Adjoint divisors on algebraic curves

(with an appendix by Fabrizio Catanese) ⋆

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Abstract

Let C be a numerically connected curve lying on a smooth algebraic surface. We show that if $\mathcal{A}$ is an ample invertible sheaf satisfying some technical numerical hypotheses then $\omega_C \otimes \mathcal{A}$ is normally generated. As a corollary we show that the sheaf $\omega_C^{\otimes 2}$ on a numerically connected curve C of arithmetic genus $p_a \geq 3$ is normally generated if ω_C is ample and does not exist a subcurve $B \subset C$ such that $p_a(B) = 1 = B.(C - B)$.

Key words: algebraic curve, invertible sheaves, normal generation, Picard scheme
MSC: Primary 14H45, 14C20; Secondary 14H20, 14C22

Introduction

Let C be a curve (possibly non reduced or reducible) lying on a smooth algebraic surface, ω_C be the dualizing sheaf of C, and let $\mathcal{H}$ be an invertible sheaf on C. We recall the notion of normal generation as introduced by Mumford in [21] .

Definition *$\mathcal{H}$ is said to be* k-normal *if the multiplication map*

$$\rho_k : (H^0(C, \mathcal{H}))^{\otimes k} \longrightarrow H^0(C, \mathcal{H}^{\otimes k})$$

is surjective. $\mathcal{H}$ is said to be normally generated *if the maps ρ_k are surjective for all $k \in \mathbb{N}$.*

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This corresponds to say that the graded ring $R(C,\mathcal{H}) = \bigoplus_{k\geq 0} H^0(C,\mathcal{H}^{\otimes k})$ is generated in degree 1. Notice that if $\mathcal{H}$ is ample on C then $\mathcal{H}$ turns out to be normally generated if and only if $\mathcal{H}$ is very ample and the associated embedded scheme $\varphi_{|\mathcal{H}|}(C) \subset \mathbb{P}(H^0(C,\mathcal{H})^\vee)$ is projectively Cohen–Macaulay. The study of normal generation of divisors on smooth curves goes back to Castelnuovo at the end of the nineteenth century (see [5]) and during last century several authors studied the ring $R(C,\mathcal{H})$ in the smooth case. Castelnuovo's theorem (from the early 1890's) says that on a smooth curve of genus g an invertible sheaf of degree $\geq 2g+1$ is normally generated.

In this paper we will study the ring $R(C,\mathcal{H})$ for a numerically connected curve contained in a smooth algebraic surface S. Some of the motivations of our analysis can be found in several aspects of the theory of algebraic surfaces. For instance taking the resolution of a normal surface singularity $\pi : S \to X$ (cf. [19]) or the *relative canonical algebra* of a fibration $f : S \to B$. In particular a conjecture of Reid (cf. [23]) states that if $f : S \to B$ is a relatively minimal fibration of curves of genus ≥ 2 then the relative canonical algebra $R(f) = \bigoplus_{n\geq 0} f_*(\omega_{S/B}^{\otimes n})$ is generated in degree ≤ 3 (the *1-2-3 conjecture*); since the fibration is minimal the conjecture follows iff one shows that for all the fibers F the canonical ring $R(F,\omega_F)$ is generated in degree ≤ 3.

In [6] were studied particular cases of $R(C,\mathcal{H})$, in which C is a curve of low (arithmetical) genus and $\mathcal{H}$ is a very ample invertible sheaf. In the same reference is given a complete classification and a detailed description of the geometry of the cases $p_a(C) = 1, \deg\mathcal{H} = 3,4$ and $p_a(C) = 2, \deg\mathcal{H} = 5$. More recently in [17] and in [18] was studied the degree bound of the generators of the canonical ring $R(C,\omega_C)$ and it was shown that the 1-2-3 conjecture is true except in some very special cases. The purpose of this paper is to study the ring $R(C,\mathcal{H})$ for every curve C of arbitrary genus under some numerical assumption on $\mathcal{H}$, paying particular attention to adjoint divisors $\mathcal{H} \cong \omega_C \otimes \mathcal{A}$ ($\mathcal{A}$ ample) and to the canonical ring $R(C,\omega_C)$. Our first result, which generalize Castelnuovo's theorem, is the following (for remaining definitions and notation we refer to §1.)

Theorem A *Let C be a numerically connected curve contained in a smooth algebraic surface and let $\mathcal{H} \overset{\text{num}}{\sim} \mathcal{F}' \otimes \mathcal{G}'$, where $\mathcal{F}', \mathcal{G}'$ are invertible sheaf such that*

$$\deg \mathcal{G}'_{|B} \geq \ p_a(B) \quad \forall \text{ subcurve } \ B \subseteq C \tag{1}$$

$$\deg \mathcal{F}'_{|B} \geq \ p_a(B)+1 \quad \forall \ \text{ subcurve } \ B \subseteq C \tag{2}$$

Then $\mathcal{H}$ is normally generated on C.

Theorem A follows by the classical arguments of Castelnuovo theory and two vanishing theorems for divisors of low degree on C (cf. Theorems 3.1 and 3.2). Specifically Thm. 3.1 states that

If $\mathcal{G}$ is a "general" invertible sheaf such that $\deg \mathcal{G}_{|B} \geq p_a(B)$ $\forall$ subcurve $B \subseteq C$, then $H^1(C, \mathcal{G}) = 0$

while Thm. 3.2 states that

If $\mathcal{F}$ is a "general" invertible sheaf such that $\deg \mathcal{F}'_{|B} \geq p_a(B) + 1$ $\forall$ subcurve $B \subseteq C$, then the linear system $|\mathcal{F}|$ is base point free.

(where "general" means that the corresponding class in the Picard scheme is in general position). These results are achieved by an extension of the classical Abel-Jacobi map, following essentially the ideas introduced by Altman and Kleiman in [1]. Thm. 3.2 is also based on the analysis given by Catanese in the Appendix, where you can find a detailed study of the group $\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta)$ for ζ and Δ 0-dimensional subschemes of C.

As an application of Theorem A we get the following result on the normal generation of adjoint divisors.

Theorem B *Let C be a numerically connected curve contained in a smooth algebraic surface and let $\mathcal{H} \overset{\text{num}}{\sim} \omega_C \otimes \mathcal{A}$, with $\mathcal{A}$ an ample invertible sheaf such that*

$$\deg \mathcal{H}_{|B} \geq 2p_a(B) + 1 \quad \forall \textit{ subcurve } \ B \subseteq C$$

Then $\mathcal{H}$ is normally generated on C except possibly in the following cases:

(a) $C = n\Gamma, \deg \mathcal{A}_{|\Gamma} = 1, \Gamma^2$ *even;*
(b) $C = \Gamma_1 + 2\Gamma_2, \deg \mathcal{A}_{|\Gamma_i} = 1, \Gamma_1 \cdot \Gamma_2 = 1$*;*
(c) For all irreducible $\Gamma \subset C$ $\deg \mathcal{A}_{|\Gamma} = 1, \Gamma \cdot (C - \Gamma) = 2$

Here the exceptional cases appear since we apply Theorem A to prove this result, but we believe it should be possible to treat these particular configurations with different methods.

Taking $\mathcal{A} \cong \omega_C$, as a corollary of the foregoing theorem we obtain a result recently proved by Konno in [17] on the 1-2-3 conjecture.

Theorem C *Let C be a numerically connected curve contained in a smooth algebraic surface. Assume $p_a(C) \geq 3$, ω_C ample on C and furthermore that there does not exist a subcurve $B \subset C$ such that $p_a(B) = 1 = B.(C - B)$.*

Then $\omega_C^{\otimes 2}$ is normally generated and the canonical ring $R(C, \omega_C)$ is generated in degree ≤ 3.

Notice that under the condition $p_a(B) = 1 = B.(C - B)$ the sheaf $\omega_C^{\otimes 2}$ is ample but not very ample, thus the above numerical hypothesis turns out to be necessary.

The paper is organized as follows: in §1 we recall some background results for projective curves; in §2 we analyze the Picard scheme and the Hilbert scheme for a curve on a smooth algebraic surface; in §3 we prove two vanishing theorems for divisor of low degree; in §4 we prove the theorems; the Appendix, by Fabrizio Catanese, is dedicated to a detailed study of the extension classes of 0-dimensional scheme, with particular attention to the classes which correspond to invertible sheaves.

1 Notation and background results

1.1 Notation and conventions

We work over an algebraically closed field $\mathbb{K}$ of characteristic ≥ 0. By a curve C we mean a curve contained in a smooth algebraic surface S. $\mathcal{O}_C$ denotes its structure sheaf, ω_C its dualizing sheaf and $p_a(C)$ the arithmetic genus, $p_a(C) = 1-\chi(\mathcal{O}_C)$. Keeping the usual notation for effective divisor on smooth surfaces we write C as $\sum_{i=1}^s n_i\Gamma_i$ and a by subcurve $B \subseteq C$ we mean a curve $\sum m_i\Gamma_i$, with $0 \leq m_i \leq n_i$.

If C, C' are curves on S, $C \cdot C'$ denotes their intersection number as divisors on S. Following the original definition of Franchetta a curve C is said to be *numerically m-connected* if $C_1 \cdot C_2 \geq m$ for every effective decomposition $C = C_1 + C_2$. A curve C is said to be *numerically connected* if it is 1-connected.

If $\mathcal{F}$ is an invertible sheaf on C, we write $\deg \mathcal{F}_{|C}$ for the degree of $\mathcal{F}$ on C, $\deg \mathcal{F}_{|C} = \chi(\mathcal{F}) - \chi(\mathcal{O}_C)$, and by $|\mathcal{F}|$ we mean the linear system of divisors of sections of $H^0(C, \mathcal{F})$. By Serre duality we mean Grothendieck-Serre-Riemann-Roch duality theorem:

$$H^1(C, \mathcal{G}) \overset{\mathrm{d}}{=} \mathrm{Hom}(\mathcal{G}, \omega_C) \quad \text{for } \mathcal{G} \text{ a coherent sheaf,}$$

(where $\overset{\mathrm{d}}{=}$ denotes duality of vector spaces.)

Two invertible sheaves $\mathcal{F}$, $\mathcal{F}'$ are said to be numerically equivalent on $C = \sum_{i=1}^s n_i\Gamma_i$ (notation: $\mathcal{F} \overset{\text{num}}{\sim} \mathcal{F}'$) if $\deg \mathcal{F}_{|\Gamma_i} = \deg \mathcal{F}'_{|\Gamma_i}$ for all Γ_i.

Following [7] a *cluster* Z of *degree* $\deg Z = r$ is a 0-dimensional subscheme with length $\mathcal{O}_Z = \dim_k \mathcal{O}_Z = r$. By a general local transverse cut Δ_i we mean a cluster on C with support a general smooth point Q of C_{red} such that $\mathcal{O}_C(\Delta_i)$ is invertible. As a scheme, $\Delta_i \cong \mathbb{K}[x]/(x^\nu)$, where $\nu = \mathrm{mult}_Q(C)$.

1.2 Background results on projective curves

In this section we recall some useful results proved in [7] on invertible sheaves on projective curves. The main instrument in the analysis of sheaves on projective curves with several components is the following lemma (cf. [7], Lemma 2.4), which holds in a more general setup.

Lemma 1.1 (Automatic adjunction) *Let C be a pure 1-dimensional projective scheme, let $\mathcal{F}$ be a coherent sheaf on C, and $\varphi\colon \mathcal{F} \to \omega_C$ a map of $\mathcal{O}_C$-modules. Set $\mathcal{J} = \mathrm{Ann}\,\varphi \subset \mathcal{O}_C$, and write $B \subset C$ for the subscheme defined by $\mathcal{J}$. Then B is Cohen–Macaulay and φ has a canonical factorization of the form*

$$\mathcal{F} \twoheadrightarrow \mathcal{F}_{|B} \to \omega_B = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}_B, \omega_C) \subset \omega_C,$$

where $\mathcal{F}_{|B} \to \omega_B$ is generically onto.

In the following theorem we summarize the main applications to curves lying on smooth surfaces of the results proved in [7] on Cohen–Macaulay 1-dimensional projective scheme. For a general treatment see §2, §3 of [7].

Theorem 1.2 *Let C be a curve contained in an smooth algebraic surface, ω_C the canonical sheaf of C and let $\mathcal{H}$ be an invertible sheaf on C.*

(i) If $\deg \mathcal{H}_{|B} \geq 2p_a(B) - 1\ \forall\ B \subseteq C$ then $H^1(C, \mathcal{H}) = 0$.
(ii) If $\deg \mathcal{H}_{|B} \geq 2p_a(B) + 1\ \forall\ B \subseteq C$ then $\mathcal{H}$ is very ample on C.
(iii) If C is 1-connected then $H^1(C, \omega_C) \cong \mathbb{K}$.
(iv) If C is 2-connected and $C \ncong \mathbb{P}^1$ then $|\omega_C|$ is base point free.
(v) If C is 3-connected and C is not honestly hyperelliptic (i.e., there does not exist a finite morphism $\psi\colon C \to \mathbb{P}^1$ of degree 2) then ω_C is very ample.

(cf. Thm. 1.1, Thm. 3.3, Thm. 3.6 in [7]).

1.3 Koszul cohomology groups of algebraic curves

In this section we recall the notion of Koszul cohomology groups as introduced and developed by Green in [10], and we focus on some applications of Koszul cohomology to the analysis of invertible sheaves on a numerically connected curve C.

Let $\mathcal{H}$, $\mathcal{F}$ be invertible sheaves on C and let $W \subseteq H^0(C,\mathcal{F})$ be a subspace which yields a base point free system of projective dimension r.

Let $M_{\mathcal{F}}$ be the Kernel of the evaluation map $W \otimes \mathcal{O}_C \stackrel{\text{ev}}{\to} \mathcal{F}$. Twisting with $\mathcal{H} \otimes \mathcal{F}^q$ and taking exterior powers we get the following exact sequence

$$0 \to M_{p,q} \to \bigwedge^p W \otimes \mathcal{H} \otimes \mathcal{F}^q \to M_{p-1,q+1} \to 0$$

where $M_{p,q}$ denotes the sheaf $\bigwedge^p M_{\mathcal{F}} \otimes \mathcal{H} \otimes \mathcal{F}^q$. Taking cohomology we have the following commutative diagram:

$$\begin{array}{ccccc}
 & 0 & & & \\
 & \downarrow & & & \\
 & H^0(M_{p+1,q-1}) & & & \\
 & \downarrow & & & \\
 & \bigwedge^{p+1} W \otimes H^0(\mathcal{H} \otimes \mathcal{F}^{q-1}) & & & 0 \\
 & \varphi_{p,q} \downarrow & \searrow^{d_{p+1,q-1}} & & \downarrow \\
0 \to & H^0(M_{p,q}) \to & \bigwedge^p W \otimes H^0(\mathcal{H} \otimes \mathcal{F}^q) & \longrightarrow & H^0(M_{p-1,q+1}) \\
 & \downarrow & & d_{p,q} \searrow & \downarrow \\
 & H^1(M_{p+1,q-1}) & & & \bigwedge^{p-1} W \otimes H^0(\mathcal{H} \otimes \mathcal{F}^{q+1}) \\
 & \downarrow & & & \\
 & \bigwedge^{p+1} W \otimes H^1(\mathcal{H} \otimes \mathcal{F}^{q-1}) & & &
\end{array}$$

where $d_{p,q}$ are the Koszul differentials, defined as follows:

$$d_{p,q} : \bigwedge^p W \otimes H^0(C, \mathcal{H} \otimes \mathcal{F}^q) \longrightarrow \bigwedge^{p-1} W \otimes H^0(C, \mathcal{H} \otimes \mathcal{F}^{q+1})$$
$$\sum s_{i_1} \wedge s_{i_2} \wedge \ldots \wedge s_{i_p} \otimes \alpha_{i_1 i_2 \ldots i_p} \mapsto \sum s_{i_1} \wedge \ldots \hat{s}_{i_j} \ldots \wedge s_{i_{r-1}} \otimes \alpha_{i_1 \ldots \hat{\imath}_j \ldots i_p} \cdot s_{i_j}$$

(here $\{s_0, \ldots, s_r\}$ is a basis for W.) The Koszul groups $\mathcal{K}_{p,q}(W,\mathcal{F},\mathcal{H})$ are defined by $\ker d_{p,q} / \operatorname{im} d_{p+1,q-1}$. If $W = H^0(C,\mathcal{F})$ they are usually denoted by $\mathcal{K}_{p,q}(\mathcal{F},\mathcal{H})$.

For our analysis the main applications of Koszul cohomology are remark 1.3 and Propositions 1.4, 1.5 below.

Remark 1.3 *The multiplication map* $H^0(C,\mathcal{H}) \otimes H^0(C,\mathcal{F}) \to H^0(C, \mathcal{H} \otimes \mathcal{F})$ *is surjective if and only if* $\mathcal{K}_{0,1}(\mathcal{F},\mathcal{H}) = 0$.

Proposition 1.4 (Duality) *If $|\mathcal{F}|$ is a base point free system of dimension r then*

$$\mathcal{K}_{p,q}(\mathcal{F},\mathcal{H}) \overset{\mathrm{d}}{=} \mathcal{K}_{r-p-1,2-q}(\mathcal{F},\omega_C\otimes\mathcal{H}^{-1})$$

(where $\overset{\mathrm{d}}{=}$ means duality of vector space).

Proof. Consider the above commutative diagram and then replace p, q, $\mathcal{H}$ respectively with $r-p-1$, $2-q$, $\omega_C\otimes\mathcal{H}^{-1}$. Now $\bigwedge^p M_{\mathcal{F}}^* \cong \bigwedge^{r-p} M_{\mathcal{F}}\otimes\mathcal{F}$, thus by Serre duality we have

$$\begin{aligned}\mathcal{K}_{p,q}(\mathcal{F},\mathcal{H}) &\cong \operatorname{coker}\varphi_{p,q} \overset{\mathrm{d}}{=} \ker\varphi_{p,q}^* \cong\\ &\cong \ker\{H^1(\textstyle\bigwedge^{r-p} M_{\mathcal{F}}\otimes\omega_C\otimes(\mathcal{H}\otimes\mathcal{F}^q)^{-1}) \to\\ &\quad \textstyle\bigwedge^{r-p}(H^0(\mathcal{F}))\otimes H^1(\omega_C\otimes(\mathcal{H}\otimes\mathcal{F}^{q-1})^{-1})\} \cong\\ &\cong \mathcal{K}_{r-p-1,2-q}(\mathcal{F},\omega_C\otimes\mathcal{H}^{-1}).\ \square\end{aligned}$$

Proposition 1.5 (H^0-Lemma) *Let C be a numerically connected curve and let $\mathcal{F}$, $\mathcal{H}$ be invertible sheaves on C. Assume that $|\mathcal{F}|$ is a base point free system of dimension r. If either $H^1(C,\mathcal{H}\otimes\mathcal{F}^{-1})=0$, or $\omega_C\cong\mathcal{H}\otimes\mathcal{F}^{-1}$ and $r\geq 2$, then $\mathcal{K}_{0,1}(\mathcal{F},\mathcal{H})=0$, that is, the multiplication map*

$$H^0(C,\mathcal{H})\otimes H^0(C,\mathcal{F})\to H^0(C,\mathcal{H}\otimes\mathcal{F})$$

is surjective.

Proof. By duality we need to prove that $\mathcal{K}_{r-1,1}(\mathcal{F},\omega_C\otimes\mathcal{H}^{-1}) = 0$. Let $\{s_0,\ldots,s_r\}$ be a basis for $H^0(\mathcal{F})$ and $\alpha=\sum s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_{r-1}}\otimes\alpha_{i_1i_2\ldots i_{r-1}}\in\bigwedge^{r-1}W\otimes H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$ be an element in the Kernel of the Koszul map $d_{r-1,1}$. The proposition then follows since the vector space $H^0(C,\omega_C\otimes\mathcal{H}^{-1}\otimes\mathcal{F})$ is zero in the former case and $\cong\mathbb{K}$ (cf. Thm. 1.2) in the latter. $\square$

2 The Picard scheme and the Hilbert scheme of C

Let $C=\sum_{i=1}^s n_i\Gamma_i$ be a curve contained in a smooth algebraic surface. Following the papers [2,3] we define the multidegree $\mathbf{d}$ of an invertible sheaf $\mathcal{F}$ as follows.

For each i the natural inclusion map $\epsilon_i:\Gamma_i\to C$ induces a map $\epsilon_i^*:\mathcal{F}\to\mathcal{F}_{|\Gamma_i}$. We let $d_i=\deg\mathcal{F}_{|\Gamma_i}$ and we define the multidegree of $\mathcal{F}$ on C $\mathbf{d}:=(d_1,...,d_s)$. Notice that we have $\deg_C\mathcal{F}=\sum_{i=1}^s n_id_i$. If $B=\sum m_i\Gamma_i$ is a subcurve of C, by $\mathbf{d}_B$ we mean the multidegree of $\mathcal{F}_{|B}$.

By $\mathrm{Pic}^{\mathbf{d}}(C)$ we denote the Picard scheme which parametrizes the classes of invertible sheaves of multidegree $\mathbf{d}$ and by $\mathrm{Pic}^{\mathbf{0}}(C)$ the Picard scheme of invertible sheaves of multidegree $\mathbf{0} = (0, \ldots, 0)$. We recall that for every $\mathbf{d} = (d_1, ..., d_s)$ there is an isomorphism $\mathrm{Pic}^{\mathbf{d}}(C) \cong \mathrm{Pic}^{\mathbf{0}}(C)$ and furthermore $\dim \mathrm{Pic}^{\mathbf{0}}(C) = h^1(C, \mathcal{O}_C)$ (cf. e.g. [4]).

Concerning the Picard group of C and the Picard group of a subcurve $B \subset C$ we have the following relations:

Remark 2.1 *Let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be a curve contained in a smooth algebraic surface and let $B \subset C$ be a subcurve. Then the exact sequence*

$$0 \to \mathcal{O}_D(-B) \to \mathcal{O}_C \to \mathcal{O}_B \to 0$$

(where $C = D + B$) induces a surjection $\mathrm{Pic}(C) = H^1(\mathcal{O}_C^*) \twoheadrightarrow H^1(\mathcal{O}_B^*) = \mathrm{Pic}(B)$. *Furthermore* $\mathrm{Pic}^{\mathbf{d}}(C) \twoheadrightarrow \mathrm{Pic}^{\mathbf{d}_B}(B)\ \forall \mathbf{d}$.

$\mathrm{Hilb}^{\delta}(C)$ denotes the Hilbert scheme of clusters on C of degree δ. Notice that $\mathrm{CaDiv}^{\delta}(C)$ is an open subset of $\mathrm{Hilb}^{\delta}(C)$ (cf. e.g. [16, §1]). From the results on clusters contained in a smooth algebraic surfaces (cf. the papers [14] by Iarrobino and [22] by Rego) we get the following

Proposition 2.2 *If C is contained in a smooth algebraic surface S then*

$$\dim(\mathrm{Hilb}^{\delta}(C)) = \delta$$

Proof. First of all let us consider the *Punctual Hilbert scheme* of degree n at a point $x \in C$:

$$\mathrm{Hilb}^n_x(C) = \{\zeta \subset C \,|\, \deg(\zeta) = n,\ \mathrm{Supp}(\zeta) = x\}.$$

We have $\mathrm{Hilb}^n_x(C) \subseteq \mathrm{Hilb}^n_x(S) = \{\zeta \subset S \,|\, \deg(\zeta) = n,\ \mathrm{Supp}(\zeta) = x\}$, and it is well known that $\dim(\mathrm{Hilb}^n_x(S)) = n - 1$ (cf. [14] or [22, p. 221]). Next, consider all the partitions of δ, i.e., for every $h = 1, \ldots, \delta$ let

$$\underline{\mathbf{N}}^{\mathbf{h}}(\delta) = \{\underline{n} = (n_1, \ldots, n_h) \in \mathbb{N}^h \,|\, 0 < n_1 \leq \ldots \leq n_h,\ \sum n_i = \delta\}$$

and for $\underline{n} = (n_1, \ldots, n_h) \in \underline{\mathbf{N}}^{\mathbf{h}}(\delta)$ set

$$\begin{aligned}\mathrm{Hilb}^{\delta}_{\underline{n}}(C) := \{(\zeta_1, \ldots, \zeta_h) \,|\, \zeta_i \text{ is a cluster} \subset C \text{ s.t.}\\ \mathrm{Supp}(\zeta_i) = P_i \in C,\ \deg(\zeta_i) = n_i\}.\end{aligned}$$

Now, by the above argument, $\dim(\mathrm{Hilb}^{\delta}_{\underline{n}}(C)) \leq \delta$ for each $\underline{n}$ and furthermore equality certainly holds for $\underline{n} = (1, \ldots, 1)$. Thus we can conclude since we can write

$$\mathrm{Hilb}^{\delta}(C)=\bigcup_{h=1}^{\delta}\{\bigcup_{\underline{n}\in\mathbb{N}^h(\delta)}\mathrm{Hilb}^{\delta}_{\underline{n}}(C)\}\quad\Box$$

We remark, as pointed out by Kleiman and Kleppe in the irreducible case (cf. [15]), that if there exists a point x such that $\dim T_{x,C}\geq 3$ then we have $\dim(\mathrm{Hilb}^n_x(C))\geq n$. This is the principal reason of the restriction of our analysis to curves contained in smooth algebraic surfaces.

3 Vanishing theorems for general divisors of low degree

In this section we prove two vanishing theorems for invertible sheaves of low degree on a curve contained in a smooth algebraic surface, generalizing some classical results for divisors on smooth projective curves. Notice that if C is not reduced we can no longer apply the usual dichotomy "*point - effective Cartier divisor*" and we can't use instruments like the "uniform position principle". However, we can obtain similar results, via Serre duality, considering the Abel map in its general form.

The first result of this section is the following

Theorem 3.1 *Assume $C=\sum_{i=1}^{s}n_i\Gamma_i$ to be a curve contained in smooth algebraic surface. Let $\mathbf{d}=(d_1,...,d_s)\in\mathbb{N}^s$ be such that for each invertible sheaf $\mathcal{G}'$ of multidegree $\mathbf{d}$ we have*

$$\deg\mathcal{G}'_{|B}\geq\ p_a(B)\quad\forall\ subcurve\ \ B\subseteq C\tag{1}$$

Then

$$W=\{[\mathcal{G}]\in\mathrm{Pic}^{\mathbf{d}}(C)|h^1(C,\mathcal{G})\neq 0\}\subseteq\mathrm{Pic}^{\mathbf{d}}(C)$$

has dimension $<\dim\mathrm{Pic}^{\mathbf{d}}(C)$, that is,for $[\mathcal{G}]$ general in $\mathrm{Pic}^{\mathbf{d}}(C)$, $H^1(C,\mathcal{G})=0$.

Proof. We prove the thesis by an induction argument on the number of components of C.

Let C be reduced and irreducible and assume $H^1(C,\mathcal{G})\neq 0$. Then by Serre duality this corresponds to say that there exists a non-zero morphism of sheaves $\varphi:\mathcal{G}\to\omega_C$, whose cokernel defines a cluster Δ of length $\delta\leq p_a(C)-2$:

$$0\to\mathcal{G}\xrightarrow{\varphi}\omega_C\to\mathcal{O}_\Delta\to 0$$

This means that there is a morphism Ψ (the generalized Abel map)

$$\begin{aligned}\Psi : \mathrm{CaDiv}^{\delta}(C) &\longrightarrow \mathrm{Pic}^{\mathbf{d}}(C)\\ \Delta &\mapsto [\omega_C \otimes \mathcal{I}_{\Delta}]\end{aligned}$$

and we deduce from this construction that $H^1(C, \mathcal{G}) \neq 0$ if and only if the class $[\mathcal{G}] \in \mathrm{im}(\Psi)$. This conclude the proof if C is reduced and irreducible since $\dim(\mathrm{im}(\Psi)) \leq \dim \mathrm{Hilb}^{\delta}(C) = \delta < \dim \mathrm{Pic}^{\mathbf{d}}(C)$.

Next, assume that $C = \sum_{i=1}^{s} n_i \Gamma_i$ and that there exists a non-zero morphism $\varphi : \mathcal{G} \to \omega_C$. We claim that for $[\mathcal{G}]$ general φ is generically onto. Indeed, if φ were not generically onto by automatic adjunction (lemma 1.1) it would factor as $\varphi_{|B} : \mathcal{G}_{|B} \hookrightarrow \omega_B$; since by Remark 2.1 $\mathrm{Pic}(C) \twoheadrightarrow \mathrm{Pic}(B)$ and by induction we may assume $\varphi_{|B} \equiv 0$ for a general $[\mathcal{G}]$, then we would get $\varphi \equiv 0$, absurd. But now the proof works exactly as in the irreducible case since then $\mathrm{coker}(\varphi)$ defines a cluster of $\deg = \delta \leq p_a(C) - 2$, which is a Cartier divisor.

Q.E.D.

The second step of our analysis is the study of base-point free systems of low degree. In this case the fundamental tool we apply is Thm. A.1 of the Appendix by Catanese. Our result is the following

Theorem 3.2 *Assume $C = \sum_{i=1}^{s} n_i \Gamma_i$ to be a curve contained in smooth algebraic surface. Let $\mathbf{d} = (d_1, ..., d_s) \in \mathbb{N}^s$ be such that for each invertible sheaf $\mathcal{F}'$ of multidegree $\mathbf{d}$ we have*

$$\deg \mathcal{F}'_{|B} \geq \; p_a(B) + 1 \quad \forall \;\; \textit{subcurve} \;\; B \subseteq C \tag{2}$$

Then for $[\mathcal{F}]$ general in $\mathrm{Pic}^{\mathbf{d}}(C)$, $|\mathcal{F}|$ is a base-point free system.

Proof. By Theorem 3.1 for $[\mathcal{F}]$ general we may assume $H^1(C, \mathcal{F}) = 0$. Thus $|\mathcal{F}|$ is base point free if and only if $H^1(C, \mathcal{M}_x \cdot \mathcal{F}) = 0$ for all $x \in C$. By Serre duality this is equivalent to $\mathrm{Hom}(\mathcal{M}_x \cdot \mathcal{F}, \omega_C) = 0$ for every x in C.

Assume there exists a point x and non-zero morphism $\varphi : \mathcal{M}_x \cdot \mathcal{F} \to \omega_C$. Then by automatic adjunction (lemma 1.1) there exists a subcurve B such that $\varphi_{|B} : (\mathcal{M}_x \cdot \mathcal{F})_{|B} \to \omega_B$ is generically onto. Thus we get an extension

$$0 \to (\mathcal{M}_x \cdot \mathcal{F})_{|B} \to \omega_B \to \mathcal{O}_{\Delta} \to 0 \tag{3}$$

where Δ is a 0-dimensional subscheme of B of length $\delta = \delta(B) = 2p_a(B) - 1 - \deg \mathcal{F}_{|B}$. Furthermore, by Thm. A.1 the class $[\mathcal{F}]$ of $\mathcal{F}$ in $\mathrm{Pic}^{\mathbf{d}_B}(B)$ is uniquely determined by $\Delta \in \mathrm{Hilb}^{\delta}(B)$ and $x \in B$. Thus the system $|\mathcal{F}|$ has some base point if and only if $[\mathcal{F}] \in \mathrm{im}(\Phi)$ where Φ is the morphism

$$\begin{aligned} \Phi : \mathrm{Hilb}^{\delta}(B) \times B &\longrightarrow \mathrm{Pic}^{\mathbf{d}_B}(B) \\ (\Delta,\ x) &\mapsto [\mathcal{F}] \end{aligned}$$

which associates to a couple $(\Delta, x) \in \mathrm{Hilb}^{\delta}(B) \times B$ the unique class $[\mathcal{F}]$ in $\mathrm{Pic}^{\mathbf{d}_B}(B)$ satisfying (3). Then we conclude by a pure dimension count since $\dim \mathrm{Pic}^{\mathbf{d}_B}(B) > \delta + 1 \geq \dim(\mathrm{im}(\Phi))$.

Q.E.D.

4 Normally generated adjoint divisors

This section is devoted to the proofs of Theorems A, B and C stated in the introduction.

Proof of Theorem A. For all $k \in \mathbb{N}$ we have to show the surjectivity of the maps

$$\rho_k : (H^0(C, \mathcal{H}))^{\otimes k} \longrightarrow H^0(C, \mathcal{H}^{\otimes k})$$

For $k = 0, 1$ it is obvious since C is numerically connected. For $k \geq 3$ we use induction applying proposition 1.5 to the sheaves $\mathcal{H}^{\otimes(k-1)}$ and $\mathcal{H}$.

Now we treat the case $k = 2$. By hypothesis $\mathcal{H} \overset{\text{num}}{\sim} \mathcal{F}' \otimes \mathcal{G}'$, where $\mathcal{G}'$ satisfies condition (1) and $\mathcal{F}'$ satisfies condition (2). Let $\mathbf{d_1}$ be the multidegree of $\mathcal{G}'$ and $\mathbf{d_2}$ be the multidegree of $\mathcal{F}'$. Take a general $\mathcal{G} \cong \mathcal{O}_C(\Delta)$ in $\mathrm{Pic}^{\mathbf{d_1}}(C)$, so that $\mathcal{F} := \mathcal{H}(-\Delta)$ is in general position. Then, by Theorems 3.1 and 3.2, $|\mathcal{F}|$ is a base point free system and $H^1(C, \mathcal{F}) = H^1(C, \mathcal{H} \otimes \mathcal{F}^{-1}) = 0$. Thus we can simply proceed as in the smooth case (cf. for example [21, Thm. 6]): from the exact sequence

$$0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{O}_\Delta \to 0$$

we get the following commutative diagram

$$\begin{array}{ccccc} H^0(\mathcal{F}) \otimes H^0(\mathcal{H}) & \hookrightarrow & H^0(\mathcal{H}) \otimes H^0(\mathcal{H}) & \twoheadrightarrow & H^0(\mathcal{O}_\Delta) \otimes H^0(\mathcal{H}) \\ r_1 \downarrow & & \rho_2 \downarrow & & r_3 \downarrow \\ H^0(\mathcal{F} \otimes \mathcal{H}) & \hookrightarrow & H^0(\mathcal{H}^{\otimes 2}) & \twoheadrightarrow & H^0(\mathcal{O}_\Delta \otimes \mathcal{H}) \end{array}$$

which allows us to conclude that ρ_2 is surjective since r_1 is onto by Prop. 1.5 and r_3 is onto because $|\mathcal{H}|$ is base point free.

Q.E.D. for Theorem A

Proof of Theorem B. The theorem follows if we find a decomposition $\mathcal{H} \overset{\text{num}}{\sim} \mathcal{F}' \otimes \mathcal{G}'$, with $\mathcal{F}'$ and $\mathcal{G}'$ as in Theorem A. We will show that such a decomposition does in fact exist if the exceptional cases (a), (b) and (c) do not occur.

To this aim, let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be our curve, and for all $i = 1, \ldots, s$ set $\gamma_i = \deg \omega_{C|\Gamma_i}$, $d_i = \deg \mathcal{H}_{|\Gamma_i}$. By our assumptions for all $i = 1, \ldots, s$ there exists an integer δ_i such that $\gamma_i \leq 2\delta_i \leq d_i$. Taking δ_i local transverse cuts on each irreducible component Γ_i, then we find an invertible sheaf $\mathcal{G}''$ of multidegree $(\delta_1, \ldots, \delta_s)$ such that

$$\deg \omega_{C|B} \leq \deg \mathcal{G}''^{\otimes 2}_{|B} \leq \deg \mathcal{H}_{|B}$$

i.e., by 1-connectedness $\deg \mathcal{G}''_{|B} \geq p_a(B)$ for all $B \subset C$, except possibly for $B = C$. Set $\mathcal{F}'' := \mathcal{H} \otimes \mathcal{G}''^{-1}$. Then we have $\deg \mathcal{F}''^{\otimes 2}_{|\Gamma_i} \geq \deg \mathcal{H}_{|\Gamma_i} \ \ \forall i$, whence $\deg \mathcal{F}''_{|B} \geq p_a(B) + 1$ for all $B \subseteq C$.

Case (i). If there exists an index h s.t. $\gamma_h < 2\delta_h \leq d_h$ then we simply take $\mathcal{F}' = \mathcal{F}''$ and $\mathcal{G}' = \mathcal{G}''$ since then the inequality $\deg \mathcal{G}'_{|C} \geq p_a(C)$ holds too.

Case (ii). Otherwise for all $i = 1, \ldots s$ $\deg \omega_{C|\Gamma_i}$ is even and $\deg \mathcal{A}_{|\Gamma_i} = 1$. In particular C is 2-connected and since case (c) does not hold there exists an irreducible component Γ_h of multiplicity n_h such that $\Gamma_h \cdot (C - \Gamma_h) \geq 4$. Taking $\mathcal{A}_h$ a general transverse cut on this component, we let $\mathcal{G}' := \mathcal{G}'' \otimes \mathcal{A}_h$, $\mathcal{F}' := \mathcal{F}'' \otimes \mathcal{A}_h^{-1}$, and we infer that $\mathcal{G}'$ and $\mathcal{F}'$ respectively satisfy condition (1) and condition (2) of Theorem A.

For $\mathcal{G}'$ this is obvious. About $\mathcal{F}'$ we have

$$\deg \mathcal{F}'_{|B} = \frac{\deg \omega_{C|B}}{2} + \deg \mathcal{A}_{|B} - \deg \mathcal{A}_{h|B}$$

Thus, if $B = C$ the required inequality holds since $\deg \mathcal{A}_{|B} - \deg \mathcal{A}_{h|B} \geq 2$ because case (a) and (b) do not occur, while if $B \subset C$ and $B \neq m_h \Gamma_h$ it holds because C is 2-connected and $\deg \mathcal{A}_{|B} - \deg \mathcal{A}_{h|B} \geq 1$. Finally, if $B = m_h \Gamma_h$ $(1 \leq m_h \leq n_h)$ it holds thanks to our choice of Γ_h.

Q.E.D. for Theorem B

Proof of Theorem C. The hypotheses of the theorem imply

$$\deg \omega_C^{\otimes 2}{}_{|B} = \deg \omega_B^{\otimes 2} + 2B \cdot (C - B) \geq 2p_a(B) + 1 \quad \forall B \subseteq C.$$

Furthermore the exceptional cases (a),(b),(c) of the above theorem do not occur by degree considerations. Thus ${\omega_C}^{\otimes 2}$ is normally generated by Theorem B. In particular we get the surjection

$$H^0(C, \omega_C^{\otimes 2}) \otimes H^0(C, \omega_C^{\otimes 2}) \twoheadrightarrow H^0(C, \omega_C^{\otimes 4}).$$

We conclude therefore that $R(C, \omega_C)$ is generated in degree ≤ 3 since for every integer $k \geq 5$ $H^0(C, \omega_C^{\otimes 2}) \otimes H^0(C, \omega_C^{\otimes k-2}) \twoheadrightarrow H^0(C, \omega_C^{\otimes k})$ by Prop. 1.5.

Q.E.D. for Theorem C

A Invertible sheaves and extension classes of 0-dimensional schemes

By

Fabrizio Catanese [1]

Let C be a curve over an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$ (C is thus projective and pure of dimension 1, but possibly reducible or nonreduced) and let $(\Delta, \mathcal{O}_\Delta)$, $(\zeta, \mathcal{O}_\zeta)$ be 0-dimensional subschemes of C.

As usual $\mathcal{I}_\zeta, \mathcal{I}_\Delta$ will denote the ideal sheaves of ζ, respectively Δ.

The following is the main result of this appendix

Theorem A.1 *Assume that* $, \zeta, \Delta$ *are* 0*-dimensional subschemes of a curve* C*, and that we are given two exact sequences*

$$0 \to \mathcal{I}_\zeta \mathcal{F} \to \mathcal{L} \to \mathcal{O}_\Delta \to 0$$

$$0 \to \mathcal{I}_\zeta \mathcal{F} \to \mathcal{L}' \to \mathcal{O}_\Delta \to 0$$

where $\mathcal{F}, \mathcal{L}, \mathcal{L}'$ *are* $\mathcal{O}_C$*-invertible sheaves. Then* $\mathcal{L}, \mathcal{L}'$ *are isomorphic.*

The above theorem gives thus an answer to a fundamental question stemming from the analysis of base points of linear systems, and is applied in the paper by Franciosi in the case where $\mathcal{I}_\zeta = \mathcal{M}_z$, the maximal ideal of a point z.

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Notice that if $e = e(\mathcal{L}), e' = e'(\mathcal{L}') \in \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F})$ are the respective extension classes associated to $\mathcal{L}, \mathcal{L}'$ through the above exact sequences, a sufficient condition in order that $\mathcal{L}, \mathcal{L}'$ be isomorphic, is the existence of automorphisms $\alpha \in \mathrm{Aut}(\mathcal{O}_\Delta), \beta \in \mathrm{Aut}(\mathcal{I}_\zeta\mathcal{F})$ so that $e = \beta e'\alpha$.

Therefore, in this section we will study the group $\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F})$, recalling some well known facts for the sake of clarity, and paying particular attention to the extension classes which arise from invertible sheaves.

Going back to the exact sequence defining Δ:

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_C \to \mathcal{O}_\Delta \to 0$$

let's apply $\mathcal{H}om_{\mathcal{O}_C}(-, \mathcal{I}_\zeta\mathcal{F})$ taking account that $\mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F}) = 0$.

We get

$$0 \to \mathcal{I}_\zeta\mathcal{F} \to \mathcal{H}om_{\mathcal{O}_C}(\mathcal{I}_\Delta, \mathcal{I}_\zeta\mathcal{F}) \to \mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F}) \to 0$$

and by the "local-to- global" spectral sequence of Ext (cf. [9, pp.263-264]) it follows

$$\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F}) \cong H^0(\mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F}))$$

Moreover, the previous exact sequence yields a homomorphism

$$H^0(\mathcal{H}om_{\mathcal{O}_C}(\mathcal{I}_\Delta, \mathcal{I}_\zeta\mathcal{F})) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F}))$$

with cokernel contained in $H^1(\mathcal{I}_\zeta\mathcal{F})$.

We observe however that, given any invertible sheaf $\mathcal{G}$, and an extension as above, we can tensor it with $\mathcal{G}$, so that our original problem is independent of the choice of $\mathcal{F}$. Choose then $\mathcal{F}$ sufficiently ample so that, by Serre's Theorem B(n), the cohomology group $H^1(\mathcal{I}_\zeta\mathcal{F})$ vanishes, and we have shown

Lemma A.2 *Extensions $0 \to \mathcal{I}_\zeta\mathcal{F} \to \mathcal{L} \to \mathcal{O}_\Delta$ as above are classified by the group $H^0(\mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F}))$.*

In view of the isomorphism $\mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta\mathcal{F}) \cong \mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta)$ it suffices to consider now the case $\mathcal{F} = \mathcal{O}_C$.

Let us now recall (cf. e.g. [12, pp.722-723]) the meaning of the surjection of sheaves

$$\mathcal{H}om_{\mathcal{O}_C}(\mathcal{I}_\Delta, \mathcal{I}_\zeta) \to \mathcal{E}xt^1_{\mathcal{O}_C}(\mathcal{O}_\Delta, \mathcal{I}_\zeta).$$

For a homomorphism $\varphi : \mathcal{I}_\Delta \to \mathcal{I}_\zeta$ the associated extension class is given as follows.

The homomorphism φ induces an embedding $\Phi : \mathcal{I}_\Delta \to \mathcal{I}_\Delta \oplus \mathcal{I}_\zeta$, defined as $\Phi := \mathrm{id} \oplus \varphi$; take then the exact sequence

$$0 \to \mathcal{I}_\Delta \oplus \mathcal{I}_\zeta \to \mathcal{O}_C \oplus \mathcal{I}_\zeta \to \mathcal{O}_\Delta \to 0$$

and factor out $\Phi(\mathcal{I}_\Delta)$: finally you get

$$0 \to ((\mathcal{I}_\Delta \oplus \mathcal{I}_\zeta)/\Phi(\mathcal{I}_\Delta)) \cong \mathcal{I}_\zeta \to (\mathcal{O}_C \oplus \mathcal{I}_\zeta)/\Phi(\mathcal{I}_\Delta) \to \mathcal{O}_\Delta \to 0$$

and we define $\mathcal{L}_\varphi := (\mathcal{O}_C \oplus \mathcal{I}_\zeta)/\Phi(\mathcal{I}_\Delta)$.

The first question : when does such an extension yield an invertible sheaf? is answered by the following

Proposition A.3 *Let C be an affine curve and let Δ, ζ be 0-dimensional subschemes of C.*

Let $\varphi : \mathcal{I}_\Delta \to \mathcal{I}_\zeta$ be a homomorphism and let $\mathcal{L}_\varphi$ be the corresponding (local) extension

$$0 \to \mathcal{I}_\zeta \to \mathcal{L}_\varphi \to \mathcal{O}_\Delta \to 0$$

Then $\mathcal{L}_\varphi$ is invertible if and only if

- *$\mathcal{I}_\zeta$ is invertible at the points $x \notin \mathrm{Supp}(\Delta)$*
- *φ is a local isomorphism at the points $x \in \mathrm{Supp}(\Delta)$.*

Proof. Since the rank of $\mathcal{L}_\varphi$ at each generic point is 1, $\mathcal{L}_\varphi$ is invertible if and only if for every point x in C $\dim_{\mathbb{K}_x} \mathcal{L}_\varphi/\mathcal{M}_x\mathcal{L}_\varphi = 1$ ($\mathbb{K}_x$ being the residue field at x).

Case (i): $x \notin \mathrm{Supp}(\Delta)$.

Then, locally at x, we have the isomorphism $\mathcal{L}_\varphi \cong \mathcal{I}_\zeta$. Then $\mathcal{L}$ is invertible if and only if $\mathcal{I}_\zeta$ is also invertible.

Case (ii): $x \in \mathrm{Supp}(\Delta)$.

In this case $\mathcal{I}_\Delta \subset \mathcal{M}_x$, whence $\dim \mathcal{L}_\varphi/\mathcal{M}_x\mathcal{L}_\varphi = 1$ if and only if

$$\dim(\mathcal{O}_C \oplus \mathcal{I}_\zeta)/(\mathcal{M}_x \oplus (\mathcal{M}_x\mathcal{I}_\zeta + \varphi(\mathcal{I}_\Delta)) = 1$$

i.e., if and only if $\dim \mathcal{I}_\zeta/(\mathcal{M}_x\mathcal{I}_\zeta + \varphi(\mathcal{I}_\Delta)) = 0$. By Nakayama's lemma, this is equivalent to $\varphi(\mathcal{I}_\Delta) = \mathcal{I}_\zeta$.

The proposition then follows by the following

Claim A.4 *$\varphi(\mathcal{I}_\Delta) = \mathcal{I}_\zeta$ if and only if φ is an isomorphism.*

Proof. We have $\mathrm{Ker}(\varphi) \subseteq \mathcal{I}_\Delta$ and furthermore $\mathrm{Ker}(\varphi)$ has 0-dimensional support since at each generic points ξ the morphism $\varphi : \mathcal{O}_{C,\xi} \to \mathcal{O}_{C,\xi}$ is onto if and only if it is injective (this can be easily seen since $\mathcal{O}_{C,\xi}$ is a finite dimensional $K(\xi)$-vector space).

If there were a point $x \in \mathrm{Supp}(\mathrm{Ker}(\varphi))$ and a nonzero function $g \in (\mathrm{Ker}(\varphi))_x$ then for each non-zero divisor $h \in \mathcal{M}_x$ there would exist a positive integer ν such that $g \cdot h^\nu = 0$. But this would imply $g = 0$ which is absurd.

Thus $\mathrm{Ker}(\varphi) = 0$. □ *Q.E.D. for Proposition A.3*

A corollary of the above proof yields

Corollary A.5 *Let as above C be an affine curve and Δ, ζ be 0-dimensional subschemes of C.*

Given an exact sequence

$$0 \to \mathcal{I}_\zeta \to \mathcal{L} \to \mathcal{O}_\Delta \to 0$$

with $\mathcal{L}$ invertible, then the above exact sequence is locally isomorphic to the exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_C \to \mathcal{O}_\Delta \to 0.$$

Proof. We know from the proof of the previous proposition that $\mathcal{I}_\zeta$ and $\mathcal{I}_\Delta$ are locally isomorphic, so, in handling the local situation we can reduce to the case where $\zeta = \Delta$ and $\mathcal{L} = \mathcal{O}_C$.

We have therefore two exact sequences

$$0 \to \mathcal{I}_\Delta \to \mathcal{L} \to \mathcal{O}_\Delta \to 0$$

and

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_C \to \mathcal{O}_\Delta \to 0,$$

with $\mathcal{L}$ invertible.

On the other hand we have a concrete expression of $\mathcal{L}$ as

$$\mathcal{L}_\varphi := (\mathcal{O}_C \oplus \mathcal{I}_\Delta)/\Phi(\mathcal{I}_\Delta).$$

We take a homomorphism $\iota : \mathcal{O}_C \to (\mathcal{O}_C \oplus \mathcal{I}_\Delta)$ given by $(id \oplus 0)$, and we obtain a homomorphism $\Psi : \mathcal{O}_C \to \mathcal{L}$ composing it with the projection onto $\mathcal{L}_\varphi := (\mathcal{O}_C \oplus \mathcal{I}_\Delta)/\Phi(\mathcal{I}_\Delta)$.

Clearly, Ψ is an isomorphism since $\iota \oplus \Phi$ yields an automorphism of $(\mathcal{O}_C \oplus \mathcal{I}_\Delta)$; moreover, $\psi(\mathcal{I}_\Delta) \subset (\mathcal{I}_\Delta) \oplus (\mathcal{I}_\Delta)$, whence $\psi(\mathcal{I}_\Delta) \subset (\mathcal{I}_\Delta) \oplus (\mathcal{I}_\Delta)/\Phi(\mathcal{I}_\Delta)$.

It follows that $\Psi^{-1}(\mathcal{I}_\Delta) \oplus (\mathcal{I}_\Delta)/\Phi(\mathcal{I}_\Delta)$ is an ideal containing $(\mathcal{I}_\Delta)$, but with the same colength, hence it equals $(\mathcal{I}_\Delta)$.

We can rephrase what proved insofar as follows: the isomorphism Ψ carries $(\mathcal{I}_\Delta)$ to itself and induces automorphisms $\alpha \in \mathrm{Aut}(\mathcal{O}_\Delta)$, and $\beta \in \mathrm{Aut}(\mathcal{I}_\zeta)$ such that $e = \beta e_\phi \alpha$, e_ϕ, e being the respective extension classes associated to the two given exact sequences. *Q.E.D. for Cor. A.5*

We are now ready to prove the main result of the appendix

Proof of Theorem A.1. From corollary A.5 follows that any extension

$$0 \to \mathcal{I}_\zeta \mathcal{F} \to \mathcal{L} \to \mathcal{O}_\Delta \to 0$$

is isomorphic to the extension

$$0 \to \mathcal{I}_\Delta \mathcal{L} \to \mathcal{L} \to \mathcal{O}_\Delta \to 0.$$

By our assumption and from corollary A.5 follows that we have an isomorphism

$$\tau : \mathcal{I}_\Delta \mathcal{L} \to \mathcal{I}_\Delta \mathcal{L}'$$

which is locally induced , for any choice of local trivializations of $\mathcal{L}$, resp. $\mathcal{L}'$, by a unit $t \in \mathcal{O}_C^*$.

We let now H be a sufficiently ample Cartier divisor so that $\mathcal{I}_\Delta \mathcal{L}(H)$ is generated by global sections, and we take a general section $s \in H^0(\mathcal{I}_\Delta \mathcal{L}(H))$.

Clearly, s induces a section $\sigma \in H^0(\mathcal{L}(H))$, and defining $s' := \tau(s)$, we obtain analogously a section $\sigma' \in H^0(\mathcal{L}'(H))$. However, by our local condition on τ, it follows that $div(\sigma) = div(\sigma')$: whence $\mathcal{L}(H) \cong \mathcal{L}'(H)$ or, equivalently $\mathcal{L} \cong \mathcal{L}'$.

Q.E.D. for Theorem A.1

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