Structures preserved by the QR-algorithm

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The shifted QR-algorithm

- Given a matrix $A \in \mathbb{C}^{n\times n}$.
  We want to compute the eigenvalues, eigenvectors of A.
- Initialization of the shifted QR-algorithm: $A^{(0)} = A$.
  QR-step: given $A^{(\nu)}$, we compute

$$A^{(\nu)} - \lambda I = QR \tag{1}$$

$$A^{(\nu+1)} = RQ + \lambda I, \tag{2}$$

with $\lambda \in \mathbb{C}$ the *shift*, Q unitary and R upper triangular.

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  with $\lambda \in \mathbb{C}$ the *shift*, Q unitary and R upper triangular.
- (1) and (2) imply the similarity relations

$$A^{(\nu+1)} = Q^H A^{(\nu)} Q$$

$$A^{(\nu+1)} = R A^{(\nu)} R^{-1}.$$

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- Similarity relations

$$A^{(\nu+1)} = Q^H A^{(\nu)} Q \tag{3}$$

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- Preservation of structure under the shifted QR-algorithm:
  (3) $\Rightarrow$ polynomial structures
  (4) $\Rightarrow$ rank structures.

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Definition

- *A polynomial structure on $\mathbb{C}^{n\times n}$ is defined as a collection $\mathcal{P} = \{p_k\}_k$, where each p_k is a polynomial in 7 variables.*
- *A matrix A is said to satisfy the structure $\mathcal{P} = \{p_k\}_k$ if for every k,*

$$p_k(A, A^H, A^{-1}, A^{-H}, \mathrm{Herm}_k, \mathrm{Uni}_k, (\mathrm{Rk}\ r)_k) = 0,$$

  for certain
  - Herm_k *Hermitian,*
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- *$\mathcal{M}$: set of matrices satisfying $\mathcal{P}$.*

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Theorem

Polynomial structure is strictly preserved by the shifted QR-algorithm, i.e. $A^{(\nu)} \in \mathcal{M} \Leftrightarrow A^{(\nu+1)} \in \mathcal{M}$.

PROOF.

- Any unitary matrix Q can be 'pulled through'.
- $\Rightarrow$ Polynomial structures satisfied by A, must carry over to $A_Q = Q^H A Q$. And conversely, by applying the same argument to $Q A_Q Q^H = Q(Q^H A Q)Q^H = A$.
- In particular, this holds for the Q-factor of the shifted QR-algorithm, and hence for the matrices $A^{(\nu)}$ and $A^{(\nu+1)} = Q^H A^{(\nu)} Q$.

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$$Q^H p(A, A^H, A^{-1}, A^{-H}, \mathrm{Herm}, \mathrm{Uni}, \mathrm{Rk}\ r)Q$$
$$= p(A_Q, A_Q^H, A_Q^{-1}, A_Q^{-H}, \mathrm{Herm}_Q, \mathrm{Uni}_Q, (\mathrm{Rk}\ r)_Q),$$

where

$$A_Q := Q^H A Q, \quad \mathrm{Herm}_Q := Q^H(\mathrm{Herm})Q, \quad \mathrm{Uni}_Q := Q^H(\mathrm{Uni})Q,$$
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Examples of polynomial structures

- Hermitian matrices: $A - A^H = 0$, or $A - \mathrm{Herm} = 0$,
- unitary matrices: $A^H - A^{-1} = 0$, or $A - \mathrm{Uni} = 0$,
- normal matrices: $AA^H - A^H A = 0$,
- unitary plus rank r correction: $A - \mathrm{Uni} - \mathrm{Rk}\ r = 0$,
- [Bini, Gemignani, Pan]: $A - \mathrm{Herm} - \mathrm{Rk}\ 1 = 0$,
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Definition

- *We define a rank structure on* $\mathbb{C}^{n\times n}$ *as a collection* $\mathcal{R} = \{\mathcal{B}_k\}_k$ *where each* $\mathcal{B}_k$ *is a 'structure block'.*

$$\mathcal{B}_k = (i_k, j_k, r_k, \lambda_k):$$

  - i_k: *row index,*
  - j_k: *column index,*
  - r_k: *rank upper bound,*
  - $\lambda_k \in \mathbb{C}$: *shift element.*

- *A matrix* $A \in \mathbb{C}^{n\times n}$ *satisfies the structure* $\mathcal{R}$ *if for every* k,

$$\operatorname{Rank} A_k(i_k : n, 1 : j_k) \leq r_k, \quad \textit{where } A_k := A - \lambda_k I.$$

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Definition

(Continuation)

- *As a special case, $\mathcal{R}$ is called a pure rank structure if all structure blocks $\mathcal{B}_k$ have shift element $\lambda_k = 0$.*
- *$\mathcal{M}$: set of matrices which satisfy $\mathcal{R}$.*
  $\mathcal{R}_{\text{pure}}$: pure rank structure.
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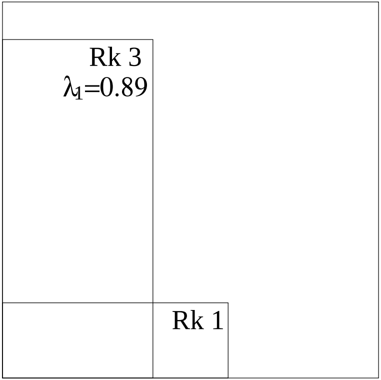
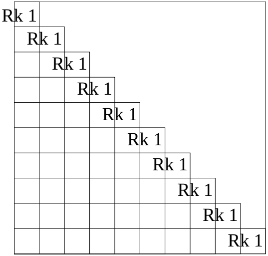
Example

Here is an example of a rank structure $\mathcal{R} = \{\mathcal{B}_1, \mathcal{B}_2\}$. The structure block $\mathcal{B}_1$ intersects the diagonal and has shift $\lambda_1 = 0.89$, while the structure block $\mathcal{B}_2$ is pure:



Example

Here is an example of a rank structure $\mathcal{R}_{\text{pure}} = \{\mathcal{B}_k\}_{k=1}^n$, yielding the class of lower semiseparable matrices:



Allowing shift elements λ_k, we get the class $\mathcal{R} = \{\mathcal{B}_k\}_{k=1}^n$ of lower semiseparable *plus diagonal* matrices.
The diagonal $\Lambda = \text{diag}(\lambda_k)_{k=1}^n$ is part of the structure.

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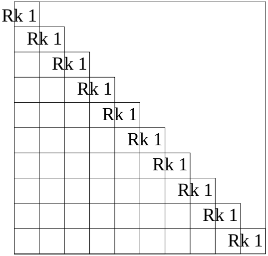

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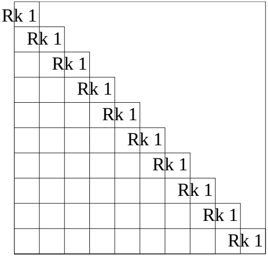

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Examples of rank structures

- Hessenberg matrices (+symmetry: tridiagonal)
- lower-semiseparable matrices (+symmetry: semiseparable)
- [Fasino] lower-semiseparable plus diagonal (+symmetry: semiseparable plus diagonal)
- Higher semiseparability ranks
- Also 'poorly ordered' structures are possible

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Theorem

(The nonsingular case:) For $A \in \mathcal{M}$ nonsingular we have

1. *rank structure is strictly preserved by applying a QR-step without shift on A;*
2. *factorizing $A = QR$, then Q satisfies the pure structure induced by $\mathcal{R}$.*

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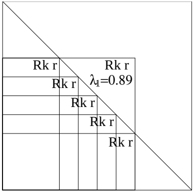

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Singular case

- We proved preservation of structure if A is nonsingular.
  What happens in the singular case?

Theorem

Let A satisfy a structure block $\mathcal{B}_k$.
By applying a QR-step without shift on A, the rank upper bound r_k of $\mathcal{B}_k$ can increase by at most $\#(\mathcal{I}_{\text{dep},A} \cap \mathcal{I}_{\text{left},k})$.

Example of $\mathcal{I}_{\text{left},k}$:

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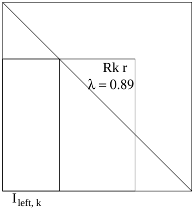

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Solution: apply a QR-step with 'suitable' choice of the QR-decomposition $A = QR$.

$$\Rightarrow \begin{cases} \text{Effectively eliminating QR-decompositions} \\ \text{Sparse Givens patterns} \end{cases}$$

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Givens transformations

- Given a matrix A, we can search a QR-decomposition by solving

$$\begin{cases} Q^H A & = & R \\ Q^H & = & (G^{(n-1)}_{n-1,n}) \dots (G^{(2)}_{2,3} \dots G^{(2)}_{n-1,n})(G^{(1)}_{1,2} \dots G^{(1)}_{n-1,n}). \end{cases}$$

$G^{(j)}_{i-1,i}$: Givens transformation acting on rows $i-1$ and i.

- For $n = 3$ this specializes to $(G^{(2)}_{2,3})(G^{(1)}_{1,2} G^{(1)}_{2,3})A = R$:

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Effectively eliminating QR-decompositions

$$\begin{cases} Q^H A &= R \\ Q^H &= (G^{(n-1)}_{n-1,n}) \dots (G^{(2)}_{2,3} \dots G^{(2)}_{n-1,n})(G^{(1)}_{1,2} \dots G^{(1)}_{n-1,n}). \end{cases}$$

Definition

A QR-decomposition $A = QR$ is called effectively eliminating if each non-trivial $G^{(j)}_{i-1,i}$ realizes a transition

$$\begin{bmatrix} 0 & \dots & 0 & a \\ 0 & \dots & 0 & b \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \dots & 0 & s \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix},$$

where $b \neq 0$ lies in the strictly lower triangular part of A.

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Effectively eliminating QR-decompositions

Example: for the matrix

$$A = \begin{bmatrix} 0 & \times & \times \\ 0 & 1 & \times \\ 0 & 1 & \times \end{bmatrix}, \quad (5)$$

we can solve $(G_{2,3}^{(2)})(G_{1,2}^{(1)} G_{2,3}^{(1)})A = R$ with

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Theorem

The effectively eliminating QR-decomposition of A is essentially unique, i.e. given $A = Q_1 R_1$ and $A = Q_2 R_2$ both effectively eliminating, we have that $Q_1 = Q_2 D$ for a certain unitary diagonal matrix D.

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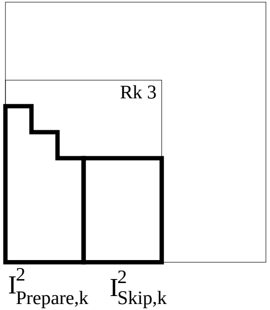
Outline

Sparse Givens patterns

Definition

Given a pure structure block $\mathcal{B}_k = (i_k, j_k, r_k)$.
We define the staircase shaped set $\mathcal{I}^2_{\mathrm{Prepare},k}$ and the rectangular shaped set $\mathcal{I}^2_{\mathrm{Skip},k}$ as illustrated.



Sparse Givens patterns

Reason for introducing $\mathcal{I}^2_{\text{Prepare},k}$, $\mathcal{I}^2_{\text{Skip},k}$:

Definition

Let $\mathcal{R}_{\text{pure}} = \{\mathcal{B}_k\}_k$ be such that $r_k =: r$ for all k.
A QR-decomposition $A = QR$ is said to satisfy the sparse Givens pattern induced by $\mathcal{R}_{\text{pure}}$ if $G^{(j)}_{i-1,i} = I_2$ for all $(i,j) \in \bigcup_k \mathcal{I}^2_{\text{Skip},k}$.

Idea:

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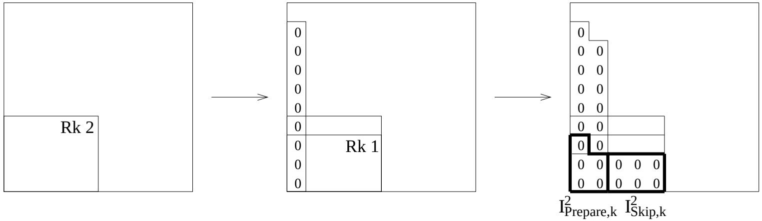

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Relative position:

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Types a, b: good behaviour (definition can be easily adapted)
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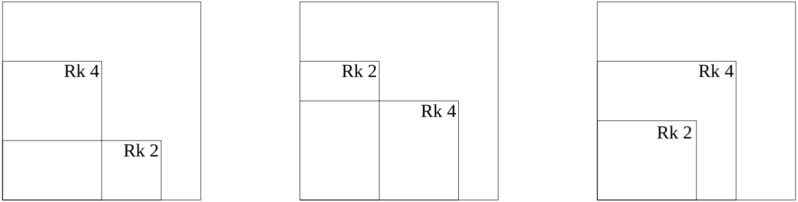

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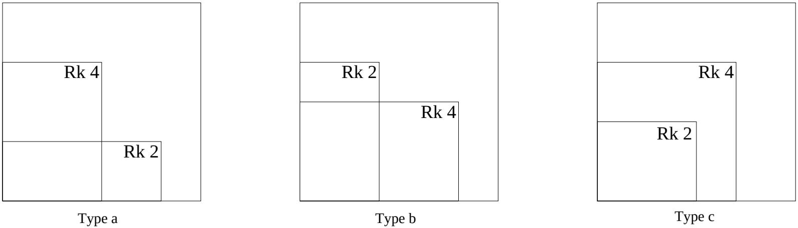

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We introduced now sparse Givens pattern induced by any pure structure $\mathcal{R}_{\text{pure}}$.

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Theorem

1. *for* $A \in \mathcal{M}_{\text{pure}}$*, we have the implication* *effectively eliminating* $\Rightarrow$ *sparse Givens pattern induced by* $\mathcal{R}_{\text{pure}}$*;*
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Preservation of structure by a QR-step:

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Given a structure $\mathcal{R}$ and its induced pure structure $\mathcal{R}_{\text{pure}}$.
Let $A \in \mathcal{M}$ be arbitrary, possibly singular.
When applying a QR-step without shift on A, we have the implications

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Example: lower semiseparable plus diagonal matrices

Example: let $\mathcal{R}$ be a lower ss+d structure, for some $\Lambda = \mathrm{diag}(\lambda_k)$.

Induced pure structure $\mathcal{R}_{\mathrm{pure}}$:

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Doing this in an effectively eliminating way: $\mathcal{R}$ is preserved.

By some additional theorems: when $A \in \mathcal{M}$ is unreduced, then (6) suffices to preserve $\mathcal{R}$.

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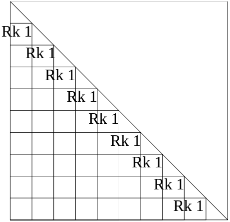

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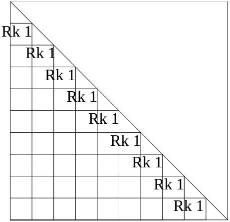

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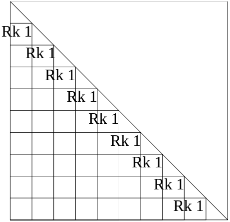

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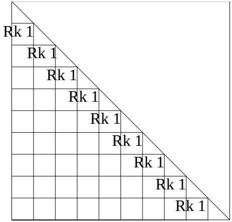

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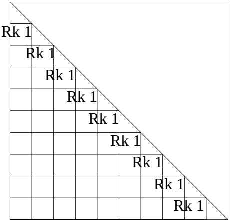

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