mathscr{D}'_{L^1}$, have similar definitions by using $\varphi\in\dot{\mathscr{D}}_{L^\infty}$ and bounded subsets of $\dot{\mathscr{D}}_{L^\infty}$. We remark that the implication $s\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty} f_k = f \Rightarrow w\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty} f_k = f$ still holds, but there are weakly convergent sequences which are not strongly convergent (for instance, the sequence $f_k(t) = \delta(t-k)$ is weakly —but not strongly— convergent to 0).

Thus for LTI systems $\mathscr{L}:\mathscr{D}'_{L^p}\to\mathscr{D}'$, we have to consider both notions of convergence. A system $\mathscr{L}$ is said to be *weakly continuous* (resp. *strongly continuous*) if it is continuous with

respect to the weak convergence (resp. strong convergence) in the input space.

Notice that, despite their name, if $\mathscr{L} : \mathscr{D}'_{L^p} \to \mathscr{D}'$ is weakly continuous then $\mathscr{L} : \mathscr{D}'_{L^p} \to \mathscr{D}'$ is strongly continuous.

Finally $\dot{\mathscr{D}}'_{L^\infty}$ denotes the space of distributions converging to 0 at infinity, i.e., of the distributions f such that $\mathscr{D}'\text{-}\lim_{|\tau|\to\infty} f(t-\tau) = 0$ (see [8], Chapter VI, Section 8). We point out that if $1 < p < q < \infty$ then there are the following inclusions

$$\mathscr{D}'_{L^1} \subset \mathscr{D}'_{L^p} \subset \mathscr{D}'_{L^q} \subset \dot{\mathscr{D}}'_{L^\infty} \subset \mathscr{D}'_{L^\infty}$$

II. Impulse response of a continuous LTI system defined on L^p and on $\mathscr{D}'_{L^p}$

Let $1 \leqslant p \leqslant \infty$, let (as usual) p' be defined by

$$1 \leqslant p' \leqslant \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

and let us consider continuous LTI systems $\mathscr{L} : \mathscr{I} \to \mathscr{D}'$ where either $\mathscr{I} = L^p$ or $\mathscr{I} = \mathscr{D}'_{L^p}$. In both cases input and output spaces verify *Assumptions* 1,2 of [7], hence by the theory developed in Section II of [7] there exists the impulse response $\Delta \in \mathscr{D}'$.

In this Section we prove that we can say much more about the nature of Δ, namely we prove that Δ is an element of $\mathscr{D}'_{L^{p'}}$.

Before proving this result, we need a lemma. The extremely technical proof we give here, is inspired by ideas and tools developed by Schwartz in Sections 7 and 8 of Chapter VI of [8].

Lemma 2.1: Let $1 \leqslant q \leqslant \infty$, and let $\Upsilon \in \mathscr{D}'$ be a distribution such that for every $\varphi \in \mathscr{D}$ it is $\Upsilon * \varphi \in \mathscr{D}'_{L^q}$. Then it is $\Upsilon \in \mathscr{D}'_{L^q}$.

Proof: The proof will be given in various steps, each with its own proof.

Let $K = [-1, 1]$. The symbol $\mathscr{D}_K$ denotes the space of C^∞ functions whose support is a subset of K. It is well known (see Chapter 7, Section 2 of [9]) that $\mathscr{D}_K$ is a Fréchet space with respect to the family of seminorms

$$p_m(\varphi) = \sup\left\{\|\varphi^{(0)}\|_\infty, \ldots, \|\varphi^{(m)}\|_\infty\right\}$$

with $m \in \mathbb{N}$. A fundamental set of open neighborhoods of 0 is given by

$$\left\{\varphi \in \mathscr{D}_K : \|\varphi^{(0)}\|_\infty, \ldots, \|\varphi^{(m)}\|_\infty \ < \ \varepsilon\right\}$$

with $m \in \mathbb{N}, \varepsilon \ > \ 0$.

For $m \in \mathbb{N}$, the symbol $\mathscr{D}^m_K$ denotes the space of C^m functions whose support is a subset of K. $\mathscr{D}^m_K$ is a Banach, hence Fréchet, space with respect to the norm

$$p(\varphi) = \sup\left\{\|\varphi^{(0)}\|_\infty, \ldots, \|\varphi^{(m)}\|_\infty\right\}$$

A fundamental set of open neighborhoods of 0 is given by

$$\left\{\varphi \in \mathscr{D}^m_K : \|\varphi^{(0)}\|_\infty, \ldots, \|\varphi^{(m)}\|_\infty \ < \ \varepsilon\right\}$$

with $\varepsilon \ > \ 0$.

Step 1. For every $\alpha, \beta \in \mathscr{D}$ it is

$$\Upsilon * \alpha * \beta \in L^q$$

Moreover the bilinear map

$$\xi : \mathscr{D}_K \times \mathscr{D}_K \to L^q$$

defined by $\xi(\alpha, \beta) = \Upsilon * \alpha * \beta$, for every fixed value of one variable is continuous with respect to the other variable.

Proof of Step 1. By the assumption on Υ, it is $\Upsilon * \alpha \in \mathscr{D}'_{L^q}$; hence by Theorem XXV, Chapter VI, Section 8 of [8] it is $\Upsilon * \alpha * \beta \in L^q$.

To prove the second statement, let $\alpha \in \mathscr{D}_K$, and let $F = \Upsilon * \alpha \in \mathscr{D}'_{L^q}$. By definition F may be written in the form $F = \sum_{h=0}^{r} f_h^{(h)}$, with $f_h \in L^q$ for every h. As a consequence, for every $\beta \in \mathscr{D}_K$ it is

$$\|\xi(\alpha,\beta)\|_q = \|F * \beta\|_q = \|\sum_{h=0}^{r} f_h * \beta^{(h)}\|_q \leqslant \sum_{h=0}^{r} \|f_h * \beta^{(h)}\|_q$$

By Young's Theorem it is $\|f_h * \beta^{(h)}\|_q \leqslant \|f_h\|_q \cdot \|\beta^{(h)}\|_1$. Hence $\|\xi(\alpha,\beta)\|_q \leqslant 2\sum_{h=0}^{r} \|f_h\|_q \cdot \|\beta^{(h)}\|_\infty$.

End of Proof of Step 1

Step 2. The bilinear map

$$\xi : \mathscr{D}_K \times \mathscr{D}_K \to L^q$$

defined by $\xi(\alpha, \beta) = \Upsilon * \alpha * \beta$ is continuous.

As a consequence there exist $m_0 \in \mathbb{N}, \varepsilon_0 \ > \ 0$ such that, defining

$$U_0 = \left\{\varphi \in \mathscr{D}_K : \|\varphi^{(0)}\|_\infty, \ldots, \|\varphi^{(m_0)}\|_\infty \ < \ \varepsilon_0\right\}$$

for every $\alpha, \beta \in U_0$ it is

$$\|\xi(\alpha,\beta)\|_q = \|\Upsilon * \alpha * \beta\|_q \leqslant 1$$

Proof of Step 2. Since $\mathscr{D}_K$ and L^q are Fréchet spaces, Step 1 allows us to apply Corollary 1 of Chapter III, Section 5 of [10] to ξ. *End of Proof of Step* 2

Let m_0, ε_0, U_0 be as in Step 2. The family

$$\frac{\varepsilon}{\varepsilon_0} U_0, \quad \varepsilon \ > \ 0$$

is a fundamental set of open neighborhoods of 0 for the topology induced by $\mathscr{D}^{m_0}_K$ in $\mathscr{D}_K$.

Step 3. Let $\varepsilon_1, \varepsilon_2 \ > \ 0$. Then for every $\alpha \in \frac{\varepsilon_1}{\varepsilon_0} U_0$, and for every $\beta \in \frac{\varepsilon_2}{\varepsilon_0} U_0$ it is

$$\|\xi(\alpha,\beta)\|_q = \|\Upsilon * \alpha * \beta\|_q \leqslant \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_0^2}$$

Proof of Step 3. Since $\frac{\varepsilon_0}{\varepsilon_1}\alpha, \frac{\varepsilon_0}{\varepsilon_2}\beta \in U_0$, by Step 2 it is

$$\|\xi(\alpha,\beta)\|_q = \|\Upsilon * \alpha * \beta\|_q = \frac{\varepsilon_1\varepsilon_2}{\varepsilon_0^2} \|\Upsilon * \left(\frac{\varepsilon_0}{\varepsilon_1}\alpha\right) * \left(\frac{\varepsilon_0}{\varepsilon_1}\beta\right)\|_q \leqslant \frac{\varepsilon_1\varepsilon_2}{\varepsilon_0^2}$$

End of Proof of Step 3

Let $\mathscr{D}^{m_0}_{(-1,1)}$ be the space of the C^{m_0} functions whose support is a subset of the open interval $(-1, 1)$.

Step 4. For every $\alpha, \beta \in \mathscr{D}^{m_0}_{(-1,1)}$ it is

$$\Upsilon * \alpha * \beta \in L^q$$

Proof of Step 4. By the assumption, there exists $\rho > 0$ such that $(\operatorname{supp} \alpha) + [-\rho, \rho] \subset K, (\operatorname{supp} \beta) + [-\rho, \rho] \subset K$. Let $\varphi \in \mathscr{D}$ be such that $\operatorname{supp} \varphi \subset [-\rho, \rho], \varphi(t) \geqslant 0$ for every $t \in \mathbb{R}$, $\int \varphi = 1$. For $j \geqslant 1$ let $\varphi_j(t) = j\varphi(jt)$, and let

$$\alpha_j = \alpha * \varphi_j, \ \beta_j = \beta * \varphi_j \in \mathscr{D}_K$$

For every $0 \leqslant h \leqslant m_0$ it is $\alpha^{(h)}, \beta^{(h)} \in C^0$, hence

$$\alpha_j^{(h)} = \alpha^{(h)} * \varphi_j \text{ converges to } \alpha^{(h)}$$

$$\beta_j^{(h)} = \beta^{(h)} * \varphi_j \text{ converges to } \beta^{(h)}$$

uniformly on K. Hence, in the space $\mathscr{D}^{m_0}_K$, α_j converges to α and β_j converges to β. As a consequence α_j, β_j are Cauchy sequences in $\mathscr{D}_K$ with respect to the topology induced by $\mathscr{D}^{m_0}_K$; by Step 3 it is easily seen that $\xi(\alpha_j, \beta_j) = \Upsilon * \alpha_j * \beta_j$ is a Cauchy sequence in L^q. Since L^q is a Banach space, there exists $f \in L^q$ such that $L^q\text{-}\lim_{j\to\infty} \Upsilon * \alpha_j * \beta_j = f$ and hence such that

$$\mathscr{D}'\text{-}\lim_{j\to\infty} \Upsilon * \alpha_j * \beta_j = f$$

Since $\mathscr{E}'\text{-}\lim_{j\to\infty} \varphi_j * \varphi_j = \delta * \delta = \delta$, it is also

$$\mathscr{D}'\text{-}\lim_{j\to\infty} \Upsilon * \alpha_j * \beta_j = \mathscr{D}'\text{-}\lim_{j\to\infty} (\Upsilon * \alpha * \beta) * (\varphi_j * \varphi_j) = \Upsilon * \alpha * \beta$$

As a consequence $\Upsilon * \alpha * \beta = f$; hence $\Upsilon * \alpha * \beta \in L^q$. *End of Proof of Step* 4

Step 5. There exist $\alpha_0, \beta_0 \in \mathscr{D}^{m_0}_{(-1,1)}$ such that

$$\delta = \alpha_0 + D^{m_0+2}\beta_0$$

Proof of Step 5. Let $\mathscr{D}_{(-1,1)}$ be the space of C^∞ functions whose support is a subset of the open interval $(-1, 1)$. Let $\gamma \in \mathscr{D}_{(-1,1)}$ be such that $\gamma(t) = 1$ for every $t \in (-1/2, 1/2)$; let $H(t)$ be the Heaviside function; and let

$$B_0(t) = \frac{t^{m_0+1}}{(m_0+1)!} H(t) \in C^{m_0}$$

Then

$$\beta_0 = \gamma B_0 \in \mathscr{D}^{m_0}_{(-1,1)}$$

$$\alpha_0 = -\sum_{h=0}^{m_0+1} \binom{m_0+2}{h} \gamma^{(m_0+2-h)} B_0^{(h)} \in \mathscr{D}_{(-1,1)} \subset \mathscr{D}^{m_0}_{(-1,1)}$$

verify the statement. *End of Proof of Step* 5

We can now prove that $\Upsilon \in \mathscr{D}'_{L^q}$. *Indeed:* Since $\delta = \alpha_0 + D^{m_0+2}\beta_0$, then

$$\delta = \delta * \delta = \alpha_0 * \alpha_0 + 2D^{m_0+2}(\alpha_0 * \beta_0) + D^{2m_0+4}(\beta_0 * \beta_0)$$

As a consequence

$$\Upsilon = \Upsilon * \delta = \Upsilon * \alpha_0 * \alpha_0 + 2D^{m_0+2}(\Upsilon * \alpha_0 * \beta_0) + D^{2m_0+4}(\Upsilon * \beta_0 * \beta_0)$$

Since $\alpha_0, \beta_0 \in \mathscr{D}^{m_0}_{(-1,1)}$, by Step 4 we obtain that Υ is a finite sum of derivatives of L^q functions; hence, by definition, $\Upsilon \in \mathscr{D}'_{L^q}$. ∎

We can now prove the result on Δ for continuous LTI system $\mathscr{L} : L^p \to \mathscr{D}'$.

Theorem 2.1: Let $\mathscr{L} : L^p \to \mathscr{D}'$ be a continuos LTI system, and let $\Delta \in \mathscr{D}'$ be its impulse response. Then $\Delta \in \mathscr{D}'_{L^{p'}}$.

Proof: Let $\varphi \in \mathscr{D}$, and let

$$\Phi_\varphi : \begin{cases} \mathscr{D}_{L^p} \to \mathbb{C} & \text{for } 1 \leqslant p < \infty \\ \dot{\mathscr{D}}_{L^\infty} \to \mathbb{C} & \text{for } p = \infty \end{cases}$$

be the linear functional defined by

$$\Phi_\varphi(f) = \langle \mathscr{L}(f), \widetilde{\varphi} \rangle$$

where, as pointed out in *Notation and conventions* (Section I), $\widetilde{\varphi}(t) = \varphi(-t)$.

Obviously for every sequence f_k converging to an f in the domain of Φ_φ it is also $L^p\text{-}\lim_{k\to\infty} f_k = f$; as a consequence, $\mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}(f_k) = \mathscr{L}(f)$, and hence $\lim_{k\to\infty} \langle \mathscr{L}(f_k), \widetilde{\varphi} \rangle = \langle \mathscr{L}(f), \widetilde{\varphi} \rangle$; hence Φ_φ is continuous. As a consequence there exists $\Gamma_\varphi \in \mathscr{D}'_{L^{p'}}$ such that

$$\Phi_\varphi(f) = \langle \Gamma_\varphi, f \rangle \text{ for every } f \in \begin{cases} \mathscr{D}_{L^p} & \text{if } 1 \leqslant p < \infty \\ \dot{\mathscr{D}}_{L^\infty} & \text{if } p = \infty \end{cases}$$

In particular, for every $\psi \in \mathscr{D}$, since by Theorem 2.1 of [7] it is $\mathscr{L}(\psi) = \Delta * \psi$, it is also

$$\langle \Delta * \psi, \widetilde{\varphi} \rangle = \langle \Gamma_\varphi, \psi \rangle$$

Observe that $\Delta * \psi \in C^\infty, \varphi \in \mathscr{D}$; hence

$$\langle \Delta * \psi, \widetilde{\varphi} \rangle = \int (\Delta * \psi)(\tau) \widetilde{\varphi}(\tau) \mathrm{d}\tau = \int (\Delta * \psi)(\tau) \varphi(0 - \tau) \mathrm{d}\tau = \big((\Delta * \psi) * \varphi\big)(0) =$$

$$\big((\Delta * \varphi) * \psi\big)(0) = \langle \Delta * \varphi, \widetilde{\psi} \rangle = \langle \widetilde{\Delta * \varphi}, \psi \rangle$$

As a consequence $\widetilde{\Delta * \varphi} = \Gamma_\varphi \in \mathscr{D}'_{L^{p'}}$, hence $\Delta * \varphi = \widetilde{\Gamma_\varphi} \in \mathscr{D}'_{L^{p'}}$.

Applying Lemma 2.1 to $\Upsilon = \Delta$ and $q = p'$, we obtain $\Delta \in \mathscr{D}'_{L^{p'}}$. ∎

Now we also prove the result on Δ for continuous LTI system $\mathscr{L} : \mathscr{D}'_{L^p} \to \mathscr{D}'$.

Theorem 2.2: Let $\mathscr{L} : \mathscr{D}'_{L^p} \to \mathscr{D}'$ be a weakly (resp. strongly) continuous LTI system, and let $\Delta \in \mathscr{D}'$ be its impulse response. Then $\Delta \in \mathscr{D}'_{L^{p'}}$.

Proof: Since weak continuity implies strong continuity, it is sufficient to prove the statement for strongly continuous systems; hence we assume that $\mathscr{L}$ is strongly continuous.

Let $\varphi \in \mathscr{D}$, and let

$$\Phi_\varphi : \begin{cases} \mathscr{D}_{L^p} \to \mathbb{C} & \text{for } 1 \leqslant p < \infty \\ \dot{\mathscr{D}}_{L^\infty} \to \mathbb{C} & \text{for } p = \infty \end{cases}$$

be the linear functional defined by

$$\Phi_\varphi(f) = \langle \mathscr{L}(f), \widetilde{\varphi} \rangle$$

Hölder's Inequality easily proves that for every sequence f_k converging to an f in the domain of Φ_φ it is also $s\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty} f_k = f$; hence, as in the Proof of Theorem 2.1, Φ_φ is continuous. As a consequence there exists $\Gamma_\varphi \in \mathscr{D}'_{L^{p'}}$ such that

$$\Phi_\varphi(f) = \langle \Gamma_\varphi, f\rangle \text{ for every } f \in \begin{cases} \mathscr{D}_{L^p} & \text{if } 1 \leqslant p < \infty \\ \dot{\mathscr{D}}_{L^\infty} & \text{if } p = \infty \end{cases}$$

Proceeding as in the Proof of Theorem 2.1, we obtain $\Delta \in \mathscr{D}'_{L^{p'}}$. ∎

Concerning LTI systems defined on $\mathscr{D}'_{L^1}$ we can say something more, which will turn out to be very useful in the next section. Our result is the following

Theorem 2.3: Let $\mathscr{L} : \mathscr{D}'_{L^1} \to \mathscr{D}'$ be a weakly continuous LTI system. Then $\Delta \in \dot{\mathscr{D}}'_{L^\infty}$.

Proof: Assume the contrary. By definition the statement

$$\mathscr{D}'\text{-}\lim_{|h|\to\infty} \Delta(t-h) = 0$$

is false; hence there exist $\varphi \in \mathscr{D}$ and a sequence $h_j \in \mathbb{R}$ with $\lim_{j\to\infty} h_j = \infty$, such that the statement

$$\lim_{j\to\infty} \langle \Delta(t-h_j), \varphi(t)\rangle = 0$$

is false.

Observe that $w\text{-}\mathscr{D}'_{L^1}\text{-}\lim_{j\to\infty} \delta(t-h_j) = 0$. Since $\mathscr{L}$ is weakly continuous, we have

$$\mathscr{D}'\text{-}\lim_{j\to\infty} \Delta(t-h_j) = \mathscr{D}'\text{-}\lim_{j\to\infty} \mathscr{L}(\delta(t-h_j)) = 0$$

and hence it is $\lim_{j\to\infty} \langle \Delta(t-h_j), \varphi(t)\rangle = 0$: absurd. ∎

III. Extension of Young's Theorem to distributions: convolution in $\mathscr{D}'_{L^p}$ spaces

In this Section we recall Schwartz' extension to $\mathscr{D}'_{L^p}$ spaces of the usual notion of convolution product defined for L^p functions. Convolution so extended is obviously commutative and, by Schwartz' results has a good behavior on strongly convergent sequences. We give here an easy proof, in a very general set up, that convolution so extended is also associative, and use this property to show that it has a good behavior even on weakly convergent sequences.

Let $1 \leqslant p, q \leqslant \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} - 1 \geqslant 0$$

and let r be defined by

$$1 \leqslant r \leqslant \infty, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

Let $f \in \mathscr{D}'_{L^p},\ g \in \mathscr{D}'_{L^q}$. By definition f and g may be written as finite sums of the form

$$f = \sum_h f_h^{(h)} \quad \text{with} \quad f_h \in L^p \text{ for every } h$$

$$g = \sum_k g_k^{(k)} \quad \text{with} \quad g_k \in L^q \text{ for every } k$$

By the classical Young's Theorem for integrable functions, for every h, k we have

- the function
$$(f_h * g_k)(t) = \int_{\mathbb{R}} f_h(t-\tau) g_k(\tau) \mathrm{d}\tau$$
is defined for almost all $t \in \mathbb{R}$
- $f_h * g_k \in L^r$
- $\|f_h * g_k\|_r \leqslant \|f_h\|_p \cdot \|g_k\|_q$

As a consequence the *convolution* of f and g may then be defined by

$$f * g = \sum_{h,k} (f_h * g_k)^{(h+k)} \in \mathscr{D}'_{L^r}$$

By [8], Chapter VI, Section 8, Thm. XXVI, this is a good definition, and it agrees with other usual settings in which convolution is already defined.

Commutativity of convolution is obvious. Concerning the behavior of convolution on strongly convergent sequences, by the above mentioned reference[1] we immediately obtain the following result.

Theorem 3.1: Let $g = s\text{-}\mathscr{D}'_{L^q}\text{-}\lim_{k\to\infty} g_k$. Then we have $f * g = s\text{-}\mathscr{D}'_{L^r}\text{-}\lim_{k\to\infty} f * g_k$.

Concerning associativity it is well known that if $f \in L^p, g \in L^q, h \in L^s$ with

$$p = q = s = 1 \qquad \text{or} \qquad p = 1, \quad s = q'$$

then

$$(f * g) * h = f * (g * h)$$

(see [11], Chapter III, Section 11).

Here we extend this result to every $f \in \mathscr{D}'_{L^p},\ g \in \mathscr{D}'_{L^q},\ h \in \mathscr{D}'_{L^s}$, where

$$\frac{1}{p} + \frac{1}{q} - 1 \geqslant 0 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{s} - 2 \geqslant 0$$

To this aim, firstly we show the origin of this condition and then we give the complete proof.

So, as above, let p, q be such that

$$\frac{1}{p} + \frac{1}{q} - 1 \geqslant 0$$

let $f \in \mathscr{D}'_{L^p}, g \in \mathscr{D}'_{L^q}$, and let r be defined by

$$1 \leqslant r \leqslant \infty \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

then

- $f * g$ is defined
- $f * g \in \mathscr{D}'_{L^r}$

Let $1 \leqslant s \leqslant \infty$ be such that

$$\frac{1}{r} + \frac{1}{s} - 1 \geqslant 0 \quad \text{i.e.} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{s} - 2 \geqslant 0$$

let σ be defined by

$$1 \leqslant \sigma \leqslant \infty, \quad \frac{1}{\sigma} = \frac{1}{r} + \frac{1}{s} - 1 = \frac{1}{p} + \frac{1}{q} + \frac{1}{s} - 2$$

[1] As it can be seen by the corresponding proof, the term *continue* in Statement 2° of Theorem XXVI, Section 8, Chapter VI of [8] must be substituted by *hypocontinue*.

and let $h \in \mathscr{D}'_{L^s}$; then

- $(f * g) * h$ is defined
- $(f * g) * h \in \mathscr{D}'_{L^\sigma}$

Since $\dfrac{1}{\sigma} = \dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{s} - 2$, we have then

$$\frac{1}{q} + \frac{1}{s} - 1 = 1 + \frac{1}{\sigma} - \frac{1}{p} \geqslant 1 - \frac{1}{p} \geqslant 0$$

Let now η be defined by

$$1 \leqslant \eta \leqslant \infty, \quad \frac{1}{\eta} = \frac{1}{q} + \frac{1}{s} - 1$$

then

- $g * h$ is defined
- $g * h \in \mathscr{D}'_{L^\eta}$

Observe that

$$\frac{1}{p} + \frac{1}{\eta} - 1 = \frac{1}{p} + \frac{1}{q} + \frac{1}{s} - 2 = \frac{1}{\sigma}$$

thus finally we have

- $f * (g * h)$ is defined
- $f * (g * h) \in \mathscr{D}'_{L^\sigma}$

Now we can prove associativity. Our proof is based on Theorem 3.1, the representation results obtained in Theorems 2.3, 3.2 and 4.1 of [7] and on usual associativity of composition of maps.

Theorem 3.2: Let f, g, h be as above. Then

$$(f * g) * h = f * (g * h)$$

Proof: Assume first that $s \neq \infty$, and consider the following LTI systems

$$\begin{array}{ll} \mathscr{H} : \mathscr{D}'_{L^1} \to \mathscr{D}'_{L^s} & \text{defined by } \mathscr{H}(\alpha) = h * \alpha \\ \mathscr{G} : \mathscr{D}'_{L^s} \to \mathscr{D}'_{L^\eta} & \text{defined by } \mathscr{G}(\beta) = g * \beta \\ \mathscr{F} : \mathscr{D}'_{L^\eta} \to \mathscr{D}'_{L^\sigma} & \text{defined by } \mathscr{F}(\gamma) = f * \gamma \end{array}$$

By Theorem 3.1, $\mathscr{H}, \mathscr{G}$, and $\mathscr{F}$ are continuous with respect to the strong convergence in their domain and codomain. As a consequence, also the LTI systems

$$\mathscr{F}\mathscr{G} : \mathscr{D}'_{L^s} \to \mathscr{D}'_{L^\sigma}, \quad \mathscr{G}\mathscr{H} : \mathscr{D}'_{L^1} \to \mathscr{D}'_{L^\eta}$$

are continuous (in the same sense). By Theorem 2.3 of [7], the impulse response of $\mathscr{G}\mathscr{H}$ is

$$(\mathscr{G}\mathscr{H})(\delta) = \mathscr{G}(\mathscr{H}(\delta)) = \mathscr{G}(h * \delta) = \mathscr{G}(h) = g * h$$

Let $\mathscr{P} : \mathscr{D}'_{L^1} \to \mathscr{D}'_{L^\eta}$ be the continuous LTI system defined by $\mathscr{P}(\alpha) = (g * h) * \alpha$. The impulse response of $\mathscr{P}$ is $g * h$. By Theorems 4.1 and 3.2 of [7] we obtain $\mathscr{G}\mathscr{H} = \mathscr{P}$. Hence

$$(\mathscr{G}\mathscr{H})(\alpha) = (g * h) * \alpha \quad \text{for every } \alpha \in \mathscr{D}'_{L^1}$$

Since $s \neq \infty$, a similar argument proves that

$$(\mathscr{F}\mathscr{G})(\beta) = (f * g) * \beta \quad \text{for every } \beta \in \mathscr{D}'_{L^s}$$

As a consequence we obtain

$$\begin{array}{l} ((\mathscr{F}\mathscr{G})\mathscr{H})(\delta) = (\mathscr{F}\mathscr{G})(h) = (f * g) * h \\ (\mathscr{F}(\mathscr{G}\mathscr{H}))(\delta) = \mathscr{F}(g * h) = f * (g * h) \end{array}$$

By associativity of composition of maps, it is $(\mathscr{F}\mathscr{G})\mathscr{H} = \mathscr{F}(\mathscr{G}\mathscr{H})$; hence $(f * g) * h = f * (g * h)$.

Assume now $s = \infty$. Since $(1/r) + (1/s) - 1 \geqslant 0$, we have $1/r \geqslant 1$; hence $r = 1$. Since $(1/p) + (1/q) - 1 = 1/r = 1$, we have $(1/p) + (1/q) = 2$; hence $p = q = 1$. As a consequence

$$f \in \mathscr{D}'_{L^1}, \quad g \in \mathscr{D}'_{L^1}, \quad h \in \mathscr{D}'_{L^\infty}$$

Observe that the result on associativity already proved applies to the ordered triplet

$$h \in \mathscr{D}'_{L^\infty}, \quad g \in \mathscr{D}'_{L^1}, \quad f \in \mathscr{D}'_{L^1}$$

Hence $(h * g) * f = h * (g * f)$.

By commutativity we have

$$\begin{array}{l} (f * g) * h = h * (f * g) = h * (g * f) \\ f * (g * h) = f * (h * g) = (h * g) * f \end{array}$$

Since the last terms of each chain of equalities are equal, then $f * (g * h) = f * (g * h)$. ∎

Thanks to associativity, we may now prove the following result concerning the behavior of convolution on weakly convergent sequences.

Theorem 3.3: Let $g = w\text{-}\mathscr{D}'_{L^q}\text{-}\lim_{k\to\infty} g_k$, and let as above $f \in \mathscr{D}'_{L^p}$, with the ulterior request that $f \in \dot{\mathscr{D}}'_{L^\infty}$ if $q = 1$ and $p = \infty$. Then we have $f * g = w\text{-}\mathscr{D}'_{L^r}\text{-}\lim_{k\to\infty} f * g_k$.

Proof: The proof will be given in various steps, each with its own proof.

Step 1. Let $1 \leqslant m \leqslant \infty$. Let $\alpha \in L^m \cap \dot{L}^\infty$, $\beta \in L^{m'} \cap \dot{L}^\infty$, where $\dot{L}^\infty$ is the linear space of the $f \in L^\infty$ null at infinity. Then $\alpha * \beta \in \dot{L}^\infty$.

Proof of Step 1. By Young's Theorem it is $\alpha * \beta \in L^\infty$.

Let $\varepsilon > 0$. There exist $-\infty < \tau_1 < \tau_2 < +\infty$ such that

$$\begin{array}{rcl} \|\beta\|_{m'} \cdot \left\|\alpha_{|_{(-\infty,\tau_1)}}\right\|_m & < & \varepsilon/3 \\ \|\beta\|_{m'} \cdot \left\|\alpha_{|_{(\tau_2,+\infty)}}\right\|_m & < & \varepsilon/3 \end{array}$$

Observe that $\alpha_{|_{(\tau_1,\tau_2)}} \in L^1(\tau_1, \tau_2)$. There exists $T > 0$ such that for every $|t| > T$ it is

$$\left\|\alpha_{|_{(\tau_1,\tau_2)}}\right\|_1 \cdot \left\|\beta_{|_{(t-\tau_2,t-\tau_1)}}\right\|_\infty < \varepsilon/3$$

For almost all t such that $|t| > T$ we have

$$|(\alpha * \beta)(t)| \leqslant \int_{-\infty}^{\tau_1} |\alpha(\tau)| \cdot |\beta(t-\tau)| \mathrm{d}\tau + \int_{\tau_2}^{+\infty} |\alpha(\tau)| \cdot |\beta(t-\tau)| \mathrm{d}\tau + \int_{\tau_1}^{\tau_2} |\alpha(\tau)| \cdot |\beta(t-\tau)| \mathrm{d}\tau$$

hence, by Hölder's Inequality we obtain

$$\begin{array}{rcl} |(\alpha * \beta)(t)| \leqslant \|\beta\|_{m'} \cdot \left\|\alpha_{|_{(-\infty,\tau_1)}}\right\|_m + \\ \|\beta\|_{m'} \cdot \left\|\alpha_{|_{(\tau_2,+\infty)}}\right\|_m + \\ \left\|\alpha_{|_{(\tau_1,\tau_2)}}\right\|_1 \cdot \left\|\beta_{|_{(t-\tau_2,t-\tau_1)}}\right\|_\infty & < & \varepsilon \end{array}$$

End of Proof of Step 1

Step 2. Let $1 \leqslant m \leqslant \infty$. Let $\alpha \in \mathscr{D}'_{L^m} \cap \mathscr{D}'_{L^\infty}$, $\beta \in \mathscr{D}_{L^{m'}} \cap \dot{\mathscr{D}}_{L^\infty}$ (remember that $\mathscr{D}'_{L^m} \subset \dot{\mathscr{D}}'_{L^\infty}$ if $m \neq \infty$, and that $\mathscr{D}_{L^{m'}} \subset \dot{\mathscr{D}}_{L^\infty}$ if $m' \neq \infty$). Then $\alpha * \beta \in \dot{\mathscr{D}}_{L^\infty}$.

Proof of Step 2. By the result on regularization in Section 8, Chapter VI of [8] it is $\alpha * \beta \in \mathscr{D}_{L^\infty}$. We must prove that, for every $n \in \mathbb{N}$ it is $(\alpha * \beta)^{(n)} \in \dot{L}^\infty$.

By Remark 3° to Theorem XXV of [8], α may be written in the form $\alpha = \sum_h \alpha_h^{(h)}$ with $\alpha_h \in L^m \cap \dot{L}^\infty$ for every h. Then

$$(\alpha * \beta)^{(n)} = \sum_h \alpha_h * \beta^{(n+h)}$$

Since every $\beta^{(n+h)} \in L^{m'} \cap \dot{L}^\infty$, by Step 1 we have $\alpha_h * \beta^{(n+h)} \in \dot{L}^\infty$; hence $(\alpha * \beta)^{(n)} \in \dot{L}^\infty$. *End of Proof of Step 2*

Step 3. Let $1 \leqslant m \leqslant \infty$. Let $F \in \mathscr{D}'_{L^m}$, $\varphi \in \mathscr{D}_{L^{m'}}$. Then

(a) $F * \varphi \in \mathscr{D}_{L^\infty}$

(b) if $m \neq 1$, for every $t \in \mathbb{R}$ it is

$$(F * \varphi)(t) = \langle F(\tau), \varphi(t - \tau) \rangle$$

(c) if $m = 1$ and moreover $\varphi \in \dot{\mathscr{D}}_{L^\infty}$, for every $t \in \mathbb{R}$ it is

$$(F * \varphi)(t) = \langle F(\tau), \varphi(t - \tau) \rangle$$

Proof of Step 3. See the regularization results in Chapter VI, Section 8 of [8]. *End of Proof of Step 3*

We can now prove that $w\text{-}\mathscr{D}'_{L^r}\text{-}\lim_{k\to\infty} f * g_k = f * g$.

First of all assume $r = 1$. Since

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r} = 1$$

we have also $p = q = 1$. Let $\varphi \in \dot{\mathscr{D}}_{L^\infty}$. Since $\varphi(\tau) = \widetilde{\varphi}(-\tau) = \widetilde{\varphi}(0 - \tau)$, by (c) of Step 3 we have

$$\langle f * g_k, \varphi \rangle = \langle (f * g_k)(\tau), \widetilde{\varphi}(0 - \tau) \rangle = ((f * g_k) * \widetilde{\varphi})(0)$$

By commutativity of convolution and by Theorem 3.2 (i.e. associativity) we have

$$((f * g_k) * \widetilde{\varphi})(0) = (g_k * (f * \widetilde{\varphi}))(0)$$

Since $f \in \mathscr{D}'_{L^1} = \mathscr{D}'_{L^1} \cap \dot{\mathscr{D}}'_{L^\infty}$, $\varphi \in \dot{\mathscr{D}}_{L^\infty} = \mathscr{D}_{L^{1'}} \cap \dot{\mathscr{D}}_{L^\infty}$, by Step 2 we have $f * \widetilde{\varphi} \in \dot{\mathscr{D}}_{L^\infty}$; hence, by (c) of Step 3 we obtain

$$(g_k * (f * \widetilde{\varphi}))(0) = \langle g_k(\tau), (f * \widetilde{\varphi})(0 - \tau) \rangle = \langle g_k, \widetilde{f * \widetilde{\varphi}} \rangle$$

As a consequence $\langle f * g_k, \varphi \rangle = \langle g_k, \widetilde{f * \widetilde{\varphi}} \rangle$. Analogously $\langle g, \widetilde{f * \widetilde{\varphi}} \rangle = \langle f * g, \varphi \rangle$. Since $w\text{-}\mathscr{D}'_{L^q}\text{-}\lim_{k\to\infty} g_k = g$, we obtain

$$\begin{aligned} \lim_{k\to\infty} \langle f * g_k, \varphi \rangle &= \lim_{k\to\infty} \langle g_k, \widetilde{f * \widetilde{\varphi}} \rangle = \\ &= \langle g, \widetilde{f * \widetilde{\varphi}} \rangle = \langle f * g, \varphi \rangle \end{aligned}$$

As a consequence

$$w\text{-}\mathscr{D}'_{L^r}\text{-}\lim_{k\to\infty} f * g_k = f * g$$

Similar arguments prove that

$$w\text{-}\mathscr{D}'_{L^r}\text{-}\lim_{k\to\infty} f * g_k = f * g$$

in the following two remaining cases: $r \neq 1$ and $q \neq 1$, $r \neq 1$ and $q = 1$. ∎

IV. Fundamental LTI systems defined on $\mathscr{D}'_{L^p}$ and their properties

Let $1 \leqslant p \leqslant \infty$. For every $\Delta \in \mathscr{D}'_{L^{p'}}$, the extension of Young's Theorem to distributions allows the construction of a concrete *fundamental* LTI system defined on $\mathscr{D}'_{L^p}$, namely the system

$$\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'$$

defined by

$$\mathscr{L}_\Delta(f) = \Delta * f \quad \text{for every } f \in \mathscr{D}'_{L^p}$$

As we will see in this Section, all fundamental LTI systems are continuous, and moreover, as we will see in Section V (again apart pathologies)

- every continuous LTI system defined on $\mathscr{D}'_{L^p}$ is a fundamental system
- every continuous LTI system defined on L^p is the restriction to L^p of a fundamental system

This is the reason of the term "*fundamental*" we have reserved them.

Let us see in details the main properties of these systems.

Concerning the image of $\mathscr{L}_\Delta$, i.e., the subset $\mathscr{L}_\Delta(\mathscr{D}'_{L^p}) \subset \mathscr{D}'$, we have the following result.

Theorem 4.1: Let $1 \leqslant q \leqslant p'$, so that

$$\mathscr{D}'_{L^1} \subset \mathscr{D}'_{L^q} \subset \mathscr{D}'_{L^{p'}}$$

and let $\Delta \in \mathscr{D}'_{L^q}$. Then

$$\mathscr{L}_\Delta(\mathscr{D}'_{L^p}) \subset \mathscr{D}'_{L^r} \quad \text{where } \frac{1}{r} = \frac{1}{q} + \frac{1}{p} - 1$$

Observe that

$$\mathscr{D}'_{L^p} \subset \mathscr{D}'_{L^r} \subset \mathscr{D}'_{L^\infty}$$

and that $\mathscr{D}'_{L^r}$ progressively decreases from $\mathscr{D}'_{L^\infty}$ to $\mathscr{D}'_{L^p}$ as q decreases from p' to 1.

In particular, independently from the q chosen,

$$\mathscr{L}_\Delta(\mathscr{D}'_{L^p}) \subset \mathscr{D}'_{L^\infty}$$

Proof: The statement is a straightforward consequence of the definition of convolution as an extension of Young's Theorem. ∎

Concerning the continuity of $\mathscr{L}_\Delta$ we have the following results.

Theorem 4.2: The following statements hold

a) $\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'$ is a strongly continuous LTI system

b) the map $\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'_{L^\infty}$ is continuous with respect to the strong convergence in both spaces

Proof: Statement b) follows by Theorem 3.1. Statement a) follows by b). ∎

Theorem 4.3: Let $1 < p \leqslant \infty$. The following statements hold

a) the system $\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'$ is weakly continuous

b) the map $\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'_{L^\infty}$ is continuous with respect to the weak convergence in both spaces

Proof: Statement b) follows by Theorem 3.3. Statement a) follows by b). ∎

Using Theorem 4.2 above, thanks to Theorem 2.3 of [7] we obtain the following result concerning the impulse response of $\mathscr{L}_\Delta$.

Theorem 4.4: The impulse response of $\mathscr{L}_\Delta$ is $\mathscr{L}_\Delta(\delta) = \Delta$.

At its turn, this result allows us to complete the analysis on weak continuity given in Thm. 4.3.

Theorem 4.5: Let $p = 1$, so that $\Delta \in \mathscr{D}'_{L^\infty}$. Then $\mathscr{L}_\Delta : \mathscr{D}'_{L^1} \to \mathscr{D}'$ is weakly continuous if and only if $\Delta \in \dot{\mathscr{D}}'_{L^\infty}$.

Furthermore if $\Delta \in \dot{\mathscr{D}}'_{L^\infty}$ then the map

$$\mathscr{L}_\Delta : \mathscr{D}'_{L^1} \to \mathscr{D}'_{L^\infty}$$

is continuous with respect to the weak convergence in both spaces.

Proof: Assume $\mathscr{L}_\Delta$ to be weakly continuous: then by Theorems 3.3 and 2.3 we get $\Delta \in \dot{\mathscr{D}}'_{L^\infty}$.

Now assume $\Delta \in \dot{\mathscr{D}}'_{L^\infty}$; by Theorem 3.3, $\mathscr{L}_\Delta : \mathscr{D}'_{L^1} \to \mathscr{D}'_{L^\infty}$ is continuous with respect to the weak convergence in both spaces. In particular $\mathscr{L}_\Delta$ is weakly continuous. ∎

Now we briefly prove an obvious result on causality. We recall that a system $\mathscr{L} : \mathscr{I} \to \mathscr{O}$ is said to be causal if for every $t_0 \in \mathbb{R}$ and for every $f, g \in \mathscr{I}$ with $\operatorname{supp}(f - g) \subset [t_0, +\infty)$, it is $\operatorname{supp}(\mathscr{L}(f) - \mathscr{L}(g)) \subset [t_0, +\infty)$. Concerning causality of $\mathscr{L}_\Delta$ we have the following result.

Theorem 4.6: $\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'$ is causal if and only if $\operatorname{supp} \Delta \subset [0, +\infty)$.

Proof: Assume $\mathscr{L}_\Delta$ causal. Then by Theorem 4.4 we have $\Delta = \mathscr{L}_\Delta(\delta)$. Obviously $\operatorname{supp}(\delta - 0) \subset [0, +\infty)$, hence $\operatorname{supp} \Delta = \operatorname{supp}(\mathscr{L}_\Delta(\delta) - \mathscr{L}_\Delta(0)) \subset [0, +\infty)$

Now assume $\operatorname{supp} \Delta \subset [0, +\infty)$. Let $t_0 \in \mathbb{R}$, and let $f, g \in \mathscr{D}'_{L^p}$ be such that $\operatorname{supp}(f - g) \subset [t_0, +\infty)$. Then $\mathscr{L}_\Delta(f) - \mathscr{L}_\Delta(g) = \Delta * (f - g)$.

Let $\varepsilon > 0$; part c) in the proof of Theorem XXV, Chapter VI, Section 8, in [8], may be refined and used to prove that

- $\Delta = \sum_h \Delta_h^{(h)}$ with $\Delta_h \in L^p$ such that $\operatorname{supp} \Delta_h \subset [-\varepsilon, +\infty)$ for every h
- $f - g = \sum_k F_k^{(k)}$ with $F_k \in L^p$ such that $\operatorname{supp} F_k \subset [t_0 - \varepsilon, +\infty)$ for every k

A classical argument proves that $\operatorname{supp}(\Delta_h * F_k) \subset [t_0 - 2\varepsilon, +\infty)$; as a consequence $\operatorname{supp}(\Delta * F) \subset [t_0 - 2\varepsilon, +\infty)$.

Since $\operatorname{supp}(\Delta * F) \subset [t_0 - 2\varepsilon, +\infty)$ for every $\varepsilon > 0$, then $\operatorname{supp}(\Delta * F) \subset [t_0, +\infty)$. ∎

Concerning the restriction of $\mathscr{L}_\Delta$ to L^p, as a corollary of Theorem 4.2 we obtain the following result.

Theorem 4.7: Let $1 \leqslant p \leqslant \infty$ and let $\mathscr{R} : L^p \to \mathscr{D}'$ be the restriction to L^p of $\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'$. Then $\mathscr{R}$ is continuous.

Proof: By Hölder's Inequality, it is easily seen that

$$f = L^p\text{-}\lim_{k\to\infty} f_k \Rightarrow f = s\text{-}\mathscr{D}'_{L^p}\text{-}\lim_{k\to\infty} f_k$$

hence, *a fortiori*, $\mathscr{R}$ is a continuous LTI system. ∎

V. Continuous LTI system defined on L^p and $\mathscr{D}'_{L^p}$: analysis by impulse response and convolution

In this Section we prove that the impulse response, as defined in Section II of [7], and the extension of convolution described in Section III allow a complete analysis of all continuous LTI systems $\mathscr{L} : L^p \to \mathscr{D}'$ and $\mathscr{L} : \mathscr{D}'_{L^p} \to \mathscr{D}'$. Indeed (again apart pathologies) we prove that in both cases we have $\mathscr{L}(f) = \Delta * f$ for every f, where Δ is the impulse response.

Firstly we consider systems defined on L^p spaces and then systems defined on $\mathscr{D}'_{L^p}$ spaces.

Theorem 5.1: Let $1 \leqslant p < \infty$, let $\mathscr{L} : L^p \to \mathscr{D}'$ be a continuous LTI system, and let $\Delta \in \mathscr{D}'_{L^{p'}}$ be its impulse response. Then for every $f \in L^p$ it is $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$.

Proof: Let $\mathscr{R} : L^p \to \mathscr{D}'$ be the restriction to L^p of $\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'$. By Theorem 4.7, $\mathscr{R}$ is a continuous LTI system.

Since for every $f \in \mathscr{D}$ it is $\mathscr{R}(f) = \Delta * f$, by Section II of [7] the impulse response of $\mathscr{R}$ is Δ. Since $\mathscr{L}$ and $\mathscr{R}$ have the same impulse response, by Theorems 3.1 and 4.1 of [7] for every $f \in L^p$ we have $\mathscr{L}(f) = \mathscr{R}(f) = \mathscr{L}_\Delta(f) = \Delta * f$. ∎

Theorem 5.2: Let $\mathscr{L} : L^\infty \to \mathscr{D}'$ be a continuous LTI system, and let $\Delta \in \mathscr{D}'_{L^1}$ be its impulse response. Then for every $f \in \mathscr{C}_0$ it is $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$.

Proof: Same argument of the Proof of Theorem 5.1; merely remember that $\Sigma(\mathscr{D}, L^\infty) = \mathscr{C}_0$ (see Section IV of [7]). ∎

We recall that in [7], Proposition 5.1, it is shown that this result is sharp.

All the following results concerning $\mathscr{D}'_{L^p}$ spaces are easy consequences of the results on impulse response and continuity for fundamental LTI systems stated in Section IV of this paper, and moreover of Theorems 3.1, 3.2 and Section IV of [7].

We omit the now straightforward proves.

Theorem 5.3: Let $1 < p < \infty$, let $\mathscr{L} : \mathscr{D}'_{L^p} \to \mathscr{D}'$ be a weakly (resp. strongly) continuous LTI system, and let $\Delta \in \mathscr{D}'_{L^{p'}}$ be its impulse response. Then $\mathscr{L} = \mathscr{L}_\Delta$. In particular for every $f \in \mathscr{D}'_{L^p}$ it is $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$.

Theorem 5.4: Let $p = 1$, let $\mathscr{L} : \mathscr{D}'_{L^1} \to \mathscr{D}'$ be a weakly continuous LTI system, and let $\Delta \in \dot{\mathscr{D}}'_{L^\infty}$ be its impulse response. Then $\mathscr{L} = \mathscr{L}_\Delta$. In particular for every $f \in \mathscr{D}'_{L^1}$ it is $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$.

Theorem 5.5: Let $p = 1$, let $\mathscr{L} : \mathscr{D}'_{L^1} \to \mathscr{D}'$ be a strongly continuous LTI system, and let $\Delta \in \mathscr{D}'_{L^\infty}$ be its impulse response. Then $\mathscr{L} = \mathscr{L}_\Delta$. In particular for every $f \in \mathscr{D}'_{L^1}$ it is $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$.

Theorem 5.6: Let $p = \infty$, let $\mathscr{L} : \mathscr{D}'_{L^\infty} \to \mathscr{D}'$ be a weakly continuous LTI system, and let $\Delta \in \mathscr{D}'_{L^1}$ be its impulse response. Then $\mathscr{L} = \mathscr{L}_\Delta$. In particular for every $f \in \mathscr{D}'_{L^\infty}$ it is $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$.

Theorem 5.7: Let $p = \infty$, let $\mathscr{L} : \mathscr{D}'_{L^\infty} \to \mathscr{D}'$ be a strongly continuous LTI system, and let $\Delta \in \mathscr{D}'_{L^1}$ be its impulse response. Then for every $f \in \dot{\mathscr{D}}'_{L^\infty}$ it is $\mathscr{L}(f) = \mathscr{L}_\Delta(f) = \Delta * f$.

As for Thm. 5.2, we recall that in [7] it is shown that this result is sharp.

All the above theorems and the continuity properties of the systems $\mathscr{L}_\Delta$ described in Section IV, allow us to draw a picture of the landscape of all continuous LTI systems defined on L^p and $\mathscr{D}'_{L^p}$:

- let $1 < p < \infty$, then

  - the continuous LTI systems defined on L^p are the restrictions to L^p itself of the $\mathscr{L}_\Delta$ with $\Delta \in \mathscr{D}'_{L^{p'}}$
  - the weakly continuous LTI systems defined on $\mathscr{D}'_{L^p}$ are the $\mathscr{L}_\Delta$ with $\Delta \in \mathscr{D}'_{L^{p'}}$
  - the strongly continuous LTI systems defined on $\mathscr{D}'_{L^p}$ are the $\mathscr{L}_\Delta$ with $\Delta \in \mathscr{D}'_{L^{p'}}$
- let $p = 1$, then
  - the continuous LTI systems defined on L^1 are the restrictions to L^1 itself of the $\mathscr{L}_\Delta$ with $\Delta \in \mathscr{D}'_{L^\infty}$
  - the weakly continuous LTI systems defined on $\mathscr{D}'_{L^1}$ are the $\mathscr{L}_\Delta$ with $\Delta \in \dot{\mathscr{D}}'_{L^\infty}$
  - the strongly continuous LTI systems defined on $\mathscr{D}'_{L^1}$ are the $\mathscr{L}_\Delta$ with $\Delta \in \mathscr{D}'_{L^\infty}$
- let $p = \infty$, then
  - among the continuous LTI systems defined on L^∞ there are the restrictions of all the $\mathscr{L}_\Delta$ with $\Delta \in \mathscr{D}'_{L^1}$, but there are others continuous LTI systems defined on L^∞ (see [1], [2], [5])
  - the weakly continuous LTI systems defined on $\mathscr{D}'_{L^\infty}$ are the $\mathscr{L}_\Delta$ with $\Delta \in \mathscr{D}'_{L^1}$
  - among the strongly continuous LTI systems defined on $\mathscr{D}'_{L^1}$ there are all the $\mathscr{L}_\Delta$ with $\Delta \in \mathscr{D}'_{L^1}$, but there are others strongly continuous LTI systems defined on $\mathscr{D}'_{L^\infty}$ (see [7]).

We remark that the pathology of $\mathscr{D}'_{L^\infty}$ may be overcome by considering the weak convergence instead that the strong one.

A way to overcome the pathology which arises considering L^∞ is described in a recent paper of Sandberg (cf. [6]) .

VI. Perspectives of applications to linear differential equations

Let $1 \leqslant p \leqslant \infty$ and let $P(D)$, $Q(D)$ be linear differential operators with constant coefficients. For every $f \in L^p$ (resp. $f \in \mathscr{D}'_{L^p}$), let $S(f) \subset \mathscr{D}'$ be the set of all the (distributional) solutions of the differential equation

$$P(D)\, x = Q(D)\, f$$

Finally, let M be the set of *all* the maps $\mathscr{L}$ from L^p (resp. $\mathscr{D}'_{L^p}$) into $\mathscr{D}'$ such that $\mathscr{L}(f) \in S(f)$ for every $f \in L^p$ (resp. $f \in \mathscr{D}'_{L^p}$).

In this Section we show, by an intentionally simple example, that the results of this paper may be used to find (again apart pathologies) the elements in M which are continuous LTI systems.

Let $P(D) = D^2 - 1, Q(D) = 1$ and let M be the set of all the maps $\mathscr{L}$ from L^p into $\mathscr{D}'$ such that, for every $f \in L^p$, $\mathscr{L}(f)$ is a solution of the differential equation

$$(D^2 - 1)\, x = f \tag{1}$$

To find the elements in M which are continuous LTI systems, first of all we prove that if such system exist, and $\mathscr{L}$ is one of them, then the impulse response Δ must be

$$\Delta(t) = -\frac{1}{2}\left(e^t H(-t) + e^{-t} H(t)\right) \tag{2}$$

Indeed: By [7], Theorem 2.1, given a sequence $\varphi_k \in \mathscr{D}$ such that $\mathscr{E}'\text{-}\lim_{k\to\infty} \varphi_k = \delta$, we have

- $\Delta = \mathscr{D}'\text{-}\lim_{k\to\infty} \mathscr{L}(\varphi_k)$
- $\mathscr{L}(\varphi) = \Delta * \varphi$ for every $\varphi \in \mathscr{D}$

Since $\mathscr{L} \in M$, for every k we have

$$(\mathscr{L}(\varphi_k))^{(2)} - \mathscr{L}(\varphi_k) = \varphi_k$$

hence

$$(\Delta * \varphi_k)^{(2)} - \Delta * \varphi_k = \varphi_k$$

i.e.

$$\left(\Delta^{(2)} - \Delta - \delta\right) * \varphi_k = 0$$

As a consequence

$$\Delta^{(2)} - \Delta - \delta = \mathscr{D}'\text{-}\lim_{k\to\infty} \left(\Delta^{(2)} - \Delta - \delta\right) * \varphi_k = 0$$

and hence $\Delta^{(2)} - \Delta = \delta$. A straightforward argument proves then that

$$\begin{aligned}\Delta(t) \in \frac{e^t - e^{-t}}{2} H(t) + \left\{\lambda e^t + \mu e^{-t} : \lambda, \mu \in \mathbb{C}\right\} = \\ = -\frac{e^t H(-t) + e^{-t} H(t)}{2} + \\ + \left\{\lambda e^t + \mu e^{-t} : \lambda, \mu \in \mathbb{C}\right\}\end{aligned}$$

Since by Theorem 2.1 we have $\Delta \in \mathscr{D}'_{L^{p'}}$, then (2) must hold. Observe that $\Delta \in L^1$.

Now, if $1 \leqslant p < \infty$, then by Theorem 5.1 for every $f \in L^p$ it must be $\mathscr{L}(f) = \Delta * f$; moreover it is immediately seen that $\Delta * f$ satisfies the differential equation (1). This proves that the unique element of M which is a continuous LTI system is the restriction to L^p of $\mathscr{L}_\Delta : \mathscr{D}'_{L^p} \to \mathscr{D}'$.

Observe that, since $\Delta \in L^1$, then for every $f \in L^p$ we have $\mathscr{L}(f) \in L^p$.

Finally, if $p = \infty$, by Theorem 5.2, we have that the system defined by $\mathscr{L}(f) = \Delta * f$ is *one* of the continuous LTI systems defined on L^∞ whose impulse response is Δ. Moreover, it is immediately seen that for every $f \in L^p$, $\Delta * f$ satisfies the differential equation (1). This proves that $\mathscr{L}$ is a continuous LTI system in M, but the pathology of L^∞ does not allow us to ensure it is the unique.

A similar argument (depending on Theorems 5.3-5.7) may be applied when M is the set of all the maps $\mathscr{L}$ from $\mathscr{D}'_{L^p}$ into $\mathscr{D}'$ such that, for every $f \in \mathscr{D}'_{L^p}$, $\mathscr{L}(f)$ is a solution of (1).

References

[1] I. Sandberg, "Continuous-time linear systems: folklore and fact," in *Circuits Syst. Signal Proc.* vol. 21, pp. 337–343, 2002

[2] ——, "Causality and the impulse response scandal," *IEEE Trans. Circuits Syst. I*, vol. 50, pp. 810–811, June 2003

[3] ——, "Notes on linear systems and impulse responses," *Circuits Syst. Signal Proc.* , vol. 23, no. 6, pp. 339–350, 2004

[4] ——, "Notes on multidimensional linear system representations," *38th Annual Conference on Information Science and Systems*, Dep. of Electrical Engineering, Princeton University, Princeton, NJ, pp. 801–806, 2004

[5] ——, "Countinuous multidimensional systems and the impulse response scandal," *Multidim. Syst. Signal Processing*, vol. 15, pp. 295–299, 2004.

[6] ——, "Bounded inputs and the representation of linear system maps," *Circuits, Syst., Signal Processing*, vol. 24, pp. 103–115, 2005.

[7] M. Ciampa. M. Franciosi, M. Poletti, “A note on impulse response for continuous, linear, time-invariant, continuous-time systems,”*IEEE Trans. Circuits Syst. I*, vol. 53, Jan. 2006.
[8] L. Schwartz, *Théorie des distributions*. Paris: Hermann, 1966.
[9] C.L. De Vito, *Functional Analysis*. New York: Academic Press, 1978.
[10] N. Bourbaki, *Espaces Vectoriels Topologiques*. Paris: Masson, 1981.
[11] A. Weil, *L’intergration dans les groupes topologiques et ses applications*. Paris: Hermann, 1951.